Lecture 2

1.3 Spectrum and functional calculus

Definition 1.13. Let A be a Banach algebra with identity. For any $a \in A$ we call the set $\mathrm{Sp}(a) = \{\lambda \in \mathbb{C} : (a - \lambda 1) \text{ is not invertible}\}$ the *spectrum* of element a. The function $R_a(\lambda) = (a - \lambda 1)^{-1}$ is called the *resolvent* of a. If A is a non-unital C^*-algebra, then *quasi-spectrum* $\mathrm{Sp}'(a)$ of element $a \in A$ is assumed to be equal to the spectrum of a as an element of the algebra A^+.

Problem 12. Show that the quasi-spectrum always contains zero.

Theorem 1.14. *The spectrum of any element is a compact non-empty set. The resolvent is an analytic function outside the spectrum (that is, for any point of the completed complex plane from the complement of the spectrum, it can be represented by a power series with coefficients from algebra, uniformly convergent on some closed disk).*

Proof. If $|\lambda| > \|a\|$, then the series $\sum_{n=0}^{\infty} \lambda^{-(n+1)} a^n$ converges in norm, and converges to $-(a - \lambda 1)^{-1}$, since $-(a - \lambda 1) \sum_{n=0}^{k} \lambda^{-(n+1)} a^n = 1 - \lambda^{-(k+2)} a^{k+1}$ converges to 1 . Thus, $R_a(\lambda)$ is analytic in the neighborhood of infinity, defined by $|\lambda| > \|a\|$, and

$$\lim_{\lambda \to \infty} \|R_a(\lambda)\| \leq \lim_{\lambda \to \infty} \frac{1}{|\lambda|} \frac{1}{1 - \frac{\|a\|}{|\lambda|}} = 0.$$

In particular, $\mathrm{Sp}(a)$ is contained in a closed disk of radius $\|a\|$ and thus, is a bounded set.

If $a - \lambda_0 1$ is invertible (that is, $\lambda_0 \notin \mathrm{Sp}(a)$) and $|\lambda - \lambda_0| < \frac{1}{\|(a - \lambda_0 1)^{-1}\|}$ then, the same way,

$$(a - \lambda 1)^{-1} = \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n (a - \lambda_0 1)^{-n-1}$$

is the Taylor series R_a in a neighborhood of λ_0, so $R_a(\lambda)$ is analytic outside $\mathrm{Sp}(a)$. The same reasoning shows, in particular, that the complement to $\mathrm{Sp}(a)$ is open, that is, $\mathrm{Sp}(a)$ is closed. Since the spectrum is bounded, it is compact.

Since $R_a(\lambda)$ is analytic, then the complex-valued function $\lambda \mapsto \varphi(R_a(\lambda))$ is also analytic for φ being an arbitrary bounded linear functional on A. If $\mathrm{Sp}(a)$ is empty, then $\varphi(R_a(\lambda))$ is an analytic function on the entire $\mathbb{C}$. However, it is bounded because $|\varphi(R_a(\lambda))| \leqslant \|\varphi\| \cdot \|R_a(\lambda)\|$ and $\|R_a(\lambda)\|$ is bounded: we estimate separately on a compact disk of radius $\lambda_0 > \|a\|$ and outside it using the estimation obtained above

$$\|R_a(\lambda)\| \leq \frac{1}{|\lambda| - \|a\|} \leq \frac{1}{|\lambda_0| - \|a\|}.$$

This means $\varphi(R_a(\lambda)) = 0$ for any φ, so $R_a(\lambda) = 0$ (see problem 13 below) for all $\lambda \in \mathbb{C}$. This is impossible (it turns out that $\lambda_1 1 - \lambda_2 1 = 0$ for $\lambda_1 \neq \lambda_2$). □

Problem 13. For an element $x \neq 0$ of a normed space X, there is a continuous linear functional φ that does not vanish on x. (This is a theorem from the standard course, a corollary of the Hahn-Banach theorem, see, for example, Corollary 2 on page 189 in [8]).

Problem 14. Let a and b be commuting elements of a Banach algebra. Then the product ab is invertible if and only if each of the elements a and b are invertible.

Theorem 1.15 (Spectral mapping theorem)**.** *If $p(z)$ is a polynomial, then* $\mathrm{Sp}(p(a)) = p(\mathrm{Sp}(a))$.

