Chapter 1

C^*-algebras

Lecture 1

1.1 Definition and first examples

Definition 1.1. An *algebra* A (over a field $\mathbb{K}$) is a ring that is a linear space over $\mathbb{K}$, and the addition in the definition of a ring and in the definition of a linear spaces are the same, and multiplications are connected by the relation $\lambda(ab) = (\lambda a)b$ for all $\lambda \in \mathbb{K}$, $a, b \in A$.

We will consider algebras only over the field of complex numbers $\mathbb{C}$.

Definition 1.2. An algebra A over $\mathbb{C}$ is called a *Banach algebra*, if the underlying linear space is a Banach space and $\|ab\| \leq \|a\|\|b\|$ for any $a, b \in A$.

Problem 1 (easy)**.** Show that in this case the multiplication is continuous (as a mapping $A \times A \to A$).

Definition 1.3. A mapping $* : A \to A$, $a \mapsto a^*$ is called an *involution* if

1. $(a^*)^* = a$;
2. $(a + b)^* = a^* + b^*$;
3. $(\lambda a)^* = \bar{\lambda} a^*$;
4. $(ab)^* = b^* a^*$;
5. $\|a^*\| = \|a\|$

for any $a, b \in A$, $\lambda \in \mathbb{C}$. A Banach algebra with involution is called *involutive Banach algebra.*

Definition 1.4. An involutive Banach algebra A is called a C^**-algebra,* if the involution satisfies the equality $\|a^* a\| = \|a\|^2$ for all $a \in A$ (this equality is called the C^**-property*).

Problem 2. Give an example of an involutive Banach algebra that is not a C^*-algebra.

Problem 3 (easy)**.** Show that property (5) of the definition 1.3 follows from properties (1 – 4) and the C^*-property.

Definition 1.5. An element $1 \in A$ is called a (left) *unit*, if $1a = a$ for any $a \in A$. A C^*-algebra A is called *unital* or *algebra with unit*, if it has a (left) unit (identity element).

Problem 4. Show that a left unit element is also a right one, which has $1^* = 1$; that the identity element is unique and that $\|1\| = 1$. It is called the *unit of algebra*.

Problem 5. Verify that the algebra $C(X)$ formed by all continuous complex-valued functions on a compact space X and the algebra $C_0(X)$ of all continuous complex-valued functions on a locally compact space X tending to 0 at infinity (that is, $f : X \to \mathbb{C}$ such that for any $\varepsilon > 0$ there exists a compact $K \subseteq X$ such that $\sup\{|f(x)| \mid x \in K\} < \varepsilon$) are commutative C^*-algebras if the *supremum-norm*: $\|f\| = \sup_{x \in X} |f(x)|$, is taken as the norm and the pointwise multiplication is taken as the multiplication. Moreover, the algebra $C(X)$ is unital.

Problem 6. Verify that the algebra $\mathbb{B}(H)$ of all bounded operators acting on a Hilbert space H is a C^*-algebra with identity. Here as a norm we take the *operator norm* $\|a\| = \sup_{h \in H,\, \|h\| \le 1} \|a(h)\|$, and the multiplication is the composition of operators.

These examples of C^*-algebras are the most important, as we will see later.

1.2 Unitalization, or attaching of a unit

If an involutive Banach algebra A does not have a unit, then it can be embedded into an involutive unital Banach algebra A as follows. Let $A^+ = A \oplus \mathbb{C}$ be a linear space (the direct sum of linear spaces). Let's define a structure of an involutive Banach algebra on A^+ by formulas

$$\begin{aligned} (a,\lambda)(b,\mu) &= (ab + \lambda b + \mu a, \lambda\mu), \\ (a,\lambda)^* &= (a^*, \bar{\lambda}), \\ \|(a,\lambda)\| &= \|a\| + |\lambda| \end{aligned} \tag{1.1}$$

for any $(a,\lambda), (b,\mu) \in A \oplus \mathbb{C}$.

