Lecture 11

Definition 3.13. Any A $*$-homomorphism between finite-dimensional C^*-algebras can be represented in the following graphical way. Let's represent A in the form k-tuple $\{(1,1)=n_1,\ldots,(1,k)=n_k\}$ corresponding $A\cong M_{n_1}\oplus\ldots\oplus M_{n_k}$, and B — in the form l-tuple $\{(2,1)=m_1,\ldots,(2,l)=m_l\}$ corresponding to $B\cong M_{m_1}\oplus\ldots\oplus M_{m_l}$. Let us represent φ using arrows between the elements of sets, and from $(1,j)$ to $(2,i)$ we draw c_{ij} arrows, the number is equal to the partial multiplicity. A sequence of such pictures for a sequence of homomorphisms $A_1\subset A_2\subset\ldots\subset A_p\subset\cdots$ is called the *Bratteli diagram of this sequence.* It is sometimes called the Bratteli diagram of an algebra, but the same algebra can be obtained from different sequences.

Problem 53. Draw Bratteli diagrams (for some defining sequences) of the following AF-algebras:

1) of the algebra of compact operators $\mathbb{K}(H)$;
2) of its unitization $\mathbb{K}(H)^+$;
3) the closure of the union of $A_p=M_{2^p}$, with embeddings $A_p\subset A_{p+1}$ of multiplicity 2 according to the formula $a\mapsto\begin{pmatrix}a&0\\0&a\end{pmatrix}$ (CAR algebra);
4) $C(K)$, where K is the Cantor set obtained from $[0,1]$ by successive by removing the middle third of the corresponding intervals. If K_p is a set, obtained at the pth step of this process, then A_p is an algebra of continuous functions constant on intervals of K_p;
5) $C(X)$, where $X:=\{0\}\cup\{\frac{1}{n}:n\in\mathbb{N}\}$, and A_k consists of all functions constant on $[0,1/2^k]$.

Lemma 3.14. *If two Bratteli diagrams coincide, then the corresponding AF-algebras are isometrically $*$-isomorphic.*

Proof. Let A_n and B_n be two sequences of finite-dimensional C^*-algebras with inclusions $\alpha_n:A_n\to A_{n+1}$, $\beta_n:B_n\to B_{n+1}$. Since the Bratteli diagrams are the same, then for each n there is an isomorphism $\varphi_n:A_n\to B_n$. Consider $\varphi_{n+1}\circ\alpha_n$ and $\beta_n\circ\varphi_n:A_n\to B_{n+1}$. They can differ only by unitary $u_{n+1}\in B_{n+1}$, that is $\beta_n\circ\varphi_n=\mathrm{Ad}_{u_{n+1}}\varphi_{n+1}\circ\alpha_n$, where $\mathrm{Ad}_{u_{n+1}}(a)=u_{n+1}a(u_{n+1})^*$.

Let's put $\psi_1=\varphi_1$, $v_1=1$. Let us define inductively $v_{n+1}=\beta_n(v_n)u_{n+1}\in B_{n+1}$, $\psi_{n+1}=\mathrm{Ad}_{v_{n+1}}\varphi_{n+1}$. Then

$$\begin{aligned}\beta_n\psi_n &= \beta_n\,\mathrm{Ad}_{v_n}\varphi_n=\mathrm{Ad}_{\beta_n(v_n)}\,\beta_n\varphi_n=\mathrm{Ad}_{\beta(v_n)}\,\mathrm{Ad}_{u_{n+1}}\varphi_{n+1}\alpha_n\\ &= \mathrm{Ad}_{\beta_n(v_n)u_{n+1}}\varphi_{n+1}\alpha_n=\psi_{n+1}\alpha_n.\end{aligned}$$

In this case $\cup_{n=1}^\infty\psi_n:\cup_{n=1}^\infty A_n\to\cup_{n=1}^\infty B_n$ is an isometric $*$-isomorphism, so the closures are also isomorphic. $\square$