Proof. For $\alpha \in \mathbb{C}$ we decompose $p(z)-\alpha$ into linear factors $p(z)-\alpha = c\prod_i(z-\beta_i)$. Then $p(a)-\alpha 1 = c\prod_i(a-\beta_i 1)$ and $p(a)-\alpha 1$ are invertible if and only if all $a-\beta_i 1$ are (applying inductively Problem 14). Then $\alpha \in \mathrm{Sp}(p(a))$ if and only if at least one of β_i belongs to $\mathrm{Sp}(a)$. On the other hand, substituting this β_{i_0} into the above representation of p, we obtain $p(\beta_{i_0}) - \alpha = c\prod_i(\beta_{i_0} - \beta_i) = 0$, that is, $\alpha = p(\beta_{i_0})$. So $\mathrm{Sp}(p(a)) \subseteq p(\mathrm{Sp}(a))$. The inverse statement is similar. □

Lemma 1.16. *Let A be a C^*-algebra (unital). Then* $\mathrm{Sp}(a^*) = \overline{\mathrm{Sp}(a)}$. *If a is unitary, then* $\mathrm{Sp}(a) \subset \mathbb{S}^1$, *where* $\mathbb{S}^1 \subset \mathbb{C}$ *is the unit sphere.*

Proof. Since $a^*(a^{-1})^* = (a^{-1}a)^* = 1^* = 1$, and similarly, in a different order, then the element is invertible if and only if we invert its conjugate. This gives the first statement.

Now let a be unitary, in particular, invertible. Then $\|a\|^2 = \|a^*a\| = 1$, so for $|\lambda| > 1$ we have $\lambda \notin \mathrm{Sp}(a)$. If $|\lambda| < 1$, then $1 - \lambda a^*$ is invertible, which means that $a(1-\lambda a^*) = a - \lambda 1$ is also invertible. Thus, $\mathrm{Sp}(a) \subset \mathbb{S}^1$. □

Definition 1.17. The number $r(a) = \sup\{|\lambda| \,:\, \lambda \in \mathrm{Sp}(a)\}$ is called the *spectral radius* of a.

Problem 15. Show that $r(a) \leq \|a\|$ for any $a \in A$.

Lemma 1.18. $r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$.

Proof. Let us expand the resolvent R_a in a neighborhood of infinity: $R_a(\lambda) = \sum_{n=0}^{\infty} \frac{a^n}{\lambda^{n+1}}$. This function is analytic for $|\lambda| > r(a)$, so for any $\rho > r(a)$, terms tends to zero: $\lim_{n\to\infty} \|\frac{a^n}{\rho^{n+1}}\| = 0$. That's why $\overline{\lim}_{n\to\infty}\|a^n\|^{1/n} \leq \rho$ for any $\rho > r(a)$, so

$$\overline{\lim}_{n\to\infty}\|a^n\|^{1/n} \leq r(a). \tag{1.4}$$

On the other hand, due to compactness of the spectrum, there is $\alpha \in \mathrm{Sp}(a)$ such that $|\alpha| = r(a)$. According to the spectral mapping theorem, $\alpha^n \in \mathrm{Sp}(a^n)$, therefore $|\alpha^n| \leq r(a^n)$. By Problem 15, $r(a^n) \leq \|a^n\|$. Combining, we get

$$r(a) = |\alpha| = (|\alpha^n|)^{1/n} \leq (r(a^n))^{1/n} \leq \|a^n\|^{1/n}$$

for all n, so

$$r(a) \leq \inf_n \|a^n\|^{1/n}. \tag{1.5}$$

Comparing (1.4) and (1.5), we conclude that $\lim_{n\to\infty} \|a^n\|^{1/n}$ exists and is equal to $r(a)$. □

1.4 Multiplicative functionals and maximal ideals of commutative Banach algebras

Recall that an algebra A is called *simple*, if it doesn't have proper (i.e., distinct from $\{0\}$ and A) ideals.

Lemma 1.19. *The algebra of complex numbers $\mathbb{C}$ is the only simple commutative algebra.*

Proof. If $A \neq \mathbb{C}$, then there is a non-scalar element $a \in A$ (that is, $a \neq \lambda 1$ for any λ). Let's take some $\alpha \in \mathrm{Sp}(a)$ and set $I = \overline{(a - \alpha 1)A}$, so that $I \neq \{0\}$ is closed (two-sided) ideal in A. For any $b \in A$, the element $(a - \alpha 1)b$ is not invertible (see Problem 14), so by Lemma 1.11 $\|(a - \alpha 1)b - 1\| \geq 1$. Therefore $1 \notin I$ and we arrive to a contradiction with the simplicity of A. □

Definition 1.20. A *multiplicative functional* on A is a nontrivial homomorphism $\varphi : A \to \mathbb{C}$. The set of all multiplicative functionals is denoted by M_A.

Problem 16. Prove that $\varphi(1) = 1$. You can use a simple general observation: image *idempotent* ($p^2 = p$) under homomorphism it is always idempotent.