We further assume that the initial algebra A is a C^*-algebra.

Problem 7. Show that this norm turns A^+ into an involutive Banach algebra, but not into a C^*-algebra. (To check completeness, use the completeness of the subspace $A \subset A^+$ of finite codimension (one), this follows from Problem 337(b) in [7].)

Definition 1.6. A subset $B \subseteq A$ is called a *subalgebra* of a (Banach) algebra A if B is an algebra with respect to the operations (and norm) of A.

A subset of $B \subseteq A$ is called (C^*-)*subalgebra* C^*-algebras A, if B is a C^*-algebra with respect to the operations and the norm of A. In particular, B is closed in A.

Definition 1.7. A subalgebra $I \subseteq A$ is called a (two-sided) *ideal*, if $aI \subseteq I$ and $Ia \subseteq I$ for any $a \in A$.

Problem 8. Prove that A is an ideal in A^+.

Lemma 1.8. *There is a norm on A^+ such that*

1) *it is equivalent to the one defined above;*

2) *on A it coincides with the original norm:*

3) *it is a C^*-norm.*

Proof. For $b = (b', \lambda) \in A^+$, consider the linear mapping $L_b : A \to A$ according to the formula $L_b(a) = ba$, $a \in A$. For L_b, we can thus define the operator norm: $\|L_b\| := \sup_{a\in A,\, \|a\|\le 1} \|L_b(a)\|$. If $b \in A$, then $\|L_b\| \le \|b\|$. Since $\|L_b(b^*)\| = \|bb^*\| = \|b\|^2$, we have $\|L_b\| = \|b\|$. For every $b = (b', \lambda) \in A^+$ we set $\|b\|_{new} = \|L_b\|$. Note that the above reasoning shows that $\|.\|_{new}$ satisfies condition 2) from the formulation of the lemma.

First of all, let's check that $\|\cdot\|_{new}$ is a norm. Obviously, the axioms of linearity and triangle are satisfied (that is, this is a *semi-norm*, or *pre-norm*), so it remains to check the nondegeneracy. Let $\|b\|_{new} = 0$ for some $b = (b', \lambda) \in A^+$. It means that $(b', \lambda)\cdot(a, 0) = (b'a + \lambda a, 0) = (0, 0)$ for any $a \in A$, so $-\frac{1}{\lambda}b' \cdot a = a$ and $-\frac{1}{\lambda}b'$ is the unit of A. But A does not have a unit. This means that $\|\cdot\|_{new}$ is the norm. Like any operator norm, the new norm is a norm of a Banach algebra, that is,

$$\|bc\|_{new} \le \|b\|_{new}\|c\|_{new} \tag{1.2}$$

for any $b, c \in A^+$ (Easy problem: check this). We do not claim yet that the involution is an isometry.

This new norm is equivalent to the previously defined norm on A^+, since $A \subset A^+$ is a subspace of codimension 1. Indeed, by the triangle inequality, we have $\|(a, \lambda)\|_{new} \le \|a\| + |\lambda| = \|(a + \lambda)\|$, so it is sufficient to show that there is a constant $c > 0$ such that $\|(a, \lambda)\|_{new} \ge c\|(a, \lambda)\|$. Let's assume the opposite: there are such pairs (a_n, λ_n) and numbers $c_n > 0$ such that $\lim_{n\to\infty} c_n = 0$ and

$$\|(a_n, \lambda_n)\|_{new} \le c_n\|(a_n, \lambda_n)\|.$$

Since none of λ_n is zero, we can assume (by dividing both sides by $\lambda_n \ne 0$ if necessary) that $\lambda_n = 1$. Applying the triangle inequality again, we get

$$\|a_n\| - 1 \le \|(a_n, 1)\|_{new} \le c_n(\|a_n\| + 1),$$

whence $\|a_n\| \le \frac{1+c_n}{1-c_n}$, and for sufficiently large n we have $\|a_n\| < 2$. But then for these n we have $\|(a_n, 1)\|_{new} \le 3c_n$, so

$$\lim_{n\to\infty} (a_n, 1) = 0. \tag{1.3}$$

Now recall that $A^+/A \cong \mathbb{C}$, and all norms on $\mathbb{C}$ are equivalent (moreover, they differ only by a constant multiple). Thus, the quotient norm on A^+/A given by $\|(a, \lambda) + A\|_{new} :=$