One should not think that AF-algebras are "small" and that C^*-subalgebras of AF-algebras are AF-algebras again. For example, $C[0,1]$ is not an AF-algebra (since its only finite-dimensional C^*-subalgebra consists of constant functions), but it is a C^*-subalgebra of the AF-algebra $C(K)$ functions on the Cantor set. Indeed, let f be a function on K that has a dense set of rational numbers in the interval $[0,1]$ as its values. For example, the restriction of the Cantor function $f : [0,1] \to [0,1]$ [8, Ch. VI, §4] on K has all rationals of the form $p/2^s$ as its values. Its spectrum is a closure of this set, that is, equal to the entire interval $[0,1]$. Thus, $C^*(1,f) \subset C(K)$ is isometrically $*$-isomorphic to $C(\mathrm{Sp}(f)) = C[0,1]$.

3.3 Multipliers

We will call a C^*-subalgebra $\mathbb{B}(H)$ *non-degenerate*, if its natural representation in the Hilbert space H is non-degenerate.

Everywhere in this section $A, B \subset \mathbb{B}(H)$.

Definition 3.15. An operator $x \in \mathbb{B}(H)$ is called *left multiplier* A if $xA \subset A$. It is called *right multiplier*, if $Ax \subset A$ and *double (or double-sided) multiplier* or simply *multiplier*, if both conditions are met.

If A is unital, then every left (right) multiplier lies in A.

Since A is weakly dense in A'', we can proceed to the closure $xA \subset A$ and get $xA'' \subset A''$. If A'' is equal to one, then $x \in A''$.

Definition 3.16. The linear mapping $\sigma : A \to A$ is called *left centralizer*, if $\sigma(ab) = \sigma(a)b$ for any $a, b \in A$. Linear mapping $\sigma : A \to A$ called *right centralizer*, if $\sigma(ab) = a\sigma(b)$ for any $a, b \in A$. Pair (σ_1, σ_2) called *double centralizer*, if σ_1 is a right centralizer, σ_2 is a left centralizer and $\sigma_1(a)b = a\sigma_2(b)$ for any $a, b \in A$.

Lemma 3.17. *Any left centralizer is bounded.*

Proof. Let us assume the opposite. Then for any $n \in \mathbb{N}$ there is an element $x_n \in A$ such that $\|x_n\| < 1/n$ and $\|\sigma(x_n)\| > n$. This means that the series $a = \sum_{n=1}^{\infty} x_n x_n^*$ converges, so $a \in A$ and $x_n x_n^* \leqslant a$. According to Lemma 1.45, x_n can be written as $x_n = a^{1/4}u_n$, where $\|u_n\| \leqslant \|a\|^{1/4}$. Therefore, for any n we have $\|\sigma(x_n)\| = \|\sigma(a^{1/4})u_n\| \leqslant \|\sigma(a^{1/4})\| \cdot \|a^{1/4}\|$. A contradiction. $\square$

Theorem 3.18. *If A is non-degenerate, then there is a bijective correspondence between left (right, double) multipliers and left (right, double) centralizers.*

Proof. If x is a left multiplier, then the mapping $A \ni a \mapsto xa \in A$ is a left centralizer. If $xa = ya$ for any $a \in A$, then $(x-y)a = 0$ for any $a \in A''$, so $x = y$ in A''.

Let σ be a left centralizer, and u_λ be an approximate unit for A. Since the directed net $\{\sigma(u_\lambda)\}$ is bounded, then it has a point of accumulation in A'' (bounded sets are weakly compact in $\mathbb{B}(H)$ and accumulation points must lie in the closure of A). Let us denote one of the accumulation points by x. For any $a \in A$, the directed net $\{u_\lambda a\}$ converges in norm to a, so that $\sigma(u_\lambda a) = \sigma(u_\lambda)a$ converges to $\sigma(a)$. Then $xa = \sigma(a) \in A$, so x is a left multiplier. If $xA = 0$, then $\sigma = 0$. Note that if $y \in A''$ is another accumulation point, then $xa = \sigma(a) = ya$ for any $a \in A$, and $(x - y)a = 0$ for any $a \in A''$ (due to the strong density of A in A''), so $x = y$ in A''. Therefore, in this case there is only one point of accumulation.

