Lecture 3

1.5 Gelfand transform

Definition 1.22. For a commutative Banach algebra A we define *Gelfand transform* $\Gamma : A \to C_0(M_A)$ by the relation $\Gamma(a) = \hat{a}$, where $\hat{a}(\varphi) := \varphi(a)$ (the necessary properties will be verified in the next lemma).

Lemma 1.23. *The Gelfand transform is a (non-strict) contraction homomorphism of algebras and the image A separates the points M_A, that is, for any two points M_A there is a function from the image with distinct values at these points.*

Proof. Functions $\hat{a}$ are continuous by the definition of the $*$-weak topology. The mapping is contractive because

$$\|\Gamma(a)\| = \sup_{\varphi \in M_A} |\varphi(a)| \le \sup_{\varphi \in M_A} \|\varphi\| \cdot \|a\| = \sup_{\varphi \in M_A} \|a\| = \|a\|.$$

The separation of points is obvious since two (multiplicative) functionals are distinct if and only if their values on some element a are distinct. If A is not unital, then we note that $\hat{a}(\tilde{0}) = 0(a) = 0$, where $M_{A^+} = M_A \cup \tilde{0}$. Therefore $\hat{a} \in C_0(M_A)$. $\square$

Problem 22. If an algebra is unital, then $\Gamma(1) = 1$.

Corollary 1.24. *Let A be a commutative unital Banach algebra. Then $a \in A$ is invertible if and only if $\hat{a}$ is invertible, and if and only if $\hat{a}(\varphi) \neq 0$ for any $\varphi \in M_A$. Therefore* $\mathrm{Sp}(a) = \mathrm{Sp}(\hat{a}) = \{\varphi(a) : \varphi \in M_A\}$ *and* $\|\hat{a}\| = r(a)$.

Proof. If a is invertible, then $\hat{a}$ is invertible by Problem 22. If a is not invertible, then consider the ideal $I = \overline{aA}$. As discussed above (see the proof of Lemma 1.19), this ideal cannot contain 1, so it is proper. Let I_M be the maximal ideal containing I (see problem 23 below), and φ be the corresponding I_M multiplicative functional. Then $\varphi(a) = 0$ and $\hat{a}$ is not invertible in $C(M_A)$.

The remaining statements are immediately obtained from what has been proven. $\square$

Problem 23. Any ideal I of a commutative Banach algebra with identity is contained in some maximum ideal. Hint: consider the union of J of all proper ideals I_α containing I partially ordered by inclusion. For each chain (a completely ordered subsystem) I_{α_τ} using, as above, by the fact that 1 does not belong to every I_{α_τ}, make sure that it does not belong to $\cup_\tau I_{\alpha_\tau}$, so this is its own ideal. Then apply Zorn's lemma.

Theorem 1.25. *Let A be a commutative C^*-algebra. Then the Gelfand transformation is an isometric $*$-isomorphism of A onto $C_0(M_A)$.*

Proof. Let us prove the theorem for a unital algebra. The necessary adaptation for the case without a unit is left to the reader as Problem 24.

Let $\varphi \in M_A$. Let us first consider the self-adjoint element $a^* = a \in A$. Let us set $u_t = \sum_{n=0}^{\infty} \frac{(ita)^n}{n!}$, $t \in \mathbb{R}$. It is easy to check by considering the partial sum and passing to the limit that $u_t^* = u_t^{-1}$, so $u_t \in A$ is unitary. Then

$$1 \geq |\varphi(u_t)| = |\sum_{n=0}^{\infty} \frac{(it\varphi(a))^n}{n!}| = |e^{it\varphi(a)}| = e^{-t\,\mathrm{Im}\,\varphi(a)}.$$

Due to the arbitrariness of $t \in \mathbb{R}$ in this estimate, we conclude that $\mathrm{Im}\,\varphi(a) = 0$, that is, $\varphi(a) \in \mathbb{R}$.

We write an arbitrary element $c \in A$ in the form $c = a + ib$, where $a = (c + c^*)/2$ and $b = (c - c^*)/2i$ are self-adjoint. According to what has been proven, $\varphi(a), \varphi(b) \in \mathbb{R}$, that means $\varphi(c^*) = \varphi(a) - i\varphi(b) = \overline{\varphi(c)}$, so the Gelfand transformation preserves involution and is thus a $*$-homomorphism.