$\inf_{a\in A} \|(a,\lambda)\|_{new}$ is equivalent to the usual norm on $\mathbb{C}$. So $\inf_{a\in A} \|(a,1)\|_{new} = \alpha > 0$, what contradicts (1.3).

Now we need to check the C^* property. By definition, for any $b \in A^+$ and any $\varepsilon > 0$ there is an element $a \in A$ such that $\|a\| = 1$ and

$$\|L_b(a)\| \geq (1-\varepsilon)\|L_b\|, \qquad \text{that is} \qquad \|ba\|_{new} = \|ba\| \geq (1-\varepsilon)\|b\|_{new}.$$

We also have

$$\|b^*b\|_{new} \geq \|a^*(b^*b)a\|_{new} = \|a^*(b^*b)a\| = \|(ba)^*(ba)\| = \|ba\|^2 \geq (1-\varepsilon)^2\|b\|^2_{new}$$

(where the first inequality is satisfied by virtue of (1.2), the next equality is satisfied by virtue of that A is an ideal in A^+, and then we apply the C^*-property in A). Passing to the limit as ε tends to zero, we obtain $\|b^*b\|_{new} \geqslant \|b\|^2_{new}$. In particular, $\|b^*\|_{new} \cdot \|b\|_{new} \geqslant \|b^*b\|_{new} \geqslant \|b\|^2_{new}$, so $\|b^*\|_{new} \geqslant \|b\|_{new}$. Therefore $\|b\|_{new} = \|(b^*)^*\|_{new} \geqslant \|b^*\|_{new}$. Thus, the involution is an isometry and, by (1.2), $\|b^*b\|_{new} \leqslant \|b^*\|_{new}\|b\|_{new} = \|b\|^2_{new}$. This means that this is a C^*-norm. □

Definition 1.9. An element $a \in A$ is called *self-adjoint*, if $a^* = a$, *unitary*, if $a^*a = aa^* = 1$ (here the algebra A is assumed to be unital), *normal*, if $a^*a = aa^*$.

Definition 1.10. In a unital C^*-algebra, an element a is called *invertible*, if there is an element $a' \in A$ such that $aa' = a'a = 1$. There is only one a' with this property (check!), called the *inverse to* a and is denoted by a^{-1}. The set of invertible elements $G(A)$ is a group.

Problem 9. Check this.

Lemma 1.11. *If $\|1-a\| < 1$, then a^{-1} exists and is equal to $a' = \sum_{n=0}^{\infty}(1-a)^n$, and the series converges in norm.*

Proof. Convergence immediately follows from the domination by a geometric progression. Next we calculate: $a'a = aa' = -(1-a)a' + a' = -\sum_{n=1}^{\infty}(1-a)^n + \sum_{n=0}^{\infty}(1-a)^n = 1$. □

Problem 10. Prove in a similar way that if $a_0 \in A$ is invertible and $\|a - a_0\| < \frac{1}{\|a_0^{-1}\|}$, then a is also invertible, and $a^{-1} = \sum_{n=0}^{\infty}[a_0^{-1}(a_0 - a)]^n a_0^{-1}$.

Corollary 1.12. *The subset $G(A) \subset A$ is an open set. Taking the inverse element $a \mapsto a^{-1}$ is a continuous map $G(A)$ into itself.*

Problem 11. Prove the corollary using the formulas established above.