A similar proof works also for right multipliers and right centralizers.

Let now x be a double multiplier. Then the mappings $\sigma_2 : a \mapsto xa$ and $\sigma_1 : a \mapsto ax$ are left and right multipliers, with $\sigma_1(a)b = (ax)b = a(xb) = a\sigma_2(b)$ for any $a, b \in A$, so x defines a double centralizer. Conversely, if (σ_1, σ_2) is a double centralizer, then, by what has been proven, σ_1 determines a right multiplier x_1, and σ_2 a left multiplier x_2. Since $ax_1b = \sigma_1(a)b = a\sigma_2(b) = ax_2b$ for any $a, b \in A$, we have $x_1 = x_2$, and $x_1 = x_2$ is a double multiplier. □

Problem 54. Let $\pi : A \to \mathbb{B}(H)$ be a degenerate representation. Let us denote by H_0 the invariant subspace $H_0 := \{\xi \in H \colon \pi(a)(\xi)0 \text{ for any } a \in A\}$. Prove that π induces a representation $\pi' : A \to \mathbb{B}(H/H_0)$, and if π was a faithful representation (an injective homomorphism), then so is π'.

Remark 3.19. Accordingly, until the end of this section we will consider non-degenerate $A \subseteq \mathbb{B}(H)$, so that (double) multipliers coincide with double centralizers.

The set of all left (right) multipliers of A is denoted by $LM(A)$ ($RM(A)$), and the set of all multipliers of A by $M(A)$.

Problem 55. Check that $RM(A) = (LM(A))^*$ and that $M(A) = LM(A) \cap RM(A)$, so that $M(A)$ is symmetric with respect to the involution.

It follows directly from the definition that all three sets are norm closed. Thus, $M(A)$ is a C^*-algebra (and the other two are, in the general case, only Banach algebras).

Problem 56. Let X be a locally compact space, and let $C_0(X)$, as before, be the C^*-algebra of continuous functions tending to 0 at infinity. Prove that the algebra $M(C_0(X)) \subset L^\infty(X)$ can be identified with the C^*-algebra $C_b(X)$ of all bounded continuous functions on X.

Example 3.20. If $A = \mathbb{K}(H)$, then $M(\mathbb{K}(H)) \subseteq \mathbb{B}(H)$, but any bounded operator is a multiplier (since $\mathbb{K}(H)$ is an ideal in $\mathbb{B}(H)$), so $M(\mathbb{K}(H)) = \mathbb{B}(H)$.

Definition 3.21. An ideal $A \subset B$ is said to be *essential* if any nonzero ideal B has a nontrivial intersection with A.

Let $A^\perp \subset B$ denote the set $A^\perp = \{x \in B \colon Ax = 0\}$.

Lemma 3.22. *An ideal $A \subset B$ is essential if and only if $A^\perp = 0$.*

Proof. Suppose that $A^\perp = 0$, but A is not essential. Then there is a nonzero ideal $J \subset B$ such that $A \cap J = \{0\}$. Let us take $j \in J$, $j \neq 0$. For any $a \in A$ we have $ja \in J \cap A$, so $ja = 0$ and $0 \neq j \in A^\perp$. A contradiction.