For a self-adjoint element ($a = a^*$) we have $\|a\|^2 = \|a^*a\| = \|a^2\|$, therefore

$$\|\hat{a}\| = r(a) = \lim_{n\to\infty} \|a^{2^n}\|^{1/2^n} = \lim_{n\to\infty} (\|a\|^{2^n})^{1/2^n} = \|a\|.$$

For an element $b \in A$ of general form we have $\|b\|^2 = \|b^*b\| = \|\widehat{b^*b}\| = \|\hat{b}^*\hat{b}\| = \|\hat{b}\|^2$, so the Gelfand transformation is an isometry (onto its image).

Therefore $\Gamma(A)$ is norm closed and involutive subalgebra with identity in $C(M_A)$ separating the points. By theorem Stone-Weierstrass[1], $\Gamma(A) = C(M_A)$. $\square$

Problem 24. Prove the theorem for a non-unital algebra.

So in particular there is an inverse mapping for the transformation Gelfand, which is also an isometric $*$-isomorphism.

Let $a \in A$ be a normal element. Let us denote by $C^*(1, a)$ (respectively, $C^*(a)$) C^*-algebra generated by 1 and a (resp., only a). Due to normality, these algebras are commutative, with the first definition supposing that A is unital.

Let us clarify that by a *C^*-algebra generated by a set* we call the minimal C^*-subalgebra A containing the set, that is, the intersection all C^*-subalgebras of A containing this set.

Problem 25. Verify that the C^*-algebra generated by a set is indeed a C^*-algebra.

Problem 26. If a is an invertible element, then the algebra $C^*(a)$ is unital. In this case $C^*(a) = C^*(1, a)$.

Corollary 1.26. *If $a^* = a$, then* $\mathrm{Sp}(a) \subset \mathbb{R}$.

Proof. As we know, $\mathrm{Sp}(a) = \mathrm{Sp}(\widehat{a})$ and $\widehat{a}^* = \widehat{a}$. But a self-adjoint function is exactly a function with real values, while the spectrum of a function is the set of all its values. $\square$

Corollary 1.27. *The algebra $C^*(1, a)$ is isometrically $*$-isomorphic to the algebra $C(\mathrm{Sp}(a))$ under a mapping that takes a to the function $z(t) = t$, $z : \mathrm{Sp}(a) \subset \mathbb{C} \to \mathbb{C}$. The algebra $C^*(a)$ is mapped onto $C_0(\mathrm{Sp}(a) \setminus \{0\})$.*

[1] The theorem is not always included in the standard course on functional analysis, so we present its proof in Section 1.6.

Proof. For a commutative C^*-algebra $C^*(1,a)$ we find $X = M_{C^*(1,a)}$. Any multiplicative functional $\varphi \in X$ is determined by its value $\varphi(a) = \lambda$ on a. Moreover, due to multiplicativity $\varphi(p(a,a^*)) = p(\lambda, \bar{\lambda})$ for any polynomial p. Thus, X is identified with the set of all possible values λ that $\varphi(a) = \hat{a}(\varphi)$ takes for $\varphi \in X$. According to Corollary 1.24, we have $\hat{a}(X) = \mathrm{Sp}(a)$. We obtain the identification $\mathrm{Sp}(a) \cong X$ using the correspondence $\mathrm{Sp}(a) \ni \lambda \mapsto \varphi_\lambda \in X$, where φ_λ is determined by the condition $\varphi_\lambda(a) = \lambda$.

This identification carries over to functions: every continuous function on X is identified with a continuous function on $\mathrm{Sp}(a)$, namely, the function $\hat{b} = \hat{b}(\varphi)$ is associated with the function argument $\lambda \in \mathrm{Sp}(a)$, specified as $\lambda \mapsto \varphi_\lambda(b)$. For example, if we take the polynomial $p(a,a^*) = b$, then the corresponding function will be $\lambda \mapsto \varphi_\lambda(p(a,a^*)) = p(\lambda,\bar{\lambda})$. In particular, the function $\hat{a}$ is mapped to $\lambda \mapsto \varphi_\lambda(a) = \lambda$, so the Gelfand transform identifies $\hat{a}$ with the identity mapping of $X \subset \mathbb{C}$. By Theorem 1.25, this mapping is an isometric $*$-isomorphism.

