Lecture 10

2.7 Finite-dimensional C^*-algebras

Consider the $*$-weak topology on A defined by the seminorm system $a \mapsto |\varphi(a)|$ for all linear functionals φ. From Lemma 2.20 and Theorem 2.22 it follows that the same topology can be obtained by using only seminorms, defined by states.

Note also that the corresponding LTS has the homothety property 2.27.

Lemma 2.32. *A finite-dimensional C^*-algebra is always unital.*

Proof. If A is finite-dimensional, then the topology of the norm coincides with the $*$-weak topology according to Theorem 2.31. Let u_n be an approximate unit of the algebra A. Then for any state φ the sequence $\varphi(u_n)$ is non-decreasing and bounded, hence has a limit. Passing to linear combinations, we obtain the cinvergence for any functional on the finite-dimensional vector space A, in particular, for functionals $\varphi_1, \dots \varphi_k$ of the dual base for some base $a_1, \dots, a_k$. Then there exists an element $a \in A$ with $\varphi_i(a) = \lim_n \varphi_i(u_n)$. Considering linear combinations of φ_i, we conclude that $\varphi(a) = \lim_n \varphi(u_n)$ for any φ. Therefore u_n converges to a in $*$-weak topology, and therefore in norm. Then $ax = xa = x$ for any $x \in A$, so $a = 1$. □

Lemma 2.33. *Let $I \subset A$ be an ideal in a finite-dimensional C^*-algebra A. Then $I = Ap$ for some central projector (=idempotent from the center) p.*

Proof. Since I is finite-dimensional, it is unital by Lemma 2.32. Let $p \in I$ be the unit of I. Then for every $x \in A$, one has $xp \in I$, so $p(xp) = xp$. Hence $px^*p = x^*p$ for any $x \in A$, whence $xp = pxp = px$ and p belongs to the center of A. Obviously, $p^2 = p$. □

Lemma 2.34. *A simple finite-dimensional C^*-algebra A is isometrically $*$-isomorphic to the matrix algebra M_n for some n.*

Proof. First of all, note that $aAb \neq 0$ for any non-zero $a, b \in A$. Indeed, AaA is a non-zero ideal (since A is unital and $0 \neq a = 1 \cdot a \cdot 1 \in A$), so by simplicity, $AaA = A$. Therefore $1 = \sum_i x_i a y_i$ and $b = \sum_i x_i a y_i b$. Hence, if $ayb = 0$ for any $y \in A$, then $b = \sum_i x_i(ay_ib) = 0$. This contradicts the assumption.

Let B be some maximal commutative subalgebra of A. Then it can be identified with $C(X) = \mathbb{C}^n = \mathbb{C} \cdot e_1 \oplus \dots \oplus \mathbb{C} \cdot e_n$ for some n, where X consists of n points, and $e_i \in B$ denotes the element corresponding to the characteristic functions at point i. Here e_i are projections with the relations $e_i e_j = 0$ for $i \neq j$ and $\sum_{i=1}^n e_i = 1$. Since $e_iAe_i \cdot e_j = e_j \cdot e_iAe_i = 0$ and B is maximal, then $e_iAe_i \subset B$. Therefore $e_iAe_i = \mathbb{C} \cdot e_i$ (since, obviously, $0 \neq e_iAe_i \ni e_i$, or you can use the statement from the beginning of the proof).

For any i, j there is $x = x_{ij} \in A$ such that $x = e_ixe_j \neq 0$, $\|x\| = 1$. Indeed, by virtue of the statement from the beginning of the proof, $e_iAe_j \neq 0$, so we have $x = e_iye_j$ with $\|x\| = 1$. In this case $e_ixe_j = e_ie_iye_je_j = e_iye_j = x$. Then $x^*x = e_jx^*e_ie_ixe_j \in e_jAe_j$, and therefore, according to what has been proven, this element has the form αe_j, $\alpha \in \mathbb{C}$. Since

x^*x is a positive element with norm equal to one, then $\alpha = 1$, so $x^*x = e_j$. Likewise, $xx^* = e_i$. Let us denote such $x = x_{ij}$ for $j = 1$ by u_i, so that $u_i = e_i x e_1 = e_i u_i e_1$. Then $u_i^* u_i = e_1$, $u_i u_i^* = e_i$, $i = 1, \ldots, n$. Let us set $u_{ij} := u_i u_j^*$. In this case, $u_i e_1 u_i^* = u_i u_i^* u_i u_i^* = e_i e_i = e_i$, So $u_{ij} u_{ji} = u_i u_j^* u_j u_i^* = u_i e_1 u_i^* = e_i$. Also $e_j u_{ji} = u_j u_j^* u_j u_i^* = u_j e_1 u_i^* = u_j u_i^* u_i u_i^* = u_{ji} e_i$, and $e_i u_{ij} = u_i u_i^* u_i u_j^* = u_i e_1 u_j^* = u_i u_j^* u_j u_j^*$.

If $x \in e_i A e_j$, that is, $x = e_i a e_j$, then $x u_{ji} = e_i a e_j u_{ji} = e_i a u_{ji} e_i \in e_i A e_i$, so $x u_{ji} = \lambda e_i$ for some $\lambda \in \mathbb{C}$. Then $x = x e_j = x u_{ji} u_{ij} = \lambda e_i u_{ij} = \lambda u_{ij}$, so for any $x \in A$ there is a number $\lambda_{ij}(x) \in \mathbb{C}$ such that $e_i x e_j = \lambda_{ij}(x) u_{ij}$. Thus, $x = \sum_{i,j} e_i x e_j = \sum_{ij} \lambda_{ij}(x) u_{ij}$. The correspondence $x \mapsto (\lambda_{ij}(x))$ defines an isomorphism $\kappa : A \to M_n$ (Problem 47). □

Problem 47. Check the bijectivity and necessary algebraic properties of κ.