Lecture 12

3.4 Hilbert C^*-modules

Definition 3.26. Let M be a Banach space (with norm $\|\cdot\|$), which at the same time is a right module over a C^*-algebra A (the action of A on M is assumed to be continuous). Let $\langle\cdot,\cdot\rangle : M \times M \to A$ be a sesquilinear form (called an *inner product*) with the properties:

1. $\langle m, na\rangle = \langle m, n\rangle a$ for all $m, n \in M$, $a \in A$;

2. $\langle m, n\rangle = \langle n, m\rangle^*$ for all $m, n \in M$;

3. $\langle m, m\rangle \in A$ is positive for all $m \in M$, and if it is equal to 0, then $m = 0$.

We call M a *Hilbert C^*-module* if $\|m\|^2 = \|\langle m, m\rangle\|$ for every $m \in M$.

Example 3.27. The algebra A is a Hilbert C^*-module over A if we define its inner product by the formula $\langle a, b\rangle := a^*b$, $a, b \in A$.

Example 3.28. The module A^n is a Hilbert C^*-module over A with an inner product, given by the formula $\langle (a_1, \dots, a_n), (b_1, \dots, b_n)\rangle := \sum_{i=1}^n a_i^* b_i$.

Lemma 3.29 (Cauchy(-Schwarz-Bunyakovsky) inequality)**.** *We have* $\|\langle n, m\rangle\|^2 \leqslant \|n\|^2\|m\|^2$ *for any* $n, m \in M$.

Proof. For all $A \in A$ we have $\langle m - na, m - na\rangle \geqslant 0$, so $\langle m, m\rangle - a^*\langle n, m\rangle - \langle m, n\rangle a + a^*\langle n, n\rangle a \geqslant 0$. Let's take $a = \frac{1}{\|n\|^2}\langle n, m\rangle$. Then

$$\langle m, m\rangle - \frac{2}{\|n\|^2}\langle m, n\rangle\langle n, m\rangle + \frac{1}{\|n\|^4}\langle m, n\rangle\langle n, n\rangle\langle n, m\rangle \geqslant 0.$$

Since $\|n\|^2 = \|\langle n, n\rangle\| \geqslant \langle n, n\rangle$, we obtain that $\langle m, m\rangle - \frac{1}{\|n\|^2}\langle m, n\rangle\langle n, m\rangle \geqslant 0$, so $\langle m, n\rangle\langle n, m\rangle \leqslant \|n\|^2\langle m, m\rangle$. From this we obtain the required inequality. □

Example 3.30. Let M consist of all sequences (a_i), $a_i \in A$ for which the series $\sum_{i=1}^\infty a_i^* a_i$ converges in A (in norm). The inner product is given by formula $\langle (a_i), (b_i)\rangle := \sum_{i=1}^\infty a_i^* b_i$. By the previous lemma, this series converges. This Hilbert C^* module is very important for applications. It is usually denoted by $l_2(A)$ and is called the *standard* Hilbert C^*-module.

A mapping $T : M \to M$ is called a (bounded) *operator* on a Hilbert C^*-module M if it is linear and A-linear (that is, $T(ma) = T(m)a$ for any $m \in M$, $a \in A$). If $M = A$, then this definition coincides with the definition of left centralizers.

Definition 3.31. An operator T is called *adjointable* if there is an operator S such that $\langle m, T(n)\rangle = \langle S(m), n\rangle$ for any $m, n \in M$. In this case, S is called the *adjoint operator* for T and is denoted by $T^\bigstar$. Let $\mathbb{B}_A^\bigstar(M)$ denote the set of operators admitting an adjoint.

Unlike Hilbert spaces, in Hilbert C^*-modules there are bounded operators that do not admit an adjoint.

Problem 60. Construct an example of an operator that does not admit an adjoint.

Problem 61. Prove that $\|x\| = \sup_{y\in B_1(M)} |\langle x, y\rangle|$, where $B_1(M) \subset M$ is the unit ball.

Theorem 3.32. *The algebra $\mathbb{B}_A^{\bigstar}(M)$ is a C^*-algebra.*

Outline of proof. The following points are the key ones.

1) The involution $\bigstar : \mathbb{B}_A^{\bigstar}(M) \to \mathbb{B}_A^{\bigstar}(M)$ is an isometry. Indeed, by Problem 61,

$$\|T\| = \sup_{x\in B_1(M)} \|Tx\| = \sup_{x,y\in B_1(M)} \|\langle Tx, y\rangle\| =$$

$$= \sup_{x,y\in B_1(M)} \|\langle x, T^{\bigstar} y\rangle\| = \sup_{y\in B_1(M)} \|T^{\bigstar} y\| = \|T^{\bigstar}\|.$$

2) The norm satisfies the C^*-property. This follows from the equality

$$\|T^{\bigstar}T\| \geqslant \sup_{x\in B_1(M)} \|\langle T^{\bigstar}Tx, x\rangle\| = \sup_{x\in B_1(M)} \|\langle Tx, Tx\rangle\| = \|T\|^2$$

(in the opposite direction is a general property of the operator norm, taking into account item 1).

