Lecture 13

A discussion of the program and the list of problems

Program:

1. C*-algebras: definition and examples.
2. Unitalization of a C*-algebra.
3. Spectrum of an element of a C*-algebra, its properties.
4. Commutative C*-algebras. The space of maximal ideals. The Gelfand transform.
5. Gelfand's theorem about commutative C*-algebras.
6. The Stone-Weierstrass theorem.
7. C*-algebra generated by a normal element. The functional calculus for normal operators.
8. Positive elements, its properties.
9. Approximate units, their existence.
10. Ideals, factor-algebras, hereditary subalgebras.
11. Automatic continuity of *-homomorphisms.
12. Von Neumann algebras. Bicommutant theorem.
13. Topologically irreducible representations.
14. Positive functionals, states.
15. GNS-construction.
16. Realization of C*-algebras as operator algebras (Gelfand-Naimark theorem).
17. Jordan decomposition.
18. Finite dimensional linear topological spaces. Uniqueness of Hausdorff topology.
19. Finite-dimensional C*-algebras, their unitality and structure.
20. Non-degenerate representations.
21. The algebra of compact operators and its properties.
22. AF-algebras, a description of homomorphisms of finite-dimensional algebras, Bratteli diagrams.

23. The algebras of multipliers and centralizers.

24. Hilbert C*-modules and algebras related to them.

25. The Calkin algebra and its properties.

Additional list of problems (to formulated at lectures)

1. Let A be a C^*-algebra, $a \in A$, $p, q \in A$ — orthogonal projections (i.e. self-adjoint idempotents with $pq = 0$). Show that if a is positive and $pap = 0$, then $paq = 0$.

2. Let A be a C^*-algebra, $a \in A$. Let us denote by aAa the set of all elements of the form aba, where $b \in A$, and by $\overline{aAa}$ the closure of this set. A C^*-subalgebra $B \subset A$ *is hereditary* if the conditions $0 \leq a \leq b$ and $b \in B$ imply that $a \in B$.

   (a) Check that $\overline{aAa}$ is a C^*-subalgebra for any $a \in A$.

   (b) Let $p \in A$ be a projection. Verify that pAp is closed.

   (c) Show that pAp is hereditary for any projector p.

   (d) Show that $\overline{aAa}$ is hereditary for any positive $a \in A$.

3. Let $X \subset \mathbb{R}$ be the set of points $1, 1/2, 1/3, \ldots$ and 0. Let $C(X, M_2)$ be the set of all continuous functions on X with values in the matrix algebra M_2. Let $B_1 = \{f \in C(X, M_2) : f(0)$ is diagonal$\}$, $B_2 = \{f \in C(X, M_2) : f(0)$ has the form $\left(\begin{smallmatrix} * & 0 \\ 0 & 0 \end{smallmatrix}\right)\}$.

   (a) Show that $C(X, M_2)$, B_1, B_2 are C^*-algebras.

   (b) Find all (two-sided, closed) ideals in $C(X)$, $C(X, M_2)$, B_1, B_2.

4. Let A be a C^*-algebra, $J \subset A$ be an ideal, $a \in A$ is a self-adjoint element. Show that there exists an element $j \in J$ such that $\|[a]\| = \|a - j\|$, where $[a] \in A/J$ is the class $a + J$ of element a. Hint: decompose $a - \|[a]\| \cdot 1 = a_+ - a_-$ with positive a_+, a_- and show that $a_+ \in J$.

5. Let A be a C^*-algebra, $a \in A$ be a self-adjoint element. Show that if the spectrum $\sigma(a)$ is an infinite set, then A is infinite-dimensional.

6. Describe the GNS construction for the C^*-algebra $C[0, 1]$ and for a positive linear functional φ

   (a) $\varphi(f) = f(0)$,

   (b) $\varphi(f) = \frac{1}{2}(f(0) + f(1))$,

   (c) $\varphi(f) = \int_0^1 f(x)\, dx$,

   where $f \in C[0, 1]$.

7. Describe the GNS construction for the C^*-algebra M_n of complex $n \times n$-matrices and for a positive linear functional φ

(a) $\varphi(A) = a_{11}$,

(b) $\varphi(A) = \mathrm{tr}(A)$,

where $A = (a_{ij})_{i,j=1}^n \in M_n$.

8. Let π, σ be representations of a C^*-algebra A on the Hilbert spaces H_π and H_σ, and let a partial isometry $U : H_\pi \to H_\sigma$ satisfy the equality $\sigma(a)U = U\pi(a)$ for any $a \in A$. Show that the image (resp. orthogonal complement to the kernel) of U is an invariant subspace for $\sigma(A)$ (resp. for $\pi(A)$). (U is a partial isometry if U^*U and UU^* are projections)

9. (a) Let $M_n(A)$ be the set of all $n \times n$-matrices with coefficients from a C^*-algebra A. Show that on $M_n(A)$ there exists a C^*-norm.

   (b) Let A be a C^*-algebra with norm $\|\cdot\|$, and let $\|\cdot\|'$ be another norm on A, equivalent to the first norm. Show that if $\|\cdot\|'$ is a C^*-norm, then these norms coincide. Deduce from this the uniqueness of C^*-norm on $M_n(A)$.

