Lecture 5

1.9 Ideals, factors and homomorphisms

Under an *ideal of C^*-algebra* we will always mean norm-closed two-sided ideal (for maximal ideals in in the commutative case this happens automatically).

Lemma 1.44. *Every ideal I in a C^*-algebra is self-adjoint: $I = I^*$.*

Proof. If $I \subset A$ is an ideal, then $B := I \cap I^* \subseteq A$ is a C^*-subalgebra. In this case, $B \supset I \cdot I^*$. Let (u_λ) is an approximate unit in B, and $j \in I$. Then

$$\lim_{\lambda\in\Lambda} \|j^* u_\lambda - j^*\|^2 = \lim_{\lambda\in\Lambda} \|u_\lambda(jj^* u_\lambda - jj^*) - (jj^* u_\lambda - jj^*)\| \leqslant 2 \lim_{\lambda\in\Lambda} \|jj^* u_\lambda - jj^*\| = 0.$$

Since $u_\lambda \in I$, then $j^* u_\lambda \in I$, so $j^* \in I$, since I is closed. □

The following technical lemma is often used.

Lemma 1.45. *If $x^*x \leqslant a$ is in A, then there is an element $b \in A$ such that $\|b\| \leqslant \|a\|^{1/4}$ and $x = ba^{1/4}$.*

Proof. Let us put $b_n := x(a + \frac{1}{n}1)^{-1/2} a^{1/4}$ (this element lies in A, even if A does not have a unit, but in this case it is convenient for us to carry out calculations in A^+). Let also

$$d_{nm} := \left(a + \frac{1}{n}1\right)^{-1/2} - \left(a + \frac{1}{m}1\right)^{-1/2}, \qquad f_n(t) := t^{3/4}\left(t + \frac{1}{n}\right)^{-1/2}.$$

Then the sequence of functions $\{f_n(t)\}$ converges to $f(t) := t^{1/4}$ uniformly on $[0, \|a\|]$, since due to $u^2 + v^2 \geqslant 2uv$ we have

$$\left(t^{1/4}\left(1 - \frac{t^{1/2}}{(t+1/n)^{1/2}}\right)\right)^2 = t^{1/2} \frac{t + 1/n + t - 2t^{1/2}(t+1/n)^{1/2}}{t+1/n} <$$

$$< t^{1/2} \frac{t + 1/n + t - 2t}{t + 1/n} = t^{1/2} \frac{1/n}{t + 1/n} = \frac{2\sqrt{t/n}}{t + 1/n} \cdot \frac{1}{2\sqrt{n}} \leqslant \frac{1}{2\sqrt{n}}.$$

We have

$$\begin{aligned} \|b_n - b_m\|^2 &= \|x d_{nm} a^{1/4}\|^2 = \|a^{1/4} d_{nm} x^* x d_{nm} a^{1/4}\| \leqslant \|a^{1/4} d_{nm} a d_{nm} a^{1/4}\| = \\ &= \|d_{nm} a^{3/4}\|^2 = \|f_n(a) - f_m(a)\|^2 = \sup_{t\in[0,\|a\|]} |f_n(t) - f_m(t)|. \end{aligned}$$

Thus, since f_n is a Cauchy sequence, so is b_n. Let us put $b := \lim_{n\to\infty} b_n$. Then $ba^{1/4} = \lim_{n\to\infty} b_n a^{1/4} = \lim_{n\to\infty} x(a + \frac{1}{n}1)^{-1/2} a^{1/2} = x$. □

Problem 32. Verify that the last limit is indeed x. This can be done in a similar way to the calculation for f_n in the proof, using $x^*x \leqslant a$.

Definition 1.46. A subalgebra $B \subset A$ is called *hereditary* if for any positive $b \in B$ and $a \in A$, from the condition $0 \leqslant a \leqslant b$ it follows that $a \in B$.

Problem 33. Prove that a positive element of an arbitrary C^*-subalgebra is a positive element of the entire algebra.

Lemma 1.47. *Let $I \subset A$ be an ideal and $j \in I$ a positive element. If $a^*a \leqslant j$, then $a \in I$. In particular, any ideal is a hereditary subalgebra.*

Proof. Let us represent $a = bj^{1/4}$ in accordance with Lemma 1.45. Moreover, $j^{1/4} \in C^*(j) \subset I$, and therefore $a \in I$. □

If $I \subset A$ is an ideal, then we can define Banach factor algebra A/I with norm $\|a+I\| := \inf_{j \in I} \|a + j\|$. This is an involutive algebra: since I is self-adjoint, then $\|(a + I)^*\| = \|a^* + I\| = \|a + I\|$. To be short we will denote $a + I$ by $\dot{a} \in A/I$.

