**Lecture 10**

## 3.2 AF-algebras

**Definition 3.6.** Let us call a $C^*$-algebra an *AF-algebra* (approximately finite-dimensional), if it is the closure of the union of an increasing sequence of its finite-dimensional $C^*$-subalgebras.

**Problem 49.** Prove that the matrix algebra $M_n$ is simple for any $n$ (this does not follow from Lemma 2.34, from which one can deduce that $M_n$ is simple for some $n$). *Hint:* for any ideal $I \neq \{0\}$ consider a matrix from it with $a_{ij} \neq 0$. By multiplying on the left and right by matrices with 1's in one place and zeros in the rest, you obtain a matrix of $I$ with a single nonzero element $a_{ij}$. Multiplying by permutation matrices, get similar matrices with all possible $i, j$. Their linear combinations give the entire $M_n$ algebra.

**Problem 50.** Deduce from Problem 49 and Lemma 2.34 that the image of the matrix algebra $M_n$ under a $*$-homomorphism is either a zero algebra or an algebra isomorphic to $M_n$.

**Problem 51.** Prove the following almost obvious fact: if $p$ and $q$ are projections of the same rank in $M_n$, then there exists a unitary matrix $u$ such that $q = u^*pu$.

**Lemma 3.7.** *Let $\varphi : M_n \to M_k$ be a non-zero $*$-homomorphism, so that $p := \varphi(1_n)$ is a self-adjoint projection, where $1_n$ is the unit of $M_n$. Then* $\mathrm{rk}(p) = \mathrm{Trace}(p)$ *is divided by* $n = \mathrm{rk}(1_n) = \mathrm{Trace}(1_n)$.

*Proof.* Consider some one-dimensional orthogonal (self-adjoint) projection $e \in M_n$. Then $\varphi(e)$ is a self-adjoint projection in $M_k$. Its rank does not depend on the choice of $e$, since any other $e'$ is equal to $u^*eu$ (by problem 51), where $u$ is unitary, so

$$\mathrm{Trace}(\varphi(e')) = \mathrm{Trace}(\varphi(u^*eu)) = \mathrm{Trace}(\varphi(u^*)\varphi(e)\varphi(u)) =$$

$$= \mathrm{Trace}(\varphi(u)\varphi(u^*)\varphi(e)) = \mathrm{Trace}(\varphi(uu^*e)) = \mathrm{Trace}(\varphi(e)).$$

If this (one for all) rank is zero, then $\varphi$ would be zero. This means it is equal to $c \geqslant 1$. Let us now consider an orthonormal basis $e_1, \dots, e_n$ (for example, canonical) in $\mathbb{C}^n$ and denote the corresponding one-dimensional orthoprojections by $[e_i]$, so $[e_j]\,[e_i] = 0$ for $i \neq j$. Then, since $\varphi([e_i])\varphi([e_j]) = \varphi([e_i e_j]) = 0$ for $i \neq j$, we get

$$\mathrm{Trace}(p) = \mathrm{rk}(\varphi(1_n)) = \mathrm{rk}(\varphi([e_1] \oplus \cdots \oplus [e_n])) = \mathrm{rk}(\varphi([e_1])) + \cdots + \mathrm{rk}(\varphi([e_n])) = cn.$$

□

**Definition 3.8.** The ratio $c := \frac{\mathrm{rk}(p)}{n}$ will be called *multiplicity* of $\varphi$.

Along with the standard left action of $M_n$ on $\mathbb{C}^n$, we consider the left action of $M_n$ on itself by multiplication, so the canonical expansion

$$M_n \cong \underbrace{\mathbb{C}^n \oplus \cdots \oplus \mathbb{C}^n}_{n \text{ times}} = M_n[e_1] \oplus \cdots \oplus M_n[e_n]$$

is a decomposition into simple modules (=irreducible representations), where $[e_i] \in M_n$ is an orthogonal projection onto the basis vector $e_i$ of the standard basis. Another way to write it is $[e_i] = e_i \otimes e_i^*$ (considering matrices as endomorphisms), where $e^*$ is the Hermitian conjugate functional for $e$, so $[e_i]v = (e_i \otimes e_i^*)v = e_i(e_i, v)$. For different vectors we get the matrix unit $e_{ij} = e_i \otimes (e_j)^*$, so $[e_i] = e_{ii}$.