Conversely, let A be essential, but $A^\perp \neq 0$. Then there is an element $x \in A^\perp$ such that $x \neq 0$. Consider the ideal BxB (the closure of the set of all linear combinations of elements of the form $\sum_i b_i x b'_i$, $b_i, b'_i \in B$) and take an arbitrary $y \in BxB \cap A$. As it is known (for example, from Lemma 1.45), any element of the C^*-algebra admits a decomposition into the product of two of its elements, so we can write $y = z \cdot a$, where $z, a \in BxB \cap A$. We write $z = \sum_i b_i x b'_i$, so $y = za = \sum_i b_i x(b'_i a) = 0$, since $b'_i a \in A$, hence $x b'_i a = 0$, because $x \in A^\perp$. Therefore, $BxB \cap A = 0$ and we arrive to a contradiction. □

Lemma 3.23. *Let $A \subset B$ be an essential ideal. Then there is an embedding $B \subset M(A)$ that is identical on A.*

Proof. Consider $b \in B$. Then b defines the left and right centralizer A (since A is an ideal) $\sigma_2 : a \mapsto ba$ and $\sigma_1 : a \mapsto ab$, and $\sigma_1(a)a' = (ab)a' = a(ba') = a\sigma_2(a')$ for any $a, a' \in A$, so b defines a double centralizer, and hence a multiplier. So the mapping $\pi : B \to M(A)$ is defined, identical on A. This mapping is obviously a $*$-homomorphism. It remains to check whether π is injective. If $b \in \operatorname{Ker}\pi$, then $\sigma_1 = 0$ and $\sigma_2 = 0$ (see proof of theorem 3.18), so $bA = 0$, $Ab = 0$ and $b \in A^\perp$, which means $b = 0$. □

Note that the correspondence $A \mapsto M(A)$ is not a functor. For example, for $A = \mathbb{K}(H)$ and $B = A^+$, consider the embedding $A \subset B$. Wherein $M(A) = \mathbb{B}(H)$, and $M(B) = B$. Obviously, the embedding does not continue to these multiplier algebras. However, in some cases the transition to multipliers has some functorial properties.

Lemma 3.24. *Let $\varphi : A \to B$ be a surjective $*$-homomorphism of two C^*-algebras. Then it continues to a $*$-homomorphism $\bar\varphi : M(A) \to M(B)$.*

Proof. Let $\sigma \in LM(A)$ be a left centralizer. For any $b \in B$ we set $\bar\varphi(\sigma)(b) := \varphi(\sigma(\varphi^{-1}(b)))$. It is necessary check that the map is well defined, that is, its independence of the choice of a representative in $\varphi^{-1}(b) \subset A$. Due to linearity, it suffices to check that σ maps $\operatorname{Ker}\varphi$ to itself. Let $a \in \operatorname{Ker}\varphi$. Let us represent it in the form $a = a_1 \cdot a_2$, $a_1, a_2 \in \operatorname{Ker}\varphi$. Then $\varphi(\sigma(a)) = \varphi(\sigma(a_1)a_2) = 0$, since $\operatorname{Ker}\varphi$ is an ideal.

Thus, a left centralizer σ defines the mapping $\bar\varphi(\sigma)$, which is the left centralizer of B. The same is done for right and double centralizers (Problem 57). □

Problem 57. Prove that $\bar\varphi(\sigma)$ is a left centralizer. Construct an extension of a right centralizer in a similar way. Check that for a double centralizer we get a double centralizer.

Problem 58. Check that in the situation of the previous lemma the extension $\bar\varphi$ is also surjective.

Problem 59. Prove that a representation $\pi : A \to \mathbb{B}(H)$ is non-degenerate if and only if for some approximate unit u_λ of the algebra A the following condition is satisfied: for any vector $\xi \in H$ there is a λ such that that $u_\lambda(\xi) \neq 0$.

Lemma 3.25. *Let $B \subset A$ be an algebra and its C^*-subalgebra with a common approximate unit u_λ. Then $M(B) \subset M(A)$.*

Proof. If A is non-degenerate, then B is also non-degenerate by Problem 59. So $M(B) \subset B'' \subset A''$. For any $x \in A$, $y \in M(B)$, $yx = \lim y u_\lambda x \in A$, and similarly, $xy \in A$, so y is a multiplier of A. □