**Lecture 4**

## 1.7 Positive elements

**Definition 1.32.** A self-adjoint element $a$ in a unital $C^*$-algebra $A$ is called *positive,* if $\mathrm{Sp}(a) \subset [0, \infty)$. If $A$ is not unital, then $a$ is called *positive,* if it is positive in $A^+$.

Positivity is written as $a \geqslant 0$. For two self-adjoint elements $a, b \in A$ we say that $a \geqslant b$ if $a - b \geqslant 0$.

**Problem 29.** Show that if $a \geqslant 0$ and $0 \geqslant a$, then $a = 0$; and also that $-\|a\|1 \leqslant a \leqslant \|a\|1$ for every self-adjoint $a$.

Now let's look at applications of continuous functional calculus to positivity.

**Corollary 1.33.** *Let $a \in A$ be a positive element. Then there exists a unique positive square root $b$ of $a$, that is, $b \geqslant 0$ such that $b^2 = a$.*

*Proof.* The function $f(z) = \sqrt{z}$ is defined and continuous on $[0, \infty)$, so $b = f(a)$ is defined. It is self-adjoint and even positive (since $f$ maps $[0, \infty)$ to itself) and $b^2 = f(a)^2 = a$ (by corollary 1.28). If $c$ is another positive square root of $a$, then $c = f(c^2) = f(a) = b$. □

**Corollary 1.34.** *Let $a \in A$ be a self-adjoint element. Then there are positive elements $a_+, a_- \in A$, such that $a = a_+ - a_-$ and $a_+ a_- = 0$.*

*Proof.* Let us define a continuous function $f : \mathbb{R} \to [0, +\infty)$, putting $f(x) = x$ for $x \geq 0$ and $f(x) = 0$ for $x < 0$. Let's denote $g(x) = f(-x)$. These functions satisfy $f(x) - g(x) = x$ and $f(x)g(x) = 0$. It remains to put $a_+ = f(a)$, $a_- = g(a)$. □

**Corollary 1.35.** *For a self-adjoint element $a \in A$ the following conditions are equivalent:*

(i) $a \geqslant 0$;

(ii) $a = b^2$ *for some self-adjoint* $b$;

(iii) $\|\mu 1 - a\| \leqslant \mu$ *for every* $\mu \geqslant \|a\|$;

(iv) $\|\mu 1 - a\| \leqslant \mu$ *for some* $\mu \geqslant \|a\|$.

*Proof.* By Corollary 1.33, from (i) it follows (ii). Moreover, (iii) implies (iv) by evident reasons.

Let us show that (ii) implies (iii). By assumption, $a = f(b)$, where $f(x) = x^2$. Moreover, the norm of $f$ on $\mathrm{Sp}(b)$ is equal to $\|a\|$, so $0 \leqslant \mu - x^2 \leq \mu$ for any $\mu \geqslant \|a\|$ and $x \in \mathrm{Sp}(b)$. Since $\mu 1 - a = (\mu - f)(b)$, then $\|(\mu - f)(b)\|$ equals the norm of $\mu - f$ on $\mathrm{Sp}(b)$, which does not exceed $\mu$.

Let us finally show that (iv) implies (i). Since, for some $\mu \geqslant \|a\|$, we have $\|\mu - x\| = \sup_{x \in \mathrm{Sp}(a)} |\mu - x| \leqslant \mu$, then no $x \in \mathrm{Sp}(a)$ can be negative. □

**Corollary 1.36.** *If $a$ and $b$ are positive, then $a+b$ is positive.*

*Proof.* Let us choose some $\mu \geqslant \|a\|$, $\nu \geqslant \|b\|$. Then $\|a+b\| \leqslant \mu+\nu$ and the item (iii) of Corollary 1.35 implies

$$\|(\mu+\nu)1-(a+b)\| = \|(\mu 1-a)+(\nu 1-b)\| \leqslant \|\mu 1-a\|+\|\nu 1-b\| \leqslant \mu+\nu.$$

Therefore, by item (iv) of Corollary 1.35, $a+b$ is positive. □

**Problem 30.** If $0 \leqslant a \leqslant b$, then $\|a\| \leqslant \|b\|$. *Hint:* As we know, $b \leqslant \|b\| \cdot 1_A$. Thus $a \leqslant \|b\| \cdot 1_A$, i.e. $\mathrm{Sp}(\|b\| \cdot 1_A - a) \subset [0,+\infty)]$. From $\mathrm{Sp}(\mu 1_A - a) = \mu - \mathrm{Sp}(a)$ deduce that $\|a\| = \max\{\lambda \in \mathrm{Sp}(a)\}$ equals $\min\{\mu\colon \mathrm{Sp}(\mu \cdot 1_A - a) \subset [0,+\infty)\}$. This implies the statement.

The following Proposition 1.38 is almost obvious for operators in a Hilbert space, but is very nontrivial for elements of a $C^*$-algebra. We will need the following result about spectra of products.