**Lemma 3.9.** *Any irreducible left module $M$ in $M_n$ has the form $M_n(g \otimes f^*) = \mathbb{C}^n \otimes f^*$, where $g, f$ are some unit (can be taken to be unit) vectors.*

*Proof.* For the left action, the module $M_n(g \otimes f^*) = \mathbb{C}^n \otimes f^*$ is isomorphic to $\mathbb{C}^n$ with the standard action, and therefore is irreducible. Therefore, if $g \otimes f^* \in M$, then $M = M_n(g \otimes f^*)$. It remains to show that $M$ contains an element of the form $g \otimes f^*$. But this form describes any operator of rank 1. Indeed, if $a$ is an operator of rank 1, then we must take as $f$ the unit vector perpendicular to its kernel, and $g = a(f)$. Finally, if $M$ is nonzero and $0 \neq b \in M$, then choose $f \neq 0$ from its image. Then $(f \otimes f^*)b$ is an operator of rank 1 from $M$. □

**Theorem 3.10.** *Let $\varphi$ be a (unital) $*$-automorphism of the $C^*$-algebra $M_n$. Then it is inner: $\varphi(a) = vav^*$ for any $a$, where $v \in M_n$ is unitary.*

*Proof.* Note that $\varphi$ is an isomorphism between $M_n$, considered as a left module over $M_n$ with the standard action $a \cdot x$, and $M_n$, considered as a module with the action $a * x = \varphi(a) \cdot x$, since $\varphi(a \cdot x) = \varphi(a) \cdot \varphi(x) = a * \varphi(x)$. Since $\varphi(M_n) = M_n$, then the invariant and irreducible modules for both actions are the same (the latter are described by Lemma 3.9), and $\varphi(\mathbb{C} \otimes e_i) = \mathbb{C} \otimes h_i$, moreover, since the automorphism takes a direct sum to a direct sum, then $h_i$ form a basis in $\mathbb{C}^n$ and, thus, an isomorphism $u : \mathbb{C}^n \to \mathbb{C}^n$ is defined by $u : e_i \to h_i$ (even if we assume $\|h_i\| = 1$, then $u$ is uniquely defined only up to multiplication by a diagonal matrix of complex numbers modulo one). Thus, $\varphi(e_{ij}) = r_{ij} \otimes h_j$, where $r_{ij} \in \mathbb{C}^n$ are some elements. Similar reasoning with right modules shows that $\varphi(e_{ij}) = g_i \otimes (s_{ij})^*$, where $v : \mathbb{C}^n \to \mathbb{C}^n$, $v : e_i \to g_i$, is an isomorphism, and $s_{ij} \in \mathbb{C}^n$ are some elements. From these two relations we obtain that $\varphi(e_{ij}) = \lambda_{ij} g_i \otimes (h_j)^*$, where $\lambda_{ij}$ are some numbers. Moreover (see the proof of Lemma 3.7) $\varphi([e_i]) = \lambda_{ii} g_i \otimes (h_i)^*$ is a self-adjoint projection, so $h_i = \mu g_i$ and $g_i = (\lambda_{ij}\overline{\mu})(g_i \otimes (g_i)^*)(g_i) = (\lambda_{ij}\overline{\mu})g_i$ and $\lambda_{ij}\overline{\mu} = 1$. Thus, we can assume that $h_i = g_i$, $u = v$ and (new) $\lambda_{ij}$ satisfy $\lambda_{ii} = 1$ for any $i$. In this case, $g_i$ form an orthonormal basis (see the proof of Lemma 3.7), so $u : e_i \mapsto g_i$ is unitary. Since the homomorphism $\varphi$ preserves the equalities $e_{ij}e_{ji} = e_{ii} = [e_i]$ and $e_{ij}^* = e_{ji}$, we obtain that $\lambda_{ij}\lambda_{ji} = 1$, $\overline{\lambda_{ij}} = \lambda_{ji}$, so, in particular, $\lambda_{ij}$ are complex numbers of norm 1.