10. Let φ be a state on a C^*-algebra A. Suppose that for some self-adjoint element $a \in A$ one has the equality $\varphi(a^2) = \varphi(a)^2$. Show that it follows from this that $\varphi(ab) = \varphi(ba) = \varphi(a)\varphi(b)$ for any $b \in A$.

11. Let $A = c$ be the C^*-algebra of all convergent sequences of complex numbers, $c = \{(a_n)_{n\in\mathbb{N}} : a_n \in \mathbb{C}; \lim_{n\to\infty} a_n \text{ exists}\}$. Let us consider it as a C^*-subalgebra of the algebra $\mathbb{B}(l_2)$ of bounded operators in the Hilbert space l_2 of square-integrable sequences. Find the first and second commutant, A' and A'', and (independently) the weak closure of A in $\mathbb{B}(l_2)$.

12. (a) Show that the weak topology is strictly weaker than the strong topology.

    (b) Let $P \subset \mathbb{B}(H)$ be the set of all (self-adjoint) projections on a Hilbert space. Show that if $p_\lambda \to p$ weakly converges, where $p_\lambda \in P$ and $p \in P$, then $p_\lambda \to p$ strongly converges.

    (c) Show that the strong limit of a sequence of (self-adjoint) projections is a projection.

    (d) Find an example of a weakly convergent net $p_\lambda \to p$ with $p_\lambda \in P$ and $p \notin P$.

13. Let $H_n \subset H$ be the subspace of a Hilbert space H generated by the first n vectors of an orthonormal basis. In the set of all sequences $(m_1, m_2, \ldots)$, where $m_k \in \mathbb{B}(H_n) \subset \mathbb{B}(H)$, consider the subset A of all sequences such that

    - $\sup_k \|m_k\| < \infty$;
    - the sequences $(m_1, m_2, \ldots)$ and $(m_1^*, m_2^*, \ldots)$ are convergent in the strong topology.

    Show that A is a C^*-algebra and that the mapping $(m_1, m_2, \ldots) \mapsto$ $s\text{-}\lim_{k\to\infty} m_k \in \mathbb{B}(H)$ is a surjective $*$-homomorphism of $A \to \mathbb{B}(H)$.

14. Let A be a commutative C^*-algebra and let π be its irreducible representation on a Hilbert space H. Show that $\dim H = 1$

15. Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis of a Hilbert space H, and let the operators a, b be given by the equalities $ae_i = e_{2i}$; $be_i = e_{2i-1}$. Let $E = C^*(a, b) \subset \mathbb{B}(H)$ be the C^*-algebra generated by a and b.

    (a) Check the boundedness of a and b and prove the equalities $a^*a = b^*b = 1$, $aa^* + bb^* = 1$.

    (b) Prove that E is not isomorphic to the complete group C^*-algebra $C^*(G)$ for any group G.

16. Consider $C[0, 1]$ as a C^*-subalgebra in $\mathbb{B}(H)$, where $H = L^2([0, 1])$ (continuous functions act on H by multiplication).

    (a) Check that $C[0, 1] \cap \mathbb{K}(H) = 0$;

    (b) Let φ be a linear functional on $C[0, 1]$ defined by the equality $\varphi(f) = f(0)$, $f \in C[0, 1]$. Find a sequence of $\{e_n\}_{n\in\mathbb{N}}$ vectors of unit length weakly converging to zero in H such that $\varphi(f) = \lim_{n\to\infty}\langle fe_n, e_n\rangle$ for any function $f \in C[0, 1]$.

17. Operators a, b in a Hilbert space H are called *compalent* if there exists a unitary operator $u \in \mathbb{B}(H)$ such that $u^*au - b \in \mathbb{K}(H)$. Show that if self-adjoint operators a, b are compalent then their essential spectra coincide.

18. Show that any AF C^*-algebra without unity has an approximative unity consisting of an increasing sequence of projections.

19. (a) Show that $C[0, 1]$ is not an AF-algebra.

    (b) Construct an injective $*$-homomorphism $C[0, 1]$ into the AF-algebra $C(K)$ of continuous functions on the Cantor set K. Hint: construct a function f on K that takes all rational values from $[0, 1]$ and show that $C^*(f)$ is isometrically $*$-isomorphic $C(\mathrm{Sp}(f)) = C[0, 1]$.

20. Let $A_n = M_{2^n}(\mathbb{C}) \oplus M_{2^n}(\mathbb{C})$, and let the embedding $\alpha_n : A_n \to A_{n+1}$ be given by the formula $\alpha_n : \left(\begin{smallmatrix} a_1 & 0 \\ 0 & a_2 \end{smallmatrix}\right) \mapsto \left(\begin{array}{cc|cc} a_1 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 \\ \hline 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & a_2 \end{array}\right)$, where $a_1, a_2 \in M_{2^n}(\mathbb{C})$.

    (a) Find the Bratteli diagram for the AF algebra $A = \overline{\cup_{n=1}^{\infty} A_n}$;

    (b) Find whether A is unital.

21. Find $M(A)$, where $A = \{f \in C([0, 1]; M_2) : f(0)_{11} = f(0)_{12} = f(0)_{21} = 0; f(1) = 0\}$ (here M_2 is the algebra of two-dimensional matrices).