**Lemma 1.37.** *If $a,b \in A$, then $\mathrm{Sp}(ab)\cup\{0\} = \mathrm{Sp}(ba)\cup\{0\}$.*

*Proof.* Let $0 \neq \lambda \notin \mathrm{Sp}(ab)$. This means that $(ab-\lambda 1) = -\lambda(1-\lambda^{-1}ab)$ is invertible, so there exists an element $u \in A$ such that $(1-\lambda^{-1}ab)u = 1$. Let us set $v = 1+\lambda^{-1}bua$. Then

$$(1-\lambda^{-1}ba)v = (1-\lambda^{-1}ba)(1+\lambda^{-1}bua) = 1-\lambda^{-1}ba+\lambda^{-1}bua-\lambda^{-2}babua =$$

$$= 1-\lambda^{-1}ba+\lambda^{-1}b(1-\lambda^{-1}ab)ua = 1-\lambda^{-1}ba-\lambda^{-1}ba = 1,$$

so $\lambda \notin \mathrm{Sp}(ba)$. □

**Proposition 1.38.** *The element $a^*a$ is positive for every $a \in A$.*

*Proof.* Since $a^*a$ is self-adjoint, we can write $a^*a = b_+ - b_-$ by Corollary 1.34. Let $c := \sqrt{b_-}$, $t := ac$. notice, that $f(0) = 0$ for $f(x) = \sqrt{x}$, so $c$ is approximated by polynomials in $b_-$ without a free term, which means $cb_+ = 0$. We have:

$$-t^*t = -c(b_+ - b_-)c = b_-^2. \tag{1.6}$$

Therefore, $-t^*t$ is positive.

Let us write $t$ in the form $t = x+iy$, where $x$ and $y$ are self-adjoint elements of $A$ (that is $x = (t+t^*)/2$, $y = (t-t^*)/2i$). Then $t^*t+tt^* = 2(x^2+y^2)$ is positive by Corollary 1.36. By the same consequence, we see that the element

$$tt^* = (t^*t+tt^*)-t^*t = (t^*t+tt^*)+b_-^2$$

is also positive, that is, $\mathrm{Sp}(tt^*) \subset [0,\infty)$. By Lemma 1.37, $\mathrm{Sp}(t^*t) \subset [0,\infty)$, so $t^*t$ is positive. But it is also negative by (1.6), so $t^*t = 0$ for problem 29. This means $b_- = 0$, since positive square root is unique. □

**Corollary 1.39.** *If $b \leqslant c$, then $a^*ba \leqslant a^*ca$ for any $a \in A$.*

*Proof.* Since $c-b$ is positive, then $c-b=d^2$ for some self-adjoint $d$; That's why $a^*(c-b)a = a^*d^2a = (da)^*(da) \geqslant 0$. □

**Corollary 1.40.** *If $a$ and $b$ are invertible and $0 \leqslant a \leqslant b$, then $b^{-1} \leqslant a^{-1}$.*

*Proof.* Let us first consider the special case $b = 1$. Then $\mathrm{Sp}(a) \subseteq [0,1]$. According to the spectral mapping theorem, we see that $\mathrm{Sp}(a^{-1}) \subseteq [1,\infty)$, so $a^{-1} \geqslant 1$. Let's pass to the general case. Since, by Corollary 1.39, $b^{-1/2}ab^{-1/2} \leqslant 1$, then $b^{1/2}a^{-1}b^{1/2} \geqslant 1$ by the first part of the proof. Multiplying this inequality by $b^{-1/2}$ on both sides and again applying corollary of 1.39, we obtain $a^{-1} \geqslant b^{-1}$. □

## 1.8 Approximate identity

Let $\Lambda$ be a directed set. A family $(u_\lambda)_{\lambda\in\Lambda}$ elements of the $C^*$-algebra $A$ is called an *approximate identity (unit),* if $\lim_\Lambda \|xu_\lambda - x\| = 0$ for $x \in A$ (and therefore $\lim_\Lambda \|u_\lambda x - x\| = 0$). If $A$ is unital, then we can take $u_\lambda = 1$ for any $\lambda$, so this concept is of interest only for non-unital algebras. In the definition of an approximate unit also include the conditions $0 \leqslant u_\lambda \leqslant 1$ and $u_\lambda \leqslant u_\mu$ for all $\lambda \leqslant \mu$ from $\Lambda$.

**Theorem 1.41.** *Every $C^*$-algebra has an approximate unit.*

*Proof.* Let $\Lambda = \{a \in A \mid a \geqslant 0; \|a\| < 1\}$. An order on $\Lambda$ is given by $\leqslant$. Let us show that the set of elements $\Lambda$, indexed tautologically ($a$ has index $a$), is an approximate unit.