Consider the matrix $\Lambda = \|\lambda_{ij}\|$. Passing, if necessary, from vectors $g_i$ to $(\lambda_{1i})^{-1}g_i = \overline{\lambda_{1i}}g_i$ for $i = 2, \ldots, n$, we can consider $\lambda_{1i} = \lambda_{i1} = 1$ for $i = 2, \ldots, n$. At the same time, the image $\varphi(a)$ of a matrix $a = \|a_{ij}\|$ can be written down with respect to the orthonormal basis $\{g_i\}$ as $\|\lambda_{ij}a_{ij}\|$, and if the matrix $a$ was unitary, then $\varphi(a)$ must also be unitary. Let some $\lambda_{ij} \neq 1$ (that is possible only for $i \neq j$, $i \neq 1$). Taking for these $i, j$ the unitary matrix $a$ with $a_{1i} = 1/\sqrt{2}$, $a_{1j} = 1/\sqrt{2}$, $a_{ii} = 1/\sqrt{2}$, $a_{ij} = -1/\sqrt{2}$ (and the rest in these rows and columns, of course, is formed with zeros), we obtain the orthogonality condition for these rows in the form $0 = (1/\sqrt{2}\cdot 1)\overline{(1/\sqrt{2}\cdot 1)} + (-1/\sqrt{2}\cdot\lambda_{ij})\overline{(1/\sqrt{2}\cdot 1)} = (1-\lambda_{ij})/2$, so $\lambda_{ij} = 1$. Contradiction.

So, for any $i, j$ we get that $\varphi(e_i \otimes (e_j)^*) = v(e_i) \otimes (v(e_j))^* = v \cdot (e_i \otimes (e_j)^*) \cdot v^*$. Since any matrix is a linear combination of such matrix units, by linearity we obtain the required equality $\varphi(a) = vav^*$. □

**Lemma 3.11.** *Let the homomorphism $\varphi$ be unital under the conditions of Lemma 3.7. Then $\varphi$ is determined by multiplicity up to a unitary equivalence (conjugation) in $M_k$.*

*Proof.* Let $f_i^1, \dots, f_i^c$ be an orthonormal basis in the image of the projection $\varphi(e_i)$, $s = 1, \dots, n$. Denoting by $[f_i^j]$ the corresponding one-dimensional pairwise orthogonal projections, we have $\varphi(e_i) = [f_i^1] + \cdots + [f_i^c]$. Then $\{f_i^j\}$, $i = 1, \dots, n$, $j = 1, \dots, c$, is an orthobasis $\mathbb{C}^k$, $1_k = \sum_{i,j}[f_i^j]$. If $u \in M_k$ is a unitary matrix taking $\{f_i^j\}$ to the canonical basis of $\mathbb{C}^k$, where $uf_i^j = e_{(i-1)n+j}$, then

$$[e_{(i-1)n+j}]x = e_{(i-1)n+j}(e_{(i-1)n+j}, x) = uf_i^j(uf_i^j, x) = uf_i^j(f_i^j, u^*x) = u[f_i^j]u^*x \qquad (3.1)$$

for any $x \in \mathbb{C}^k$. That is why

$$\varphi([e_i]) = [f_i^1] + \cdots + [f_i^c] = \sum_{j=1}^{c} u^*[e_{(i-1)n+j}]u = u^* \left(\sum_{j=1}^{c}[e_{(i-1)n+j}]\right) u. \qquad (3.2)$$