3) The algebra $\mathbb{B}_A^{\bigstar}(M)$ is closed as a subalgebra of the Banach algebra B of all bounded $\mathbb{C}$-linear operators $M \to M$ (with operator norm). Really, first of all, note that the Banach algebra $\mathbb{B}_A(M)$ of all operators is closed in B as the intersection over all $x \in M$, $a \in A$, of closed sets $\mathrm{Ker}(f_{x,a})$, where $f_{x,a} : B \to M$, $f_{x,a}(T) = T(xa) - T(x)a$, is a bounded linear mapping, $\|f_{x,a}\| \leqslant 2\|x\|\,\|a\|$. Let a directed net of elements $T_\alpha \in \mathbb{B}_A^{\bigstar}(M)$ converge to $T \in \mathbb{B}_A(M)$, in particular, be a Cauchy directed net. According to item 1), the directed net $T_\alpha^{\bigstar}$ is also Cauchy directed net, and therefore has a limit $S \in \mathbb{B}_A(M)$. It is easy to see that $S = T^{\bigstar}$ and $T \in \mathbb{B}_A^{\bigstar}(M)$. □

Problem 62. Complete the proof of Theorem 3.32.

If $M = A$, then the definition of an operator admitting an adjoint coincides with the definition of a double centralizer.

Definition 3.33. The operator $\theta_{x,y}$, defined by the formula $\theta_{x,y}(z) = x\langle y, z\rangle$, called *elementary.*

Problem 63. Prove that $\theta_{x,y} \in \mathbb{B}_A^{\bigstar}(M)$ by providing an explicit formula for the adjoint.

Definition 3.34. The closure of the set of $\mathbb{C}$-linear combinations of elementary operators is denoted by $\mathbb{K}_A(M)$. Its elements are called *A-compact* operators.

Problem 64. Prove that $\mathbb{K}_A(M)$ is an ideal in $\mathbb{B}_A^{\bigstar}(M)$.

Problem 65. Prove that $\mathbb{K}_A(A) = A$. Note that if A is non-unital, then $\mathbb{K}_A(A) = A \neq \mathbb{B}_A^{\bigstar}(A) = DC(A)$ (algebra of double centralizers).

3.5 Calkin algebra

If a Hilbert space H is not separable, then $\mathbb{B}(H)$ can be quite complex, in particular, have more than just one proper ideal. For example, the set of operators with separable image is an ideal. This does not happen in the case of separable H, which we will usually restrict ourselves to.

Definition 3.35. The factor-C^*-algebra $Q(H) = \mathbb{B}(H)/\mathbb{K}(H)$ is called *Calkin algebra.*

Definition 3.36. The factor-C^*-algebra $M(A)/A$ is called *algebra of external multipliers,* or *generalized Calkin algebra.*

Lemma 3.37. *Calkin algebra is simple.*

Proof. We must prove that $\mathbb{K}(H)$ is the only ideal of $\mathbb{B}(H)$. Let an ideal $I \subset \mathbb{B}(H)$ contain at least one non-compact operator. We can assume that this operator is positive (since any operator is a linear combination of four positive ones). Let us denote it by t. Note that since $\mathbb{K}(H)$ is simple (by Corollary 3.4), then either $\mathbb{K}(H) \subseteq I$ or $\mathbb{K}(H) \cap I = \{0\}$. Since $t \notin \mathbb{K}(H)$, then we can assume that there is a number $\alpha > 0$ such that both spectral projections $p_1 = p_{[0,\alpha)}$ and $p_2 = p_{[\alpha,\infty)}$ are of infinite rank. Indeed, if there is no p_2 with the specified property, then t is compact, contrary to assumption. If $p_1 = 0$, then t is invertible, which means $I = \mathbb{B}(H)$, as required. If $p_1 \neq 0$, but its rank is finite, then we have a finite number of eigenvalues, less than α, so that for some function f continuous on the spectrum, we have $0 \neq f(t) \in \mathbb{K}(H)$. Hence, $\mathbb{K}(H) \subseteq I$. Therefore $p_1 \in I$ and $t + p_1 \in I$ is invertible and $I = \mathbb{B}(H)$. So, both projections p_1 and p_2 are of infinite rank. Let $H_i = \operatorname{Im} p_i$, $i = 1, 2$. Then $H_1 \perp H_2$ and $H = H_1 \oplus H_2$. Since H_1 and H_2 are isomorphic due to the separability of H, then there is a partial isometry $w \in \mathbb{B}(H)$ such that $w|_{H_2} = 0$ and $w|_{H_1}$ maps H_1 isometrically onto H_2. Since spectral projections commute with t, then $tp_2 \geqslant \alpha \cdot 1_{H_2}$, where 1_{H_2} is the identity operator in H_2. Also, $w^*tw \geqslant \alpha \cdot 1_{H_1}$, so $tp_2 + w^*tw \geqslant \alpha \cdot 1$. Therefore $tp_2 + w^*tw \in I$ is invertible, so $I = \mathbb{B}(H)$. $\square$