First of all, we need to check that $\Lambda$ is a directed set, that is, for any two elements $a, b \in \Lambda$ there is a $c \in \Lambda$ such that $a \leqslant c$ and $b \leqslant c$. Let $f(t) := \frac{t}{1-t}$ and $g(t) := \frac{t}{1+t}$. Moreover, the function $f$ is defined on $[0,1)$, the function $g$ is defined on $[0,\infty)$ and $g(f(t)) = t$. Let us put $x := f(a)$, $y := f(a)+f(b)$, $c := g(y)$. Since $0 \leqslant g(t) < 1$, then $c \in \Lambda$. The inequality $x \leqslant y$ implies $1+x \leqslant 1+y$, so $(1+x)^{-1} \geqslant (1+y)^{-1}$ and

$$a = 1-(1+x)^{-1} \leqslant 1-(1+y)^{-1} = c.$$

The inequality $b \leqslant c$ can be obtained similarly, so we have verified that the set $\Lambda$ is directed.

Now let's check that $\lim_\Lambda \|x - ax\| = 0$ for every $x \in A$. Since by Corollary 1.34 each element can be decomposed into a linear combination of four positive elements:

$$x = \frac{x+x^*}{2} + i\frac{x-x^*}{2i} = \left(\frac{x+x^*}{2}\right)_+ - \left(\frac{x+x^*}{2}\right)_- + i\left(\frac{x-x^*}{2i}\right)_+ - i\left(\frac{x-x^*}{2i}\right)_-, \quad (1.7)$$

then it is sufficient to verify the statement for $x \geqslant 0$. Since for $a \in \Lambda$ we have $0 \leqslant 1-a \leqslant 1$, then by Corollary 1.39, $(1-a)^{1/2}(1-a)(1-a)^{1/2} \leqslant (1-a)^{1/2}(1-a)^{1/2} = 1-a$. That's why (see Problem 30)

$$\|(1-a)x\|^2 = \|x^*(1-a)^2x\| \leqslant \|x^*(1-a)x\|,$$

and it is sufficient to verify that $\lim_\Lambda \|x(1-a)x\| = 0$ for any $x \geqslant 0$ from $A$, and without loss of generality we can assume that $\|x\| = 1$.

Similar to the previous reasoning, if $a, b \in \Lambda$ and $a \leqslant b$ then $\|x^*(1-b)x\| \leq \|x^*(1-a)x\|$, so, for $x \geqslant 0$,

$$\sup_{b \in \Lambda,\, b \geqslant a} \|x(1-b)x\| = \|x(1-a)x\|.$$

Therefore we need to show that, for any positive $x \in A$ of norm one and for any $\varepsilon > 0$, there is an element $a \in \Lambda$ such that $\|x^*(1-a)x\| < \varepsilon$. Let us put $a_n := g(nx)$, $n \in \mathbb{N}$ (see the beginning of the proof). Then $\|x(1-a_n)x\| = \|h(x)\|$, where $h(t) := t^2(1-g(nt)) = \frac{t^2}{1+nt}$. For any $t \in [0,1]$, we have $0 \leqslant h(t) \leqslant \frac{1}{n}$, so $\|h(x)\| \leqslant \frac{1}{n}$. Hence,

$$\sup_{b \in \Lambda,\, b \geqslant a_n} \|x(1-b)x\| = \|x(1-a_n)x\| \leqslant \frac{1}{n}$$

for any $n$, so $\lim_{b \in \Lambda} \|x(1-b)x\| = 0$. □

**Definition 1.42.** An approximate unit is called *countable,* if the set $\Lambda$ is countable.

**Corollary 1.43.** *A separable $C^*$-algebra has a countable approximate unit.*

*Proof.* Let us choose a dense sequence $(x_n)_{n \in \mathbb{N}}$ in $A$. Then there is an element $a_1 \in A$, $0 \leqslant a_1$, $\|a_1\| < 1$ such that $\|x_1 a_1 - x_1\| \leqslant 1$ (see the proof of the previous theorem). Suppose by induction that we have already found such $a_2, \ldots, a_n$, $a_1 \leqslant a_2 \leqslant \ldots \leqslant a_n$, such that, for all $k = 1, 2, \ldots, n$, the inequalities $\|a_k\| < 1$ and $\|x_i a_k - x_i\| \leqslant \frac{1}{k}$ are satisfied for $i = 1, 2, \ldots, k$. Now let's find an element $a_{n+1} \geqslant a_n$ with norm $\|a_{n+1}\| < 1$, such that $\|x_i a_{n+1} - x_i\| \leqslant \frac{1}{n+1}$ for $i = 1, 2, \ldots, n+1$. Because the sequence $(x_n)$ is dense in $A$, then by induction we obtain a non-decreasing sequence $(a_n)_{n \in \mathbb{N}}$ positive elements of the unit ball with $\lim_{n \to \infty} \|x a_n - x\| = 0$ for each $x \in A$. Indeed, for any $\varepsilon > 0$ we can find an element $x_i$ such that $\|x - x_i\| < \varepsilon/3$, and for $x_i$ we can find a number $j$ such that $\|x_i a_k - x_i\| < \varepsilon/3$ for all $k > j$. Then for these $k$

$$\|x a_k - x\| = \|x_i a_k - x_i + (x - x_i) a_k + x - x_i\| < \varepsilon/3 + \varepsilon/3 \|a_k\| + \varepsilon/3 \leqslant \varepsilon.$$

□

**Problem 31.** Prove that the converse is not true: an algebra with countable approximate unit does not have to be separable.