Chapter 3

Special classes of C^*-algebras

Lecture 11

3.1 C^*-algebra of compact operators

In this section we will consider C^*-subalgebras of C^*-algebra $\mathbb{K}(H)$ of compact operators on the Hilbert space H. We will say that C^*-subalgebra of the algebra $\mathbb{B}(H)$ *irreducible*, if its identical representation is irreducible.

Definition 3.1. The projection p is called *minimal*, if there is no projection $q \neq 0$, $q \neq p$ such that $qp = q$. In other words, p does not *dominate* any non-trivial projection.

Lemma 3.2. *Any nonzero C^*-algebra A consisting of compact operators contains a minimal projection e and $eAe = \mathbb{C} \cdot e$. If A is irreducible, then e is a rank 1 projection (as a projection in Hilbert space).*

Proof. Since A is nonzero, it contains a nonzero positive operator (see (1.7)), which (as is known from the basic course of functional analysis, see [5, Theorem 1, p. 360]), has a discrete spectrum (except of 0) with eigenvalues of finite multiplicities. Let us consider the spectral projection for a non-zero point of the spectrum. Since the characteristic function of this isolated point is continuous on the spectrum, then this projection belongs to A. Then among the nonzero projections dominated by it there is some projection $e \in A$ of minimal rank among the dominated (since they have finite ranks). Then e is minimal (the uniqueness of the minimal and even the equality of ranks of different minimal projections is not supposed). If eAe consists not only of $\mathbb{C} \cdot e$, then in the same way we can construct a projection dominated by e and arrive to a contradiction.

Now suppose that A is irreducible, but the rank of e is greater than 1. Let us choose a pair of nonzero orthogonal vectors ξ, η in the image e. Since for any a there is a number $\lambda \in \mathbb{C}$ such that $eae = \lambda e$, we have $(\xi, a\eta) = (e\xi, ae\eta) = (\xi, eae\eta) = \lambda(\xi, \eta)$, that is $a\eta \perp \xi$ for any $a \in A$. Considering all ξ from the image e being orthogonal to η, we see that the subspace $\overline{A\eta}$ is a proper invariant subspace. A contradiction. □

Lemma 3.3. *The only irreducible C^*-subalgebra of $\mathbb{K}(H)$ is itself.*

Proof. Let A be an irreducible C^*-subalgebra of $\mathbb{K}(H)$, and $e \in A$ a minimal projection of rank 1. Then there is a unit vector $\xi \in H$ such that $e\eta = \xi(\xi, \eta)$ for any η (we take ξ from the image of e). Due to irreducibility, for any $\eta, \zeta \in H$ there are elements $a, b \in A$ such that $a\xi = \eta$, $b\xi = \zeta$ (see Lemma 2.5). Moreover, $A \ni aeb^*$ and $aeb^*(\kappa) = a\xi(\xi, b^*\kappa) = \eta(\zeta, \kappa)$, $\kappa \in H$. Thus, A contains all operators of rank 1. Such operators generate $\mathbb{K}(H)$ (any compact operator is approximated by finite-dimensional), so $A = \mathbb{K}(H)$. □

Corollary 3.4. *The algebra $\mathbb{K}(H)$ is simple.*

Proof. Since $\mathbb{K}(H)$ is irreducible, then any non-zero ideal is also irreducible (by Lemma 2.39), so it coincides with $\mathbb{K}(H)$ (by Lemma 3.3). □

Corollary 3.5. *Let A be an irreducible C^*-subalgebra of $\mathbb{B}(H)$ containing a nonzero compact operator. Then $\mathbb{K}(H) \subseteq A$.*

Proof. Since $A \cap \mathbb{K}(H)$ is a nonzero ideal of A, it is irreducible by Lemma 2.39. By Lemma 3.3 this subalgebra of $\mathbb{K}(H)$ should coincide with the entire $\mathbb{K}(H)$. □

3.2 AF-algebras

Definition 3.6. Let us call a C^*-algebra an *AF-algebra* (approximately finite-dimensional), if it is the closure of the union of an increasing sequence of its finite-dimensional C^*-subalgebras.

Problem 49. Prove that the matrix algebra M_n is simple for any n (this does not follow from Lemma 2.34, from which one can deduce that M_n is simple for some n). *Hint:* for any ideal $I \neq \{0\}$ consider a matrix from it with $a_{ij} \neq 0$. By multiplying on the left and right by matrices with 1's in one place and zeros in the rest, you obtain a matrix of I with a single nonzero element a_{ij}. Multiplying by permutation matrices, get similar matrices with all possible i, j. Their linear combinations give the entire M_n algebra.

