Lecture 6

1.10 Topologies on $\mathbb{B}(H)$ and von Neumann algebras

Besides the norm topology, there are other useful topologies on the C^*-algebra $\mathbb{B}(H)$.

Definition 1.53. *Strong topology* is defined by a system of seminorms $a \to \|a\xi\|$, $\xi \in H$.
Weak topology is defined by a seminorm system $a \to (a\xi, \eta)$, $\xi, \eta \in H$.

Theorem 1.54. *For a linear functional $\varphi : \mathbb{B}(H) \to \mathbb{C}$ the following conditions are equivalent:*

(i) There exist $\xi_k, \eta_k \in H$, $k = 1, \dots, n$, such that $\varphi(a) = \sum_{k=1}^n (a\xi_k, \eta_k)$ for any $a \in \mathbb{B}(H)$;

(ii) φ is weakly continuous;

(iii) φ is strongly continuous.

Proof. It is obvious that (i)$\Longrightarrow$(ii) $\Longrightarrow$(iii). Let us show that (iii) implies (i).

The strong continuity of φ means that the preimage $\{a : |\varphi(a)| < 1\}$ of the open unit disk is an open set in the strong topology, that is, there are positive constants $\varepsilon_1, \dots, \varepsilon_n$ and vectors $\xi_1, \dots, \xi_n$, such that for any $a \in \mathbb{B}(H)$ the condition $\|a\xi_k\| < \varepsilon_k$ (for all $k = 1, \dots, n$) implies $|\varphi(a)| < 1$. Changing the length of these vectors if necessary, we see that in an equivalent way we can say, that there exist vectors $\xi_1, \dots, \xi_n$ such that, for any $a \in \mathbb{B}(H)$, from $\max_k \|a\xi_k\| \leqslant 1$ it follows that $|\varphi(a)| \leqslant 1$. Then

$$|\varphi(a)| \leqslant \left(\sum_{k=1}^n \|a\xi_k\|^2 \right)^{1/2}. \tag{1.9}$$

Indeed, if $|\varphi(a)|^2 > \sum_{k=1}^n \|a\xi_k\|^2$ for some a, then $|\varphi(a)| > \|a\xi_k\|$ for all k, so since the number of them is finite, one can find $\alpha \in \mathbb{R}$ such that $|\varphi(\alpha a)| > 1$, and $\max_k \|\alpha a \xi_k\| < 1$ (for example, $\alpha^{-1} := (|\varphi(a)| + \max \|a\xi_k\|)/2$). A contradiction.

Let $K := \oplus_{k=1}^n H$. The algebra $\mathbb{B}(K)$ can be identified with the algebra of $n \times n$-matrices with elements from $\mathbb{B}(H)$. Let $\rho : \mathbb{B}(H) \to \mathbb{B}(K)$ map $a \in \mathbb{B}(H)$ to a diagonal matrix with all diagonal elements equal to a.

Let us denote $\xi := \xi_1 \oplus \dots \oplus \xi_n \in K$ and note that, putting $\psi(\rho(a)\xi) = \varphi(a)$, we obtain a linear functional on the closed subspace $L \subset K$, where L is the closure of the space $L_0 := \{\rho(a)\xi \mid a \in \mathbb{B}(H)\}$. Indeed, first we need to verify that ψ is well defined on L_0: if $\rho(a)(\xi) = \rho(b)(\xi)$, then $(a-b)\xi_k = 0$ for $k = 1, \dots, n$. In particular for arbitrarily large $R > 0$ we have $\|R(a-b)\xi_k\| \leqslant 1$, and therefore $|\varphi(R(a-b))| \leqslant 1$. Therefore $|\varphi(a-b)| \leqslant 1/R$. Due to the arbitrariness of R, we obtain that $\varphi(a-b) = 0$ and thus ψ is well defined on L_0. From (1.9) we see that $|\psi(\rho(a))| \leqslant \|\rho(a)\xi\|$, so $|\psi(\zeta)| \leq \|\zeta\|$ for any $\zeta \in L_0$, and hence L. So, ψ is a bounded functional on L. By the Riesz theorem on representation of functionals, there is a vector $\eta \in L \subset K$ such that $\psi(\zeta) = (\zeta, \eta)$

for all $\zeta \in L$ (we assume the Hermitian product to be linear in the first argument), so $\varphi(a) = (\rho(a)\xi, \eta)$. Decomposing it into components $\eta = \eta_1 \oplus \ldots \oplus \eta_n$, we obtain $(\rho(a)\xi, \eta) = \sum_{k=1}^{n}(a\xi_k, \eta_k)$. $\square$

Corollary 1.55. *In $\mathbb{B}(H)$ a convex set is closed for the weak topology if and only if it is closed for the strong one.*

Proof. This immediately follows from the previous theorem, since, according to the Hahn-Banach theorem, closed convex sets are obtained as the intersection of closed half-spaces, corresponding to linear functionals. $\square$

Definition 1.56. A *von Neumann algebra* is a C^*-subalgebra $\mathbb{B}(H)$ containing unity (identity operator) and closed in the weak topology.