Theorem 1.48. *The involutive algebra A/I is a C^*-algebra.*

Proof. Only the C^* property needs to be verified. Let $(u_\lambda)_{\lambda\in\Lambda}$ be an approximate unit of I (note that ideals typically do not have a unit, and in any case, a proper ideal does not contain the unit of A, even if the latter exists). Let us first show that

$$\|\dot{a}\| = \lim_{\lambda\in\Lambda} \|a - au_\lambda\|. \tag{1.8}$$

Indeed, since $u_\lambda \in I$, then $\|\dot{a}\| \le \|a - au_\lambda\|$. To prove the reverse inequality, we choose arbitrarily $\varepsilon > 0$. Then there is an element $j \in I$ such that $\|\dot{a}\| \geqslant \|a - j\| - \varepsilon$. We have

$$\lim_{\lambda\in\Lambda} \|a - au_\lambda\| \leqslant \lim_{\lambda\in\Lambda} (\|a - au_\lambda - (j - ju_\lambda)\| + \|j - ju_\lambda\|) = \lim_{\lambda\in\Lambda} \|a - au_\lambda - (j - ju_\lambda)\|.$$

Writing in A^+, where the equality $a - au_\lambda - (j - ju_\lambda) = (a - j)(1 - u_\lambda)$ holds, we obtain the estimation $\|(a - j)(1 - u_\lambda)\| \leqslant \|a - j\| < \|\dot{a}\| + \varepsilon$. Due to arbitrariness of $\varepsilon > 0$ we obtain (1.8).

Now, calculating in A^+, we find the estimation

$$\begin{aligned} \|\dot{a}^*\dot{a}\| &= \lim_{\lambda\in\Lambda} \|a^*a(1 - u_\lambda)\| \geqslant \lim_{\lambda\in\Lambda} \|(1 - u_\lambda)a^*a(1 - u_\lambda)\| = \\ &= \lim_{\lambda\in\Lambda} \|(a(1 - u_\lambda)\|^2 = \|\dot{a}\|^2. \end{aligned}$$

The inverse inequality $\|\dot{a}^*\dot{a}\| \leqslant \|\dot{a}\|^2$ is true in any involutive Banach algebra. □

Definition 1.49. Let A and B be C^*-algebras. A $*$-*homomorphism* from A to B is any homomorphism φ preserving the involution: $\varphi(a^*) = \varphi(a)^*$. If both algebras are unital, φ is called *unital*, if $\varphi(1_A) = 1_B$.

Problem 34. Let $\varphi : A \to B$ be a $*$-homomorphism of non-unital algebras. Prove that there is a unique unital $*$-homomorphism $\varphi^+ : A^+ \to B^+$, extending φ. Note: The only way to determine φ^+ is the requirement to be unital: $\varphi^+(1) = 1$.

Problem 35. Let $\varphi : A \to B$ be a $*$-homomorphism of algebras, with A non-unital, and B unital. Prove that there is a unique unital $*$-homomorphism $\varphi^{(+)} : A^+ \to B$, extending φ. *Hint:* the same as above.

Theorem 1.50. *Let $\varphi : A \to B$ be a nonzero $*$-homomorphism. Then $\|\varphi\| = 1$ (in particular, it is continuous) and $\varphi(A)$ is a C^*-subalgebra of B. If φ is injective, then it is isometric (on the image).*

Proof. If the algebra A is non-unital, then we will consider φ^+ from Problem 34 or $\varphi^{(+)}$ from problem 35. If the algebra A is unital, then we can assume that B is unital too (if not — then we attach a unity without requiring the homomorphism to be unital). Then $\varphi(1_A) = p$ is a self-adjoint idempotent ($p^2 = p$), the space $B_p := pBp$ is a subalgebra of B (see problem 36) with identity $p = p \cdot 1_B \cdot p$, and φ, considered as a homomorphism in B_p, is unital.

Thus, in the proof we can restrict ourselves to the case of a unital homomorphism $\varphi : A \to B$ of unital algebras.

To distinguish the spectrum of an element in A and B, we will write Sp_A (resp., Sp_B) for the spectrum of elements in A (resp., in B).

Let $a = a^* \in A$. Then $\mathrm{Sp}_B(\varphi(a)) \subset \mathrm{Sp}_A(a)$, since φ is a unital $*$-homomorphism of algebras and $\|\varphi(a)\| = r(\varphi(a)) \leqslant r(a) = \|a\|$. For an element $a \in A$ of general form, we have $\|\varphi(a)\|^2 = \|\varphi(a^*a)\| \leqslant \|a^*a\| = \|a\|^2$, so $\|\varphi\| \leqslant 1$, that is, φ is continuous and does not increase the norm.

Suppose now that φ is injective but not isometric. Then there is an element $a \in A$ such that $\|\varphi(a)\| < \|a\|$. This means $\|\varphi(b)\| < \|b\|$ for $b := a^*a$. Let us denote $\|\varphi(b)\| =: r$ and $\|b\| =: s$. Let h be a continuous real function that satisfies the conditions $q(t) = 0$ for $t \in [0, r]$ and $h(s) = 1$. Then $\|\varphi(h(b))\| = \|h(\varphi(b))\| = \sup_{\lambda \in \mathrm{Sp}_B(\varphi(b))} |h(\lambda)| = 0$, while $\|h(b)\| = \sup_{\lambda \in \mathrm{Sp}_A(b)} |h(\lambda)| \geqslant 1$. A contradiction with injectivity. (The commutation condition is obvious for polynomials, h_n, uniformly approximating h, and in the limit we obtain it for h.)