Thus,

$$u\varphi(a)u^* = \begin{pmatrix} \varphi_1(a) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \varphi_c(a) \end{pmatrix} \text{ (block diagonal matrix)}, \qquad (3.3)$$

where $\varphi_i : M_n \to M_n$ is a non-zero homomorphism, and therefore an isomorphism ($i = 1, \dots, c$). Therefore, we can apply Theorem 3.10 to it and find a unitary $v_i \in M_n$, such that $\varphi_i(a) = v_i^* a v^i$. Denoting $v = v_1 \oplus \cdots \oplus v_c$ (a unitary element from $M_k$), we obtain that $vu\varphi(a)u^*v^* = S_c(a)$ for any $a \in M_n$, where $S_c : M_n \to M_k$, $k = cn$, is the standard homomorphism of multiplicity $c$:

$$S_c(a) = \begin{pmatrix} a & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a \end{pmatrix} \text{ (block diagonal matrix with } c \text{ blocks equal to } a),$$

as desired. □

**Lemma 3.12.** *Let $\varphi$ be a unital $*$-homomorphism of a finite-dimensional $C^*$-algebra $A = M_{n_1} \oplus \ldots \oplus M_{n_k}$ into a finite-dimensional $C^*$-algebra $B = M_{m_1} \oplus \ldots \oplus M_{m_l}$. Then $\varphi$ is given (up to unitary equivalence in $B$) by some $l \times k$-matrix $C = (c_{ij})$ with nonnegative elements, and $\sum_{j=1}^{k} c_{ij}n_j = m_i$.*

*Proof.* Let $\epsilon_i : B \to M_{m_i}$ be the canonical epimorphism, and $\sigma_j : M_{n_j} \to A$ the canonical embedding, $i = 1, \dots, l$, $j = 1, \dots, k$. Then $\epsilon_i \circ \varphi$ is a unital homomorphism of $A$ into $M_{m_i}$. Let $c_{ij}$ be the multiplicity of $\varphi_{ij} = \epsilon_i \circ \varphi \circ \sigma_j : M_{n_j} \to M_{m_i}$ in the sense of Definition 3.8.

Note that $\sigma_j(1_{n_j})$ are pairwise orthogonal self-adjoint projections in $A$ with their sum equal to one, so $p_{ij} := \varphi_{ij}(1_{n_j})$ are pairwise orthogonal self-adjoint projections in $M_{m_i}$, also with their sum equal to one. Therefore, as before, their direct sum is unitarily equivalent with the help of $u_i$ in $M_{m_i}$ to the sum of the corresponding diagonal projections $u_i p_{ij} u_i^*$. Applying Lemma 3.11 to each of $a \mapsto u_i \varphi_{ij}(a) u_i^*$, we find that $\epsilon_i \circ \varphi$ is unitarily equivalent (in $M_{m_i}$) to the homomorphism $\mathrm{id}_{n_1}^{c_{i1}} \oplus \ldots \oplus \mathrm{id}_{n_k}^{c_{ik}} = S_{c_{i1}} \oplus \cdots \oplus S_{c_{ik}}$. Comparison of dimensions gives the equality $\sum_{j=1}^{k} c_{ij} n_j = m_i$. Since $\varphi$ is determined by the direct sum $\epsilon_i \circ \varphi$, $i = 1, \dots, l$, the statement is proven. □

**Problem 52.** Suppose that in the previous lemma we exclude the requirement of unitality. Prove an analogue of the lemma in this case. Namely, take instead of $B$ the subalgebra $\varphi(A) = p\varphi(A)p$ in $B$, where $p = \varphi(1_A)$, apply the previous lemma to $\varphi : A \to \varphi(A)$ and obtain the statement of the lemma with inequalities $\sum_{j=1}^{k} c_{ij} n_j \leqslant m_i$ instead of equalities.