$$f(x) = x^3$$

M $(U_\alpha, \varphi_\alpha)$ N $(\tilde{U}_\beta, \tilde{\varphi}_\beta)$

$M = \mathbb{R}, Id$ $N = \mathbb{R}, \cancel{f} Id$

Two non-compatible structures on the same top. space.

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{Id} & \mathbb{R} \\ \varphi_1 \downarrow \; t \mapsto t & & \varphi_2 \downarrow \; t \mapsto t^3 \end{array}$$

diffeomorphism $\iff$ the structures are comp.

$$\varphi_2 \circ \mathrm{Id} \circ \varphi_1^{-1}(t) = t^3$$

$$\varphi_1 \circ \mathrm{Id}^{-1} \circ \varphi_2^{-1}(t) = \sqrt[3]{t}$$

and these smooth structures are distinct (not compatible).

But! These manifolds (as distinct manifolds) $\mathbb{R}, \varphi_1$ and $\mathbb{R}, \varphi_2$ are diffeomorphic! Indeed

Consider $f: t \longmapsto \sqrt[3]{t}$

Then:

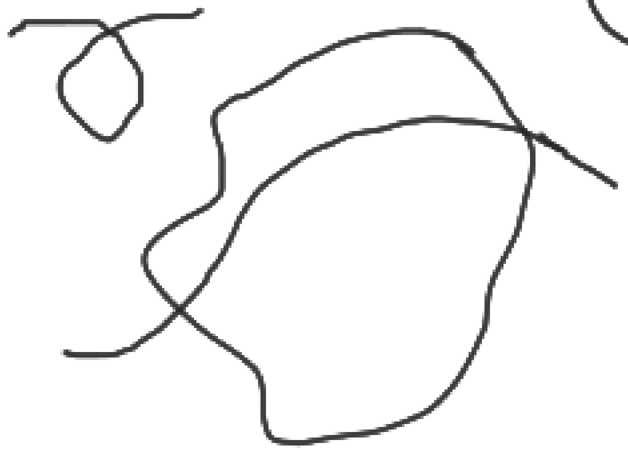
$$\begin{tikzcd} \cdot \arrow[rr, bend left, "f"] \arrow[d, "\varphi_1"'] & & \cdot \arrow[d, "\varphi_2"] \\ \cdot & & \cdot \end{tikzcd}$$

$$\varphi_2 \circ f \circ \varphi_1^{-1}(t) = \varphi_2 f(t) =$$

$$= \left(\sqrt[3]{t}\right)^3 = t$$

$$\varphi_1 \circ f^{-1} \circ \varphi_2^{-1}(t) = \left(\sqrt[3]{t}\right)^3 = t$$

this is a diffeo.



U_α, φ_α

$$\varphi_\alpha \circ \gamma : (-1,1) \longrightarrow \mathbb{R}^n$$

$$(x_\alpha^1(t), \dots, x_\alpha^n(t))$$

$$\gamma, (U_\alpha, \varphi_\alpha) \longrightarrow \left(\frac{dx_\alpha^1(t)}{dt}, \dots, \frac{dx_\alpha^n(t)}{dt}\right)\Bigg|_{t=0} = (\xi_\alpha^1, \dots)$$