In the case of a general (not necessarily injective) $*$-homomorphism, note that that $I = \operatorname{Ker} \varphi$ is closed since φ is continuous, so I is an ideal in A. Therefore φ induces an injective $*$-homomorphism $\dot\varphi : A/I \to B$ by the rule $\dot\varphi(\dot a) = \varphi(a)$. Then, by what has been proven, $\dot\varphi$ is isometric, and $\varphi(A) = \dot\varphi(A/I)$ is closed in B, so it is a C^*-subalgebra. Since φ is non-zero, then there is $a \in A$ with $\varphi(a) \neq 0$. Since $\dot\varphi$ is isometric, we have the equality $\|\dot a\| = \|\dot\varphi(\dot a)\| = \|\varphi(a)\|$. Moreover, for any $\varepsilon > 0$ there is an element $c \in A$ such that $\dot c = \dot a$ and $\|c\| < \|\dot a\| + \varepsilon$. Thus, $\|\varphi(c)\| > \|c\| - \varepsilon$. Since ε is arbitrary, we obtain $\|\varphi\| \geqslant 1$, so $\|\varphi\| = 1$. □

Problem 36. Prove that the algebra B_p is closed, first obtaining the equality $pBp = \operatorname{Ker}(L_{1-p}) \cap \operatorname{Ker}(R_{1-p})$, where L_{1-p} and R_{1-p} are the linear operators of left and right multiplication by $1 - p$ in B, given by $L_{1-p} : b \mapsto (1-p)b$ and $R_{1-p} : b \mapsto b(1-p)$.

Problem 37. Develop the result of the previous problem by verifying the decomposition into a direct sum of closed subspaces $B = pBp \oplus pB(1-p) \oplus (1-p)Bp \oplus (1-p)B(1-p)$. Moreover, if we write down the quadruple (a, b, c, d), representing an element of a given direct sum in the form of a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then the multiplication in B passes under this isomorphism to the matrix multiplication according to the standard rule.

Problem 38. Derive from Theorem 1.50 the statement $\varphi(f(a)) = f(\varphi(a))$ for any normal a and f, which is continuous on the appropriate set (not only for a polynomial).

Problem 39. Obtain a proof of Theorem 1.50 via a reduction to a map of commutative subalgebras.

Corollary 1.51. *Let $I \subset A$ be an ideal, and $B \subset A$ be a C^*-subalgebra. Then $I + B$ coincides with the C^*-subalgebra $C^*(I, B)$ generated by I and B.*

Proof. It is obvious that $I + B \subset C^*(I, B)$ is an involutive subalgebra. Let $q : A \to A/I$ be the $*$-homomorphism of factorization. We know from the previous theorem that $q(B)$ is closed in A/I, so $I + B = q^{-1}(q(B))$ is closed in A. This means that $I + B$ is a C^*-algebra, contained in $C^*(I, B)$. □

So far we were very careful when considering spectrum of an element in a C^*-algebra and its C^*-subalgebra. The next lemma shows that this is not so important.

Lemma 1.52. *Let $B \subset A$ be a unital C^*-subalgebra of a unital C^*-algebra, $1_A = 1_B$, and $a \in B$. Then $\mathrm{Sp}_B(a) = \mathrm{Sp}_A(a)$.*

Proof. Obviously, if an element has an inverse in B, then so does A, whence $\mathrm{Sp}_A(a) \subset \mathrm{Sp}_B(a)$. The reverse inclusion follows from the statement: if a is invertible into A, then its inverse belongs to B. To prove this, consider first the case $a = a^*$. Then the C^*-algebra $C = C^*(a, a^{-1})$ generated by a and a^{-1}, is a commutative unital C^*-subalgebra of A, and therefore it is isomorphic to some algebra of functions $C(X)$. Let $\hat{a}$ denote the image of a under this isomorphism. Then $0 \notin \mathrm{Sp}_{C(X)}(\hat{a}) \subset \mathbb{R}$. Let us choose polynomials p_n such that $p_n(t)$ converges uniformly to t^{-1} on $\mathrm{Sp}_{C(X)}(\hat{a})$. Then $\widehat{a^{-1}} = \lim_{n\to\infty} p_n(\hat{a})$, so $a^{-1} = \lim_{n\to\infty} p_n(a) \in C^*(a) \subset B$.

For a general element a, if a^{-1} exists in A, then $a^{-1}(a^*)^{-1} = (a^*a)^{-1} \in B$ as proven. That is why $a^{-1} = (a^*a)^{-1}a^* \in B$. □

Problem 40. Show with an example that, without the condition $1_A = 1_B$, the previous proposition does not hold.