Lecture 12

Theorem 3.10. *Let φ be a (unital) $*$-automorphism of the C^*-algebra M_n. Then it is inner: $\varphi(a) = vav^*$ for any a, where $v \in M_n$ is unitary.*

Proof. Note that φ is an isomorphism between M_n, considered as a left module over M_n with the standard action $a \cdot x$, and M_n, considered as a module with the action $a * x = \varphi(a) \cdot x$, since $\varphi(a \cdot x) = \varphi(a) \cdot \varphi(x) = a * \varphi(x)$. Since $\varphi(M_n) = M_n$, then the invariant and irreducible modules for both actions are the same (the latter are described by Lemma 3.9), and $\varphi(\mathbb{C} \otimes e_i) = \mathbb{C} \otimes h_i$, moreover, since the automorphism takes a direct sum to a direct sum, then h_i form a basis in $\mathbb{C}^n$ and, thus, an isomorphism $u : \mathbb{C}^n \to \mathbb{C}^n$ is defined by $u : e_i \to h_i$ (even if we assume $\|h_i\| = 1$, then u is uniquely defined only up to multiplication by a diagonal matrix of complex numbers modulo one). Thus, $\varphi(e_{ij}) = r_{ij} \otimes h_j$, where $r_{ij} \in \mathbb{C}^n$ are some elements. Similar reasoning with right modules shows that $\varphi(e_{ij}) = g_i \otimes (s_{ij})^*$, where $v : \mathbb{C}^n \to \mathbb{C}^n$, $v : e_i \to g_i$, is an isomorphism, and $s_{ij} \in \mathbb{C}^n$ are some elements. From these two relations we obtain that $\varphi(e_{ij}) = \lambda_{ij} g_i \otimes (h_j)^*$, where λ_{ij} are some numbers. Moreover (see the proof of Lemma 3.7) $\varphi([e_i]) = \lambda_{ii} g_i \otimes (h_i)^*$ is a self-adjoint projection, so $h_i = \mu g_i$ and $g_i = (\lambda_{ij}\overline{\mu})(g_i \otimes (g_i)^*)(g_i) = (\lambda_{ij}\overline{\mu}) g_i$ and $\lambda_{ij}\overline{\mu} = 1$. Thus, we can assume that $h_i = g_i$, $u = v$ and (new) λ_{ij} satisfy $\lambda_{ii} = 1$ for any i. In this case, g_i form an orthonormal basis (see the proof of Lemma 3.7), so $u : e_i \mapsto g_i$ is unitary. Since the homomorphism φ preserves the equalities $e_{ij} e_{ji} = e_{ii} = [e_i]$ and $e_{ij}^* = e_{ji}$, we obtain that $\lambda_{ij}\lambda_{ji} = 1$, $\overline{\lambda_{ij}} = \lambda_{ji}$, so, in particular, λ_{ij} are complex numbers of norm 1.

Consider the matrix $\Lambda = \|\lambda_{ij}\|$. Passing, if necessary, from vectors g_i to $(\lambda_{1i})^{-1} g_i = \overline{\lambda_{1i}} g_i$ for $i = 2, \dots, n$, we can consider $\lambda_{1i} = \lambda_{i1} = 1$ for $i = 2, \dots, n$. At the same time, the image $\varphi(a)$ of a matrix $a = \|a_{ij}\|$ can be written down with respect to the orthonormal basis $\{g_i\}$ as $\|\lambda_{ij} a_{ij}\|$, and if the matrix a was unitary, then $\varphi(a)$ must also be unitary. Let some $\lambda_{ij} \neq 1$ (that is possible only for $i \neq j$, $i \neq 1$). Taking for these i, j the unitary matrix a with $a_{1i} = 1/\sqrt{2}$, $a_{1j} = 1/\sqrt{2}$, $a_{ii} = 1/\sqrt{2}$, $a_{ij} = -1/\sqrt{2}$ (and the rest in these rows and columns, of course, is formed with zeros), we obtain the orthogonality condition for these rows in the form $0 = (1/\sqrt{2} \cdot 1)(\overline{1/\sqrt{2} \cdot 1}) + (-1/\sqrt{2} \cdot \lambda_{ij})(\overline{1/\sqrt{2} \cdot 1}) = (1 - \lambda_{ij})/2$, so $\lambda_{ij} = 1$. Contradiction.