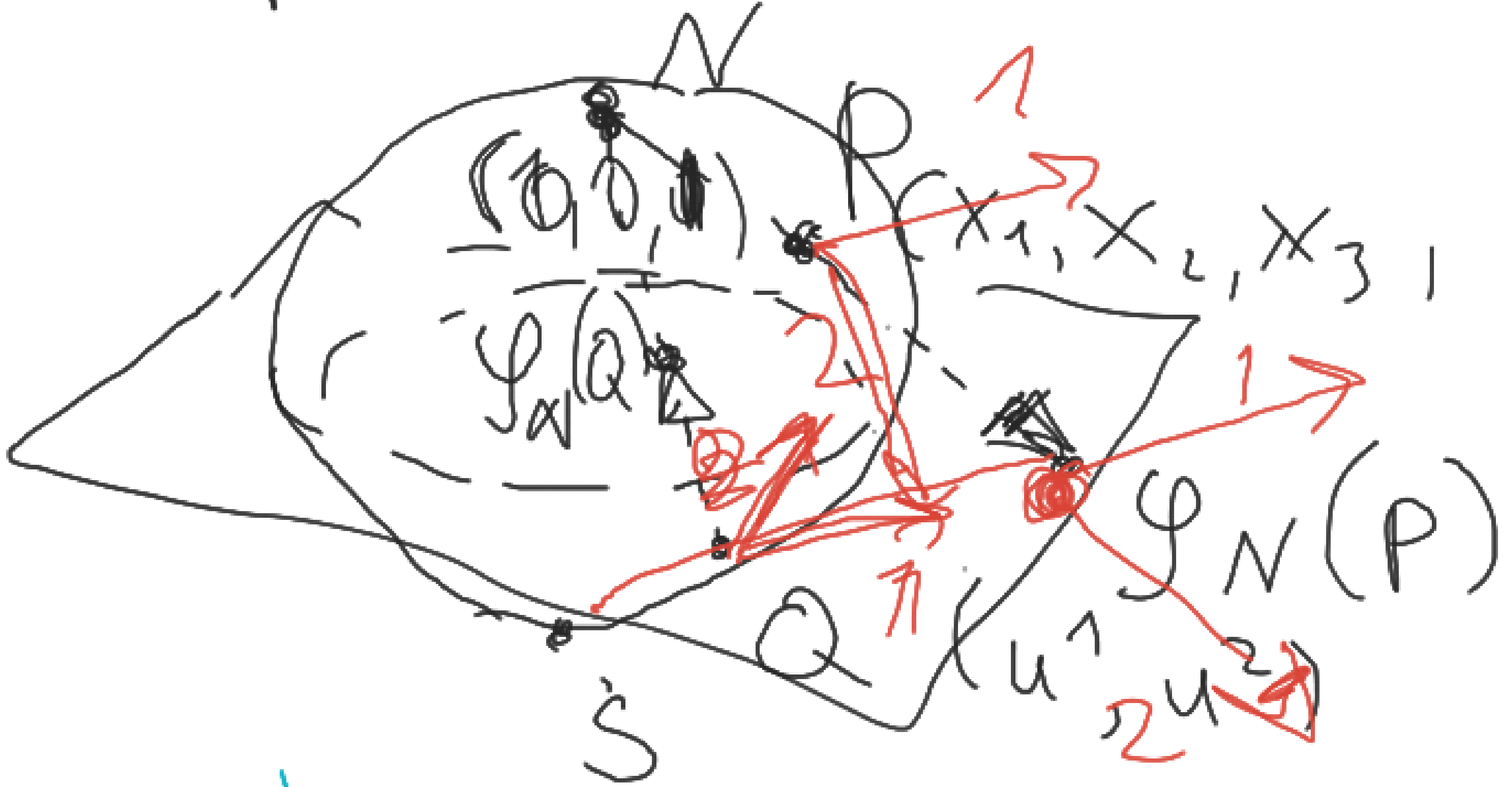
$$\gamma, (U_\beta, \varphi_\beta) \longrightarrow (\xi_\beta^1, \dots \xi_\beta^n) = \left(\frac{dx_\beta^1(t)}{dt}, \dots, \frac{dx_\beta^n(t)}{dt}\right)\Bigg|_{t=0} =$$

$$= \left(\frac{d}{dt}\left(x_\beta^1\left(x_\alpha^1(t), \dots, x_\alpha^n(t)\right)\right), \dots\dots \right) =$$

$$= \left(\frac{\partial x_\beta^1}{\partial x_\alpha^k} \cdot \underbrace{\frac{dx_\alpha^k(t)}{dt}}_{=\xi_\alpha^k}, \dots\dots\right)\Bigg|_{t=0};$$

$$\boxed{\xi_\beta^i = \frac{\partial x_\beta^i}{\partial x_\beta^k} \cdot \xi_\beta^k}$$

Complex-anal. structure on S^2



$$\varphi_N : S^2 \setminus N \to \mathbb{R}^2$$
$$U_N \qquad V_N$$
$$\varphi_S : S^2 \setminus S \xrightarrow{\sim} \mathbb{R}^2$$
$$U_S \qquad V_S$$

$$(0,0,1) + t(x_1, x_2, x_3 - 1), \quad 1 + t(x_3 - 1) = 0$$

$$u_1 = \frac{x_1}{1-x_3}, \quad u_2 = \frac{x_2}{1-x_3}. \qquad t = \frac{1}{1-x_3}$$

$$\varphi_N^{-1}(u_1,u_2)$$

$$(0,0,1)+t(u_1,u_2,-1)$$

$$(tu_1)^2+(tu_2)^2+(1-t)^2=1$$

$$t^2(u_1^2+u_2^2+1)-2t=0$$

$$t=\frac{2}{u_1^2+u_2^2+1}$$

$$x_1=\frac{2u_1}{u_1^2+u_2^2+1}$$

$$x_2=\frac{2u_2}{u_1^2+u_2^2+1}$$

$$x_3=1-\frac{2}{u_1^2+u_2^2+1}=\frac{u_1^2+u_2^2-1}{u_1^2+u_2^2+1}$$

$$\varphi_{S}\,\varphi_{N}^{-1}(u_1,u_2)=$$

$$=\varphi_S\left(\frac{2u_1}{u_1^2+u_2^2+1},\frac{2u_2}{u_1^2+u_2^2+1},\frac{u_1^2+u_2^2-1}{u_1^2+u_2^2+1}\right)$$

$$V_1=\frac{\frac{2u_1}{u_1^2+u_2^2+1}}{1+\frac{u_1^2+u_2^2-1}{u_1^2+u_2^2+1}}\,,\qquad V_2=\frac{\frac{2u_2}{u_1^2+u_2^2+1}}{1+\frac{u_1^2+u_2^2-1}{u_1^2+u_2^2+1}}$$