If a is invertible, then by Problem 26, $C^*(a)$ isometrically $*$-isomorphic to $C(\mathrm{Sp}(a))$. If a is not is invertible, then $C^*(a)$ does not have unity (see Problem 27). It corresponds under the constructed mapping for $C^*(1,a) \cong C^*(a)^+$ to the ideal $C(\mathrm{Sp}'(a))$ consisting of functions, tending to 0. □

Problem 27. Prove that if a is not invertible, then $C^*(a)$ does not have a unit. Hint: if there is a unit, then it has to be approximated by a polynomial in a and a^*, which cannot be an invertible element.

Corollary 1.28 (continuous functional calculus)**.** *Let a be a normal element of a unital C^*-algebra A, and f is a continuous function on* $\mathrm{Sp}(a)$*. Then the element $f(a) \in A$ is defined as the inverse image of f under the Gelfand transformation: $\Gamma = \Gamma_a : C^*(1,a) \to C(\mathrm{Sp}(a))$, $f(a) := (\Gamma_a)^{-1}(f)$. If $0 \in \mathrm{Sp}(a)$ and $f(0) = 0$, then $f(a) \in C^*(a)$. Moreover, $f(\mathrm{Sp}(a)) = \mathrm{Sp}(f(a))$ and if g is a continuous function on $f(\mathrm{Sp}(a))$, then $g(f(a)) = (g \circ f)(a)$.*

Proof. Everything has already been proven except the last statement. Let's first consider the polynomial $p(\lambda,\bar{\lambda})$ as $f = f(\lambda)$. Then $\Gamma(p(a,a^*))$ is a function $\lambda \mapsto p(\lambda,\bar{\lambda})$, so $\mathrm{Sp}(p(a,a^*))$ coincides with the set of values of this function, $\{\mu : \mu = p(\lambda,\bar{\lambda}), \lambda \in \mathrm{Sp}(a)\}$. Approximating f by polynomials, we get $f(\mathrm{Sp}(a)) = \mathrm{Sp}(f(a))$ (Problem 28).

Similarly, consider the polynomial $q(\lambda,\bar{\lambda})$ as g. It's easy to see that

$$q(f(a)) = \lim_\alpha (q(p_\alpha(a)) = \lim_\alpha (q \circ p_\alpha)(a) = (q \circ f)(a).$$

Now we approximate g by polynomials and use the isometricity of the inverse Gelfand transform. □

Problem 28. Prove that $f(\mathrm{Sp}(a)) = \mathrm{Sp}(f(a))$ in the proof above, approximating f by polynomials, and correctly stating what it means that the image is continuous under a uniform approximation, and using the isometricity of the Gelfand transform.

Corollary 1.29. *If a is a normal element, then $\|a\| = r(a)$.*

Proof. $\|a\| = \|\hat{a}\| = \sup_{\varphi \in M_A} |\hat{a}(\varphi)| = \sup_{\lambda \in \mathrm{Sp}(a)} |\lambda| = r(a)$. □

1.6 Addition: Stone-Weierstrass theorem

Let us first consider the algebra $C_{\mathbb{R}}(X)$ over $\mathbb{R}$ formed by all continuous real-valued functions on a compact Hausdorff space X.

Theorem 1.30. *Let $A \subseteq C_{\mathbb{R}}(X)$, where X is a compact Hausdorff space, is a closed subalgebra*[2] *such that A separates the points X and contains $1 \in C_{\mathbb{R}}(X)$ (and hence all constant functions). Then $A = C_{\mathbb{R}}(X)$.*

Proof. First of all, we note that the condition for separating points can be strengthened, namely: for any x and y from X and any u and v from $\mathbb{R}$ there is a function $g \in A$ such that $g(x) = u$ and $g(y) = v$. Indeed, since there is $f \in A$ with the property $u' = f(x) \neq f(y) = v'$, then g can be taken equal

$$g = \frac{u - v}{u' - v'} \cdot f + \frac{u'v - v'u}{u' - v'} \cdot 1.$$

For $f, g \in A$ we define continuous functions $f \vee g$, $f \wedge g$, $\gamma(g)$ as

$$(f \vee g)(s) = \max\{f(s), g(s)\}, \qquad (f \wedge g)(s) = \min\{f(s), g(s)\}, \qquad \gamma(g)(s) = |g(s)|.$$

