PRB. 15.12.

$$\begin{matrix} E \\ \pi \downarrow \\ M \end{matrix} \qquad p \in M \qquad E_p \supset E'_p \text{ subspace}$$

$E' := \bigcup_{p \in M} E'_p$, $\quad \pi' : E' \to M$, $\pi' = \pi|_{E'}$, THEN E' is a subbundle OF RANK ℓ $\Leftrightarrow$ $\forall p \in M$ $\exists U \ni p$ and $\sigma_1 : U \to E$, $\vdots$, $\sigma_\ell : U \to E$ smooth sect. of E over U.

such that for $\forall q \in U : \{\sigma_1(q), \ldots, \sigma_\ell(q)\}$ is a base of E'_q. (In particular, they are contained in E'_q).

$\Rightarrow$ It is evident: consider a v.bundle chart

$$\begin{tikzcd} E'|_U \arrow[rr, "\Phi", "\cong"'] \arrow[dr] & & U \times \mathbb{K}^\ell \arrow[dl] \\ & U & \end{tikzcd} \qquad \sigma_i(q) = \Phi^{-1}(q, e_i).$$

(a FRAME field over U).

$\Leftarrow$ Define trivializations:

$$\Phi' : E'|_U \cong U \times \mathbb{K}^\ell$$

$$\Phi'(e'_p) = (p, \lambda^1, \ldots, \lambda^\ell), \text{ where } e'_p = \lambda^i \sigma_i(p).$$

The transition maps will be formed by change of base matrices, smoothly dependent on p.

PRB 15.20. $t \in [0,\ell]$, $\begin{matrix} E \\ \pi \downarrow \\ [0,\ell] \end{matrix}$ with connection $\mathcal{H}$.

$\frac{\partial}{\partial t}$ is a vector field on $[0,\ell]$, and $\tilde{\partial}$ is its horiz. lift

1). $\gamma : [0,a] \to E$ of $\tilde{\partial}$ $\Rightarrow$ $\pi \circ \gamma$ is an integral curve of $\frac{\partial}{\partial t}$. (Indeed, $\dot{\gamma}_t = \tilde{\partial}_{\gamma(t)}$

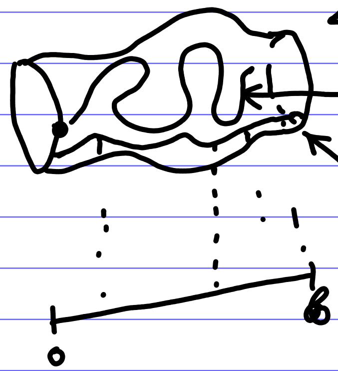
$$(\pi \circ \gamma)^\bullet_t = (d\pi)_{\gamma(t)} \dot{\gamma}_t = (d\pi)_{\gamma(t)} \tilde{\partial}_{\gamma(t)} = \frac{\partial}{\partial t}\Big|_{\pi\gamma(t)})$$

But the integral curve of $\frac{\partial}{\partial t}$ is $t \mapsto t + \text{const}$.

So $\pi\gamma(t) = t + \text{const}$ $\underset{(1)}{\equiv}$ $\varphi(t)$ s.t. $\frac{\partial \varphi}{\partial t} = \frac{d\varphi}{dt} = 1$

but $\pi\gamma(0) = 0 \Leftarrow \gamma(0) \in E_0$, so const$=0$

$\pi\gamma(t) = t$

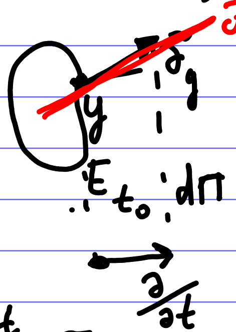


← vector bundle, but I can not draw it reasonably

γ – impossible

this γ is possible.

2). $\forall t_0 < b \;\; \exists\, \varepsilon = \varepsilon(t_0) > 0$ s.t. all integral curves of $\tilde{\partial}$ originating at a point in E_{t_0} are defined at least on $[t_0, \varepsilon)$.



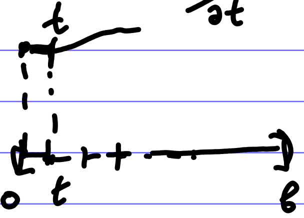
~~$\mathcal{H}_y$~~ is one-dimensional.

int curve is a solution of SODE a Cauchy problem.

$\exists$ a unique sol. locally.

for $\left(-\frac{\partial}{\partial t}\right)$ this is the same.

3).



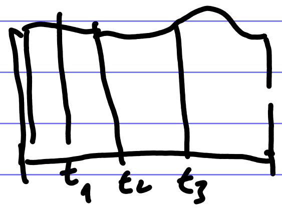
$t \to \varepsilon(t)$, $[t, \varepsilon(t))$

$\{[0, \varepsilon(0)),\ (t, \varepsilon(t))$ for $t > 0\}$

an open covering of $[0, b]$

$\to$ find a finite covering.