The simplest examples are $\mathbb{C}$ and $\mathbb{B}(H)$ (in fact, the first algebra is a special case of the second).

Definition 1.57. For the set $S \subseteq \mathbb{B}(H)$ we denote by S' its *commutant*, that is, the set of all operators $a \in \mathbb{B}(H)$ such that $as = sa$ for every $s \in S$.

Problem 41. Verify that

- If S is self-adjoint, then so is S'.
- The commutant of any set is a unital algebra.
- The commutant of any set is weakly closed.
- Thus, S' is the von Neumann algebra for any self-adjoint set S.
- If $S_1 \subset S_2$, then $S_1' \supset S_2'$.
- Always $S \subset S''$.
- Therefore $S' = S'''$, $S'' = S''''$, etc.

Theorem 1.58 (von Neumann bicommutant theorem)**.** *Let A be a C^*-subalgebra of $\mathbb{B}(H)$ containing the identity operator. Then the following conditions are equivalent.*

(i) $A = A''$;

(ii) A *is weakly closed;*

(iii) A *is strongly closed.*

Proof. Since A is a convex subset, then (ii) and (iii) are equivalent as a consequence 1.55. Since A'' is weakly closed, then (ii) follows from (i). It remains to show that (iii) implies (i).

For a vector $\xi \in H$ we denote by p the projection onto the closure V of the linear subspace formed by vectors $a\xi$, $a \in A$.

Thus, $p\eta = \eta$ for $\eta \in V$. Since $1 \in A$, then $\xi \in V$, so $p\xi = \xi$. Therefore $pap\zeta = pa\eta = a\eta = ap\zeta$ for any $\zeta \in H$, where we denote $\eta = p\zeta \in V$. So $pap = ap$ for any $a \in A$. From here $pa = (a^*p)^* = (pa^*p)^* = pap$ and we get $ap = pa$, that is $p \in A'$. Let $b \in A''$. Then $pb = bp$, so $pb\xi = bp\xi = b\xi$ and $b\xi \in V$. Thus, for every $\varepsilon > 0$ there is an element $a \in A$, for which $\|(b-a)\xi\| < \varepsilon$.

Now consider some $\xi_1, \dots, \xi_n \in H$ and define $\xi := \xi_1 \oplus \dots \oplus \xi_n \in K := H \oplus \dots \oplus H$. Let $\rho : \mathbb{B}(H) \to \mathbb{B}(K)$ be the diagonal embedding. It is easy to see that $\rho(A)'$ consists of all $n\times n$-matrices with elements from A', and $\rho(A'') = \rho(A)''$ (Problem 42). Applying the first part of the proof to this situation, we obtain that for any $b \in A''$ and every $\varepsilon > 0$ there is an element $a \in A$ such that $\|(\rho(b) - \rho(a))\xi\| < \varepsilon$. Then $\sum_{k=1}^n \|(b-a)\xi_k\|^2 = \|(\rho(b) - \rho(a))\xi\|^2 < \varepsilon^2$, so we can strongly approximate $b \in A''$ by some $a \in A$. $\square$

Problem 42. Check that $\rho(A)'$ consists of all $n\times n$-matrices with elements from A', and $\rho(A'') = \rho(A)''$.

Corollary 1.59. *If A is a von Neumann algebra, then A' is a von Neumann algebra.*

Definition 1.60. The *center* of an algebra is the set of its elements that commute with all its elements.

Corollary 1.61. *If A is a von Neumann algebra, then its center Z is also a von Neumann algebra.*

Proof. For a subalgebra $A \subseteq \mathbb{B}(H)$ we have $Z = A \cap A'$. $\square$

Let $A \subset \mathbb{B}(H)$ be a C^*-algebra containing the identity operator. Then the bicommutant theorem states that A is weakly (strongly) dense in A''. This result has the disadvantage that the approximation is done by elements with, generally speaking, an uncontrollable norm. This is overcome by the following theorem, which we present without the proof, which can be found in [10, § 4.3].

Theorem 1.62 (Kaplansky density theorem)**.** *The unit ball A is weakly (strongly) dense in the unit ball A''. The same is true for the sets of positive elements in these unit balls and for sets of unitary elements.*

Definition 1.63. If the center Z of the von Neumann algebra A consists only of scalar operators (that is, $Z = \mathbb{C}1$), then A is called a *factor*.