$v_1 = \dfrac{u_1}{u_1^2+u_2^2} \qquad v_2 = \dfrac{u_2}{u_1^2+u_2^2}$

$W = V_1 + i V_2$

$z = u_1 + i u_2$

$w = f(z) = \dfrac{1}{z}$ $\qquad \dfrac{1}{\bar z}$ instead

of φ_N : (complex conjug.) $\circ\, \varphi_N$

$\xrightarrow{\quad z \quad}$

$\mathbb{R}^2 = \mathbb{C}$

$w = \dfrac{1}{z}$ complex analytic

Suppose that you have a complex-an. manifold. Then all Jacoby matri. of Trans. func. $\boxed{\begin{pmatrix} A & B \\ -B & A \end{pmatrix}}$ (in Real coord)

(Cauchy–Riemann)

$$\frac{\partial u}{\partial x} = \left(\quad\right), \quad \left(\quad\right) = \left(\quad\right) \qquad \begin{matrix} u + iv \\ x + iy \end{matrix}$$

$\det J > 0$ You have transf. with positive Jacobians. ($\Leftrightarrow$ orient manifold)

2.27. ξ in I seien $\quad f \in C^\infty$

$$(U_\alpha, \varphi_\alpha) \longrightarrow (\xi_\alpha^1, \ldots, \xi_\alpha^n)$$

$$(x_\alpha^1, \ldots, x_\alpha^n)$$

$$J \cdot J^{-1} = E \qquad \delta_s^k$$

$$f \longmapsto \sum_{i=1}^n \frac{\partial f}{\partial x_\alpha^i} \xi_\alpha^i$$

(U_β, φ_β),

$$f \longmapsto \sum_{i=1}^n \frac{\partial f}{\partial x_\beta^i} \xi_\beta^i = \frac{\partial f}{\partial x_\alpha^k} \left(\frac{\partial x_\alpha^k}{\partial x_\beta^i} \cdot \frac{\partial x_\beta^i}{\partial x_\alpha^s} \right) \xi_\alpha^s$$

$$= \frac{\partial f}{\partial x_\alpha^s} \cdot \xi_\alpha^s$$

Differ. operator ?

1) linear

2) on germs

3) Newton-Leibnitz

In a choosen coord. system.

D_ξ

$$D_\xi(fg) = \frac{\partial(f\cdot g)}{\partial x^i_\alpha}\Big|_{p_0} \xi^i_\alpha = \left(\frac{\partial f}{\partial x^i_\alpha}\cdot g + f\cdot\frac{\partial g}{\partial x^i_\alpha}\right)\Big|_{p_0} \xi^i_\alpha =$$

$$= \left(\frac{\partial f}{\partial x^i_\alpha}\Big|_{p_0} \cdot \xi^i_\alpha\right) g(p_0) + f(p_0)\cdot\left(\frac{\partial g}{\partial x^i_\alpha}\Big|_{p_0}\right)\xi^i_\alpha$$

$$= D_\xi(f)\cdot g(p_0) + f(p_0)\cdot D_\xi(g)$$

$$\frac{\partial x^i_\beta}{\partial x^j_\alpha}\eta^j_\alpha = \frac{\partial x^i_\beta}{\partial x^j_\alpha}(\xi^j_\alpha + \xi^j_\alpha)\ldots \quad U_\gamma$$

$$\eta^i_\beta = \frac{\partial x^i_\beta}{\partial x^j_\alpha}\tilde{\xi}^j_\alpha + \frac{\partial x^i_\beta}{\partial x^j_\alpha}\tilde{\xi}^j_\alpha = \tilde{\xi}^i_\beta + \tilde{\xi}^i_\beta$$

wish:

$$\xi : U_\beta \to \; ?$$

$$\xi^k_\gamma = \frac{\partial x^k_\gamma}{\partial x^j_\alpha}\xi^j_\alpha \qquad \xi^i_\beta = \frac{\partial x^i_\beta}{\partial x^j_\alpha}\xi^j_\alpha$$

verify the tensor law

$$\xi^j_\alpha = \frac{\partial x^j_\alpha}{\partial x^i_\beta}\xi^i_\beta$$

$$\xi^k_\gamma = \left(\frac{\partial x^k_\gamma}{\partial x^j_\alpha}\frac{\partial x^j_\alpha}{\partial x^i_\beta}\right)\xi^i_\beta = \frac{\partial x^k_\gamma}{\partial x^i_\beta}\xi^i_\beta$$

$$\frac{\partial x^k_\alpha}{\partial x^i_\beta}\xi^i_\beta = \frac{\partial x^k_\alpha}{\partial x^i_\beta}\frac{\partial x^i_\beta}{\partial x^j_\alpha}\xi^j_\alpha$$

$$\delta^k_j$$