$0 = t_0, t_1, \dots, t_s$.



⊛ (see an addition below)

PRB. 15.24.

$$\begin{array}{ccc} E & & TE \\ \pi \downarrow & & d\pi \downarrow \\ M & & TM \end{array}$$

– a smooth map

A fiber bundle (do not say a vector bundle): indeed, locally: $\varphi_U : E|_U \cong U \times V$

$$TE|_U \cong \underbrace{TU}_{\text{open neigh. in } TM} \times TV \cong TU \times V$$

$$(d\pi)^{-1}(TU) = \{ v \in TE \mid d\pi(v) \in TU \}$$

$$v \in T_y E \Rightarrow d\pi(v) \in T_{\pi(y)} M$$

This means that $(d\pi)^{-1}(TU) = TE|_U$

In fact, if U is trivializing both for E and TM

Then. $TE|_U \cong TU \times V \rightsquigarrow \begin{matrix} TE \\ \downarrow \\ TM \end{matrix}$

$\cong$

$U \times W \times V \rightsquigarrow \begin{matrix} TE \\ \downarrow \\ M \end{matrix}$

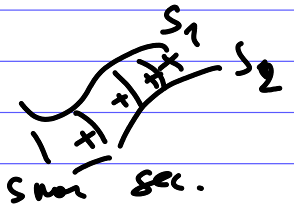
$\begin{matrix} TE \\ \downarrow \\ E \end{matrix}$

Now about the structure of a vector bundle on

To introduce it there are 2 ways:

1) Define linear structure on images of triv. maps for f.b. structure and verify that the corresp. TRANSITION maps will be fiberwise linear.

2) To define operations globally as smooth op (indep. of charts in each fiber)



$\longrightarrow$ $s_1 + s_2$ is also smooth.

$u \oplus v := (d\alpha)(u, v) \to \{u, v \in TE$ with $(d\pi)u = (d\pi)v$

where $\alpha(y_1, y_2) := y_1 + y_2$ for $(y_1, y_2) \in E + E$

$\begin{matrix} TE \ni u, v \\ \downarrow d\pi \\ TM \end{matrix}$ $\quad u \oplus v$ only if they are in the same fiber for i.e. $(d\pi)u = (d\pi)v$.

$\alpha : E \oplus E \to E$

$(d\alpha) : T(E \oplus E) \to TE$

$\qquad\qquad \| $

$\{(v, w) \in TE \times TE \mid (d\pi)v = (d\pi)w\}$

$c \odot v$

evid. smooth. So (to perform way № 2) we need to verify the axioms (8 axioms) of a linear space.

For ex.: Distributive law:

$$a \odot (v \boxplus w) = d\mu_a \circ (d\alpha)(v,w) =$$
$$= d(\mu_a \circ \alpha)(v,w) \quad \longleftarrow$$

Easier to use the 3d def.: $v = [\gamma(t)]$, $w = [\omega(t)]$

$$= [a \cdot (\gamma(t) + \omega(t))] = [a \cdot \gamma(t)] + [a \cdot \omega(t)] =$$
$$= (a \odot v) \boxplus (a \odot w).$$

On Friday at 14:00.

⊛ addition to Prb. 15.20.

While finding the curve "by stages" in 3), we do not know at which point of E_{t_i} we arrive. Thus we need an estimation (finding ε) in 2) simultaneously for all "starting" points e in E_{t_0}.

To obtain such $\varepsilon = \varepsilon(t_0)$ we first consider some inner product in E_{t_0} and the corresponding sphere $S = \{e \in E_{t_0} \mid |e| = \langle e, e\rangle^{1/2} = 1\}$. As above in 2) we find ε_e for each $e \in S$. By the same argument (from ODE) this ε_e (maximal) depends smoothly on the initial data (in particular, e). So this continuous (sufficient) function $e \longmapsto \varepsilon_e$ has minimum $0 < \varepsilon = \min_{e \in S} \varepsilon_e$, because S is a compact set. Now we wish to prove that this ε is valid for all E, not only for $S \subset E$. If $e \in E$ is an arbitrary element, then $e = a \cdot e_1$, where $e_1 \in S$ (for $|e| \neq 0$ one can take $e_1 = \frac{e}{|e|}$, $a = |e|$). We claim that $a\gamma(t)$ will be the integral curve of $\tilde{\partial}$, starting in ae, if γ was the integral curve of $\tilde{\partial}$ starting in $e \in E_{t_0}$ (and of course $a\gamma(t)$ is defined for $[t_0, t_0 + \varepsilon)$ too). Indeed, if $\dot{\gamma}|_{t_0} \in \mathcal{H}_e$, then $(a\gamma)^{\cdot}|_{t_0} = (d\mu_a)\dot{\gamma}|_{t_0} \in \mathcal{H}_{ae}$ by the property 2) of the Definition of an Ehresmann connection.