So, for any i, j we get that $\varphi(e_i \otimes (e_j)^*) = v(e_i) \otimes (v(e_j))^* = v \cdot (e_i \otimes (e_j)^*) \cdot v^*$. Since any matrix is a linear combination of such matrix units, by linearity we obtain the required equality $\varphi(a) = vav^*$. □

Lemma 3.11. *Let the homomorphism φ be unital under the conditions of Lemma 3.7. Then φ is determined by multiplicity up to a unitary equivalence (conjugation) in M_k.*

Proof. Let $f_i^1, \dots, f_i^c$ be an orthonormal basis in the image of the projection $\varphi(e_i)$, $s = 1, \dots, n$. Denoting by $[f_i^j]$ the corresponding one-dimensional pairwise orthogonal projections, we have $\varphi(e_i) = [f_i^1] + \dots + [f_i^c]$. Then $\{f_i^j\}$, $i = 1, \dots, n$, $j = 1, \dots, c$, is an orthobasis $\mathbb{C}^k$, $1_k = \sum_{i,j} [f_i^j]$. If $u \in M_k$ is a unitary matrix taking $\{f_i^j\}$ to the canonical basis of $\mathbb{C}^k$, where $u f_i^j = e_{(i-1)n+j}$, then

$$[e_{(i-1)n+j}]x = e_{(i-1)n+j}(e_{(i-1)n+j}, x) = u f_i^j (u f_i^j, x) = u f_i^j (f_i^j, u^* x) = u [f_i^j] u^* x \qquad (3.1)$$

for any $x \in \mathbb{C}^k$. That is why

$$\varphi([e_i]) = [f_i^1] + \cdots + [f_i^c] = \sum_{j=1}^{c} u^*[e_{(i-1)n+j}]u = u^* \left(\sum_{j=1}^{c} [e_{(i-1)n+j}] \right) u. \qquad (3.2)$$

Thus,

$$u\varphi(a)u^* = \begin{pmatrix} \varphi_1(a) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \varphi_c(a) \end{pmatrix} \text{ (block diagonal matrix)}, \qquad (3.3)$$

where $\varphi_i : M_n \to M_n$ is a non-zero homomorphism, and therefore an isomorphism ($i = 1, \ldots, c$). Therefore, we can apply Theorem 3.10 to it and find a unitary $v_i \in M_n$, such that $\varphi_i(a) = v_i^* a v^i$. Denoting $v = v_1 \oplus \cdots \oplus v_c$ (a unitary element from M_k), we obtain that $vu\varphi(a)u^*v^* = S_c(a)$ for any $a \in M_n$, where $S_c : M_n \to M_k$, $k = cn$, is the standard homomorphism of multiplicity c:

$$S_c(a) = \begin{pmatrix} a & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a \end{pmatrix} \text{ (block diagonal matrix with } c \text{ blocks equal to } a\text{)},$$

as desired. □

Lemma 3.12. *Let φ be a unital $*$-homomorphism of a finite-dimensional C^*-algebra $A = M_{n_1} \oplus \ldots \oplus M_{n_k}$ into a finite-dimensional C^*-algebra $B = M_{m_1} \oplus \ldots \oplus M_{m_l}$. Then φ is given (up to unitary equivalence in B) by some $l \times k$-matrix $C = (c_{ij})$ with nonnegative elements, and $\sum_{j=1}^{k} c_{ij} n_j = m_i$.*

Proof. Let $\epsilon_i : B \to M_{m_i}$ be the canonical epimorphism, and $\sigma_j : M_{n_j} \to A$ the canonical embedding, $i = 1, \ldots, l$, $j = 1, \ldots, k$. Then $\epsilon_i \circ \varphi$ is a unital homomorphism of A into M_{m_i}. Let c_{ij} be the multiplicity of $\varphi_{ij} = \epsilon_i \circ \varphi \circ \sigma_j : M_{n_j} \to M_{m_i}$ in the sense of Definition 3.8.

Note that $\sigma_j(1_{n_j})$ are pairwise orthogonal self-adjoint projections in A with their sum equal to one, so $p_{ij} := \varphi_{ij}(1_{n_j})$ are pairwise orthogonal self-adjoint projections in M_{m_i}, also with their sum equal to one. Therefore, as before, their direct sum is unitarily equivalent with the help of u_i in M_{m_i} to the sum of the corresponding diagonal projections $u_i p_{ij} u_i^*$. Applying Lemma 3.11 to each of $a \mapsto u_i \varphi_{ij}(a) u_i^*$, we find that $\epsilon_i \circ \varphi$ is unitarily equivalent (in M_{m_i}) to the homomorphism $\mathrm{id}_{n_1}^{c_{i1}} \oplus \ldots \oplus \mathrm{id}_{n_k}^{c_{ik}} = S_{c_{i1}} \oplus \cdots \oplus S_{c_{ik}}$. Comparison of dimensions gives the equality $\sum_{j=1}^{k} c_{ij} n_j = m_i$. Since φ is determined by the direct sum $\epsilon_i \circ \varphi$, $i = 1, \ldots, l$, the statement is proven. □

Problem 52. Suppose that in the previous lemma we exclude the requirement of unitality. Prove an analogue of the lemma in this case. Namely, take instead of B the subalgebra $\varphi(A) = p\varphi(A)p$ in B, where $p = \varphi(1_A)$, apply the previous lemma to $\varphi : A \to \varphi(A)$ and obtain the statement of the lemma with inequalities $\sum_{j=1}^{k} c_{ij} n_j \leqslant m_i$ instead of equalities.