Lemma 1.21. *A multiplicative functional on a commutative unital Banach algebra A has norm 1. The mapping that associates to each multiplicative functional its kernel is a bijection onto the set of* maximal ideals *of A, that is, such proper ideals that are not contained in any other ideal except the entire algebra A*

Proof. Since, according to Problem 16, $\varphi(1) = 1$, we have $\|\varphi\| \geq 1$. Let $\|\varphi\| > 1$, so there is an element $a \in A$ such that $\|a\| < 1$ and $\varphi(a) = 1$. Then the series $b = \sum_{n=1}^{\infty} a^n$ converges and $a + ab = b$. This means that $\varphi(b) = \varphi(a)(1 + \varphi(b)) = 1 + \varphi(b)$. A contradiction. We obtained $\|\varphi\| = 1$.

Since $\operatorname{Ker} \varphi$ has codimension 1, it is a maximal ideal.

Any functional φ is completely determined by its kernel and the condition $\varphi(1) = 1$ (see Problem 17 below), so the indicated correspondence is a bijection onto the image.

Let us show that it is an epimorphism. If $M \subset A$ is a maximal ideal, then $\mathrm{dist}(M, 1) = 1$, since the unit open ball with center 1 consists of invertible elements (Lemma 1.11), and M cannot contain invertible elements (if $a \in M$ is invertible, then $1 \in M$, and hence $M = A$). Then the closed ideal $\overline{M}$ (the closure of M) also does not contain 1, so by maximality $\overline{M} = M$. Consider the factor algebra A/M being simple (since otherwise M would not be maximal) commutative unital Banach algebra. Therefore, according to Lemma 1.19, $A/M \cong \mathbb{C}$. The corresponding factorization map is a non-zero homomorphism, that is, a multiplicative functional (with kernel M). □

Problem 17. Prove that any functional φ is completely determined by its kernel and condition $\varphi(1) = 1$ (factor by kernel and consider the induced functional on $\mathbb{C}$).

Since multiplicative functionals are bounded by 1, M_A is a subset of the unit ball in the dual Banach space A' (the space of all bounded linear functionals on A). The space A' can be equipped with the $*$-*weak topology*, determined by the prebase of neighborhoods of the form $U_{\varphi_0,\varepsilon,a} = \{\varphi \in A' : |(\varphi - \varphi_0)(a)| < \varepsilon\}$, $\varphi_0 \in A'$, $a \in A$, $\varepsilon > 0$. In terms of directed nets: a directed net φ_α converges to φ if $\varphi_\alpha(a)$ converges to $\varphi(a)$ for each $a \in A$. The unit ball of space A' is compact and Hausdorff with respect to the $*$-weak topology (Banach-Alaoglu theorem, see [5, Theorem 5, p. 325]).

Problem 18. Verify that M_A is $*$-weakly closed.

Therefore M_A is also $*$-weakly compact.

Problem 19. The following example describes a typical situation, as it will become clear a little further. Let $A = C[0,1]$, so the multiplicative functionals correspond to points of $[0,1]$. Namely, $\varphi(g) = g(t)$. Show that for the "ordinary" norm of the dual space for any two points $t \neq s$, the distance between φ_t and φ_s is 1. That is the set of multiplicative functionals is discrete, non-compact. There is no limit points, unlike the situation with the $*$-weak topology.

If A does not have unit, consider the Banach algebra $A^+ = A \oplus \mathbb{C}$ (with the original norm (1.2), not the C^*-norm from Lemma 1.8). In it, A is itself a maximal ideal corresponding to the multiplicative functional $\varphi_0((a,\lambda)) = \lambda$. Any other maximal ideal I of the algebra A^+ must have a proper intersection with A. Then $I \cap A$ is an ideal of codimension 1, since I had codimension 1 in A^+. Thus, $I \cap A$ is a maximal ideal in A. Since its codimension is 1, then the quotient mapping is defined being a nonzero homomorphism to $\mathbb{C}$. Conversely, if $\varphi : A \to \mathbb{C}$ is a (nonzero) multiplicative functional on A, then the formula $\widetilde{\varphi}((a,\lambda)) = \varphi(a) + \lambda$ defines a unique extension of φ to a multiplicative functional on A^+ (Problem 21). We obtain a bijective correspondence between M_A and $M_{A^+} \setminus \{\varphi_0\}$.

Problem 20. Prove that M_A is a locally compact Hausdorff space, and M_{A^+} is its one-point compactification.

Problem 21. Check that if $\varphi : A \to \mathbb{C}$ is a (non-zero) multiplicative functional on A, then the formula $\widetilde{\varphi}((a,\lambda)) = \varphi(a) + \lambda$ defines a unique extension of φ to a multiplicative functional on A^+.