According to Weierstrass's theorem on the approximation of continuous functions by polynomials, there is a sequence of polynomials p_n such that

$$|\,|\lambda| - p_n(\lambda)| \leqslant \frac{1}{n} \qquad \text{with } -n \leqslant \lambda \leqslant n.$$

Then

$$|\,|g(s)| - p_n(g)(s)| = |\,|g(s)| - p_n(g(s))| \leqslant \frac{1}{n} \qquad \text{for } -n \leqslant g(s) \leqslant n.$$

So $\gamma(g) \in A$. Therefore, $f \vee g \in A$, $f \wedge g \in A$, since

$$f \vee g = \frac{f+g}{2} + \frac{\gamma(f-g)}{2}, \qquad f \wedge g = \frac{f+g}{2} - \frac{\gamma(f-g)}{2}.$$

Let us now consider an arbitrary $F \in C_{\mathbb{R}}(X)$ and, by the remark from the beginning of the proof, find for arbitrary $x, y \in X$ a function $f_{x,y} \in A$ such that $f_{x,y}(x) = F(x)$ and $f_{x,y}(y) = F(y)$. Having temporarily fixed y, we find for each $x \in X$ a neighborhood U_x such that $f_{x,y}(u) > F(u) - \varepsilon$ for $u \in U_x$. Let us choose a finite subcover $U_{x_1}, \dots U_{x_p}$ and define $f_y = f_{x_1,y} \vee \dots \vee f_{x_p,y}$. Then $f_y(u) > F(u) - \varepsilon$ for any $u \in X$. Since $f_{x_i,y}(y) = F(y)$ for any $i = 1, \dots, p$, then $f_y(y) = F(y)$. This means that there is a neighborhood V_y of a point y such that $f_y(u) < F(u) + \varepsilon$ for $u \in V_y$. Let's choose a finite subcover $V_{y_1}, \dots, V_{y_q}$ and define $f := f_{y_1} \wedge \dots \wedge f_{y_q}$. Since every $f_{y_i}(u) > F(u) - \varepsilon$ for any $u \in X$, then $f(u) > F(u) - \varepsilon$ for any $u \in X$. On the other hand, for any $u \in X$ there is $V_{y_i} \ni u$, so $f(u) < f_{y_i}(u) < F(u) + \varepsilon$. Combining the inequalities, we obtain that $|f(u) - F(u)| < \varepsilon$ for any $u \in X$. Due to arbitrariness ε we obtain the required result. □

[2]this condition can be weakened

Theorem 1.31. *Let $A \subseteq C(X)$, where X is a compact Hausdorff space, is a closed involutive subalgebra such that A separates the points X and contains $1 \in C(X)$ (and hence all constant functions). Then $A = C(X)$.*

Proof. The involution has the form $f^*(x) = \overline{f(x)}$. Let A_R consist of real-valued functions belonging to A. Note that this is a unital subalgebra of the algebra $C_{\mathbb{R}}(X)$. Since A_R coincides with the kernel of a continuous $\mathbb{R}$-linear mapping $f \mapsto f - f^*$, then it is closed in A, and hence in $C(X)$. Therefore $A_R = A \cap C_{\mathbb{R}}(X)$ is closed in $C_{\mathbb{R}}(X)$. Finally, A_R separates the points X. Indeed, if $f(x) \neq f(y)$, where $f \in A$, then $f = f_1 + if_2$ for $f_1 = (f + f^*)/2 \in A_R$, $f_1 = (f - f^*)/2i \in A_R$, so at least one of f_1, f_2 separates x and y.

Therefore, by the previous theorem, $C_{\mathbb{R}}(X) = A_R \subset A$. Using the representation $f = f_1 + if_2$ again, but for the entire $C(X)$, we see that $\mathbb{C}$-linear combinations of elements of $C_{\mathbb{R}}(X)$ give $C(X)$ and, at the same time, give A by virtue of what has been proven. So $A = C(X)$. $\square$