Definition 3.13. Any A $*$-homomorphism between finite-dimensional C^*-algebras can be represented in the following graphical way. Let's represent A in the form k-tuple $\{(1,1) = n_1, \ldots, (1,k) = n_k\}$ corresponding $A \cong M_{n_1} \oplus \ldots \oplus M_{n_k}$, and B — in the form l-tuple $\{(2,1) = m_1, \ldots, (2,l) = m_l\}$ corresponding to $B \cong M_{m_1} \oplus \ldots \oplus M_{m_l}$. Let us represent φ using arrows between the elements of sets, and from $(1,j)$ to $(2,i)$ we draw c_{ij} arrows, the number is equal to the partial multiplicity. A sequence of such pictures for a sequence of homomorphisms $A_1 \subset A_2 \subset \ldots \subset A_p \subset \cdots$ is called the *Bratteli diagram of this sequence*. It is sometimes called the Bratteli diagram of an algebra, but the same algebra can be obtained from different sequences.

Problem 53. Draw Bratteli diagrams (for some defining sequences) of the following AF-algebras:

1) of the algebra of compact operators $\mathbb{K}(H)$;

2) of its unitization $\mathbb{K}(H)^+$;

3) the closure of the union of $A_p = M_{2^p}$, with embeddings $A_p \subset A_{p+1}$ of multiplicity 2 according to the formula $a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ (CAR algebra);

4) $C(K)$, where K is the Cantor set obtained from $[0,1]$ by successive removing the middle third of the corresponding intervals. If K_p is a set, obtained at the pth step of this process, then A_p is an algebra of continuous functions constant on intervals of K_p;

5) $C(X)$, where $X := \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$, and A_k consists of all functions constant on $[0, 1/2^k]$.

Lemma 3.14. *If two Bratteli diagrams coincide, then the corresponding AF-algebras are isometrically $*$-isomorphic.*

Proof. Let A_n and B_n be two sequences of finite-dimensional C^*-algebras with inclusions $\alpha_n : A_n \to A_{n+1}$, $\beta_n : B_n \to B_{n+1}$. Since the Bratteli diagrams are the same, then for each n there is an isomorphism $\varphi_n : A_n \to B_n$. Consider $\varphi_{n+1} \circ \alpha_n$ and $\beta_n \circ \varphi_n : A_n \to B_{n+1}$. They can differ only by unitary $u_{n+1} \in B_{n+1}$, that is $\beta_n \circ \varphi_n = \mathrm{Ad}_{u_{n+1}} \varphi_{n+1} \circ \alpha_n$, where $\mathrm{Ad}_{u_{n+1}}(a) = u_{n+1} a (u_{n+1})^*$.

Let's put $\psi_1 = \varphi_1$, $v_1 = 1$. Let us define inductively $v_{n+1} = \beta_n(v_n) u_{n+1} \in B_{n+1}$, $\psi_{n+1} = \mathrm{Ad}_{v_{n+1}} \varphi_{n+1}$. Then

$$\begin{aligned} \beta_n \psi_n &= \beta_n \mathrm{Ad}_{v_n} \varphi_n = \mathrm{Ad}_{\beta_n(v_n)} \beta_n \varphi_n = \mathrm{Ad}_{\beta(v_n)} \mathrm{Ad}_{u_{n+1}} \varphi_{n+1} \alpha_n \\ &= \mathrm{Ad}_{\beta_n(v_n) u_{n+1}} \varphi_{n+1} \alpha_n = \psi_{n+1} \alpha_n. \end{aligned}$$

In this case $\cup_{n=1}^{\infty} \psi_n : \cup_{n=1}^{\infty} A_n \to \cup_{n=1}^{\infty} B_n$ is an isometric $*$-isomorphism, so the closures are also isomorphic. ∎

One should not think that AF-algebras are "small" and that C^*-subalgebras of AF-algebras are AF-algebras again. For example, $C[0,1]$ is not an AF-algebra (since its only finite-dimensional C^*-subalgebra consists of constant functions), but it is a C^*-subalgebra of the AF-algebra $C(K)$ functions on the Cantor set. Indeed, let f be a function on K that has a dense set of rational numbers in the interval $[0,1]$ as its values. For example, the restriction of the Cantor function $f : [0,1] \to [0,1]$ [8, Ch. VI, §4] on K has all rationals of the form $p/2^s$ as its values. Its spectrum is a closure of this set, that is, equal to the entire interval $[0,1]$. Thus, $C^*(1,f) \subset C(K)$ is isometrically $*$-isomorphic to $C(\mathrm{Sp}(f)) = C[0,1]$.