Problem 50. Deduce from Problem 49 and Lemma 2.34 that the image of the matrix algebra M_n under a $*$-homomorphism is either a zero algebra or an algebra isomorphic to M_n.

Problem 51. Prove the following almost obvious fact: if p and q are projections of the same rank in M_n, then there exists a unitary matrix u such that $q = u^*pu$.

Lemma 3.7. *Let $\varphi : M_n \to M_k$ be a non-zero $*$-homomorphism, so that $p := \varphi(1_n)$ is a self-adjoint projection, where 1_n is the unit of M_n. Then* $\mathrm{rk}(p) = \mathrm{Trace}(p)$ *is divided by* $n = \mathrm{rk}(1_n) = \mathrm{Trace}(1_n)$.

Proof. Consider some one-dimensional orthogonal (self-adjoint) projection $e \in M_n$. Then $\varphi(e)$ is a self-adjoint projection in M_k. Its rank does not depend on the choice of e, since any other e' is equal to u^*eu (by problem 51), where u is unitary, so

$$\mathrm{Trace}(\varphi(e')) = \mathrm{Trace}(\varphi(u^*eu)) = \mathrm{Trace}(\varphi(u^*)\varphi(e)\varphi(u)) =$$

$$= \mathrm{Trace}(\varphi(u)\varphi(u^*)\varphi(e)) = \mathrm{Trace}(\varphi(uu^*e)) = \mathrm{Trace}(\varphi(e)).$$

If this (one for all) rank is zero, then φ would be zero. This means it is equal to $c \geqslant 1$. Let us now consider an orthonormal basis $e_1, \ldots, e_n$ (for example, canonical) in $\mathbb{C}^n$ and denote the corresponding one-dimensional orthoprojections by $[e_i]$, so $[e_j]\,[e_i] = 0$ for $i \neq j$. Then, since $\varphi([e_i])\varphi([e_j]) = \varphi([e_i e_j]) = 0$ for $i \neq j$, we get

$$\mathrm{Trace}(p) = \mathrm{rk}(\varphi(1_n)) = \mathrm{rk}(\varphi([e_1] \oplus \cdots \oplus [e_n])) = \mathrm{rk}(\varphi([e_1])) + \cdots + \mathrm{rk}(\varphi([e_n])) = cn.$$

□

Definition 3.8. The ratio $c := \frac{\mathrm{rk}(p)}{n}$ will be called *multiplicity* of φ.

Along with the standard left action of M_n on $\mathbb{C}^n$, we consider the left action of M_n on itself by multiplication, so the canonical expansion

$$M_n \cong \underbrace{\mathbb{C}^n \oplus \cdots \oplus \mathbb{C}^n}_{n \text{ times}} = M_n[e_1] \oplus \cdots \oplus M_n[e_n]$$

is a decomposition into simple modules (=irreducible representations), where $[e_i] \in M_n$ is an orthogonal projection onto the basis vector e_i of the standard basis. Another way to write it is $[e_i] = e_i \otimes e_i^*$ (considering matrices as endomorphisms), where e^* is the Hermitian conjugate functional for e, so $[e_i]v = (e_i \otimes e_i^*)v = e_i(e_i, v)$. For different vectors we get the matrix unit $e_{ij} = e_i \otimes (e_j)^*$, so $[e_i] = e_{ii}$.

Lemma 3.9. *Any irreducible left module M in M_n has the form $M_n(g \otimes f^*) = \mathbb{C}^n \otimes f^*$, where g, f are some unit (can be taken to be unit) vectors.*

Proof. For the left action, the module $M_n(g \otimes f^*) = \mathbb{C}^n \otimes f^*$ is isomorphic to $\mathbb{C}^n$ with the standard action, and therefore is irreducible. Therefore, if $g \otimes f^* \in M$, then $M = M_n(g \otimes f^*)$. It remains to show that M contains an element of the form $g \otimes f^*$. But this form describes any operator of rank 1. Indeed, if a is an operator of rank 1, then we must take as f the unit vector perpendicular to its kernel, and $g = a(f)$. Finally, if M is nonzero and $0 \neq b \in M$, then choose $f \neq 0$ from its image. Then $(f \otimes f^*)b$ is an operator of rank 1 from M. □