Remark 1.64. It should be noted that besides the continuous functional calculus for self-adjoint operators, there is a Borel functional calculus: instead of norm approximation of continuous functions by polynomials here Borel functions are approximated by polynomials, and the corresponding operators will converge in the weak topology. More precisely, let the polynomials p_i converge monotonically and pointwise to a Borel function f on the spectrum of a self-adjoint operator $a \in \mathbb{B}(H)$. Then $\{p_i(a)\}$ is a strongly convergent sequence of operators (this is a statement from the standard course, see for example [15, §§ 7 and 11]). Since all polynomials commute with the commutator of the self-adjoint operator, then for any Borel function f on the spectrum of a self-adjoint operator a, the operator $f(a)$ lies in $\{a\}''$.

Chapter 2

Representations of C^*-algebras

2.1 Definition and basic properties

Definition 2.1. A *representation* of a C^*-algebra A on a Hilbert space H is a $*$-homomorphism from A to $\mathbb{B}(H)$.

Definition 2.2. A representation of a C^*-algebra A is called *algebraically irreducible*, if there is no proper invariant linear subspace in H (when operated by operators from the image of the representation). A representation is *topologically irreducible*, if there is no proper closed invariant subspaces.

We will see soon that for C^*-algebras these two concepts coincide.

Lemma 2.3. *A representation π is topologically irreducible if and only if $\pi(A)' = \mathbb{C}1$.*

Proof. If $\pi(A)'$ contains something other than scalars, then it also contains a self-adjoint non-scalar operator (this immediately follows from the expansion of a non-scalar operator into a linear combination of two self-adjoint ones $a = \frac{a+a^*}{2} + i \cdot \frac{a-a^*}{2i}$). Using Borel functional calculus (see note 1.64) for this self-adjoint operator b, we can obtain a proper projection p in $\pi(A)'$. Namely, if an operator is nonscalar, then it has at least two distinct points in the spectrum, say, t_0 and t_1, and we need to consider a Borel function f, taking values 0 and 1, and $f(t_0) = 0$, $f(t_1) = 1$ (task 43). (You can also not use calculus, but simply take suitable spectral projections from the standard spectral theorem, that by construction have the necessary commutation properties). Then pH is a closed invariant subspace, since $p \in \pi(A)'$.

Conversely, let $L \subset H$ be a closed $\pi(A)$-invariant subspace, and $p \in \mathbb{B}(H)$ is a projection onto this subspace. Then $\pi(a)p = p\pi(a)p$ for any $a \in A$. Therefore $p\pi(a) = (\pi(a^*)p)^* = (p\pi(a^*)p)^* = p\pi(a)p = \pi(a)p$ and $p \in \pi(A)'$. Moreover, p is not a scalar. $\square$

Problem 43. Verify in the proof above that $f(b)$ is a proper projection, since $f^2 = f$ and $\mathrm{Sp}(f(b)) = \{0, 1\}$.

Problem 44. Prove a more general fact: if a self-adjoint element a in a unital C^*-algebra has $\mathrm{Sp}(a) = \{0, 1\}$, then a is a nonscalar idempotent.

Lemma 2.4. *Let π be a topologically irreducible representation of a C^*-algebra A in a Hilbert space H. Then for any $t \in \mathbb{B}(H)$, a finite-dimensional subspace $L \subset H$ and $\varepsilon > 0$, there is an element $a \in A$ such that $\|a\| \leq \|t|_L\|$ and $\|(\pi(a) - t)|_L\| < \varepsilon$.*

Proof. Since π is topologically irreducible, then by Lemma 2.3 $\pi(A)'$ coincides with scalars, hence $\pi(A)'' = \mathbb{B}(H)$. That is why $\pi(A)$ is dense in $\mathbb{B}(H)$ in the weak (strong) topology. Without loss of generality, we can assume that $\|t|_L\| = 1$. Let us put $s = tp_L$, where p_L is the projection onto L. Since L is finite-dimensional, then, by Kaplansky's density theorem, there is $b \in A$ such that $\|\pi(b)\| \leqslant 1$ and $\|(\pi(b) - s)|_L\| < \varepsilon/2$. Then there is an element $c \in A$ such that $\pi(c) = \pi(b)$ and $\|c\| < \|\pi(b)\|(1 + \varepsilon/2)$ (see Theorem 1.50). Let us put $a := \frac{c}{1+\varepsilon/2}$. Then $\|a\| \leqslant 1$ and

$$\|(\pi(a) - t)|_L\| \leqslant \|(\pi(c) - t)|_L\| + \|\pi(a) - \pi(c)\| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

$\square$