Lecture 7

Lemma 2.5. *Let π be a topologically irreducible representation of the C^*-algebra A in the Hilbert space H. Then for any $t \in \mathbb{B}(H)$, finite-dimensional subspace $L \subset H$ and $\varepsilon > 0$, there is an element $a \in A$ such that $\pi(a)|L = t|_L$ and $\|a\| \leqslant \|t\| + \varepsilon$.*

Proof. By the previous lemma, there is an element $a_0 \in A$ such that $\|a_0\| \leqslant \|t\|$ and $\|(\pi(a_0) - t)|_L\| < \varepsilon/2$. By induction we can find for each n is an element of $a_n \in A$ such that $\|a_n\| \leqslant 2^{-n}\varepsilon$ and $\|(\sum_{k=0}^{n} \pi(a_n) - t)p_L|\| < 2^{-n-1}\varepsilon$. Indeed, suppose that the elements are found for some n and all the smaller ones. Applying the previous lemma to $s = -\sum_{k=0}^{n} \pi(a_k) + t$, the same subspace L and $2^{-n-2}\varepsilon$, we find an element a_{n+1} such that $\|a_{n+1}\| \leqslant 2^{-n-1}\varepsilon$ and $\|(\sum_{k=1}^{n+1} \pi(a_k) - t)p_L\| < 2^{-n-2}\varepsilon$. Now let's put $a = \sum_{k=0}^{\infty} a_k$. Then $a \in A$ and it is evident that $\|a\| \leqslant \|t\| + \varepsilon$ and $a|_L = t|_L$. □

Theorem 2.6. *Every topologically irreducible representation of a C^*-algebra is algebraically irreducible.*

Proof. Let's assume the opposite Let $V \subset H$ be a non-closed invariant space, and $\overline{V}$ is its closure. It is also an invariant subspace (since the action is continuous), so $\overline{V} = H$. Let us take $\eta \in H \setminus V$, of norm 1 for example. Let $\xi \in V$ is a nonzero vector, and t is an operator in H such that $t\xi = \eta$. Then, by the previous lemma, there is an $a \in A$ such that $\pi(a)\xi = \eta$. Contradiction with the invariance of V. □

2.2 Positive linear functionals

Definition 2.7. Linear functional (we do not require continuity, see Lemma 2.10 below) φ on the C^*-algebra A is called *positive*, if $\varphi(a) \geqslant 0$ for any $a \geqslant 0$. If a positive linear functional is continuous and has norm 1, then it is called a *state*.

Example 2.8. If π is a representation of A in the Hilbert space H and $\xi \in H$, then the functional $\varphi(a) := (\xi, \pi(a)\xi)$ is positive. If A is unital and $\|\xi\| = 1$, then such φ is a state.

With every positive linear functional φ we can associate a sesquilinear form on A given by the formula $\langle a, b\rangle := \varphi(a^*b)$, that is, the form $\langle \cdot, \cdot \rangle$ is linear in the second argument is conjugate linear in the first argument. By definition of positivity of the functional $\langle a, a\rangle = \varphi(a^*a) \geqslant 0$ for any $a \in A$. Therefore, by the following lemma it is Hermitian symmetric: $\langle b, a\rangle = \overline{\langle a, b\rangle}$.

Lemma 2.9 (from linear algebra course)**.** *If a sesquilinear form has $\langle a, a\rangle \in \mathbb{R}$ for any a, then it is Hermitian symmetric.*

Proof. Let us write down the polarization identities

$$\langle a + b, a + b\rangle = \langle a, a\rangle + \langle a, b\rangle + \langle b, a\rangle + \langle b, b\rangle, \tag{2.1}$$

$$\langle a + ib, a + ib\rangle = \langle a, a\rangle + \langle a, ib\rangle + \langle ib, a\rangle + \langle ib, ib\rangle = \langle a, a\rangle + i(\langle a, b\rangle - \langle b, a\rangle) + \langle b, b\rangle. \tag{2.2}$$

From the first we obtain that $\langle a, b\rangle + \langle b, a\rangle$ is real, and from the second — that $\langle a, b\rangle - \langle b, a\rangle$ is imaginary. So $\overline{\langle a, b\rangle} = \langle b, a\rangle$. □

Thus, $\langle a, b\rangle$ is a positive Hermitian form and, therefore, the Cauchy-(Schwartz-Bunyakovsky) inequality holds for it: $|\langle a, b\rangle|^2 \leq \langle a, a\rangle\langle b, b\rangle$, that is, $|\varphi(a^*b)|^2 \leqslant \varphi(a^*a)\varphi(b^*b)$.

Lemma 2.10. *Positive linear functionals are continuous. If u_λ is an approximate unit for A, then $\|\varphi\| = \lim_{\lambda\in\Lambda} \varphi(u_\lambda)$. In particular, if A is unital, then $\|\varphi\| = \varphi(1)$.*

Proof. Let us first consider the unital case. If $0 \leqslant a \leqslant 1$, then since φ is positive, we obtain that $0 \leqslant \varphi(a) \leqslant \varphi(1)$. For $x \in A$ with $\|x\| \leqslant 1$ we have $0 \leqslant x^*x \leqslant 1$, so $|\varphi(x)|^2 = |\varphi(1 \cdot x)|^2 \leqslant \varphi(1) \cdot \varphi(x^*x) \leqslant \varphi(1)^2$ by the Cauchy-Schwartz-Bunyakovsky inequality. Thus, $\|\varphi\| \leqslant \varphi(1) \leqslant \|\varphi\|$.

Now consider the non-unital case. Suppose that φ is not bounded on the unit ball A. Then it is not restricted on the subset of the unit ball consisting of positive elements (since any element a is decomposable into a linear combination of four positive elements with norms not exceeding $\|a\|$, see (1.7)). Thus, for every $k \in \mathbb{N}$ there is a positive element $a_k \in A$ such that $\|a_k\| \leq 1$ and $\varphi(a_k) > 2^k$. Let us put $a := \sum_{k=1}^{\infty} \frac{a_k}{2^k} \in A$. Then for any $n \in \mathbb{N}$ we have $a \geqslant \sum_{k=1}^{n} \frac{a_k}{2^k}$ and

$$\varphi(a) \geqslant \varphi\left(\sum_{k=1}^{n} \frac{a_k}{2^k}\right) = \sum_{k=1}^{n} \frac{\varphi(a_k)}{2^k} > n,$$

that is impossible. Thus, φ is bounded in the non-unital case as well.

Let $m := \lim_{\lambda\in\Lambda} \varphi(u_\lambda^2)$, and the limit exists since the direction net is increasing and bounded by $\|\varphi\|$ from above. Then, for any $x \in A$ with $\|x\| \leqslant 1$ we have $|\varphi(x)| = \lim_{\lambda\in\Lambda} |\varphi(u_\lambda x)|$ due to the continuity of φ. Therefore, by the Cauchy-Schwartz-Bunyakovsky inequality we have $|\varphi(x)|^2 \leqslant \varphi(u_\lambda^2)\varphi(x^*x) \leqslant m\|\varphi\|$. For any $\varepsilon > 0$ we choose an element $x \in A$ such that $\|\varphi\|^2 < |\varphi(x)|^2 + \varepsilon$. Then $\|\varphi\|^2 < m\|\varphi\| + \varepsilon$. Hence, $\|\varphi\|^2 \leqslant m\|\varphi\|$ and $\|\varphi\| \leqslant m$. Since for any $\varepsilon > 0$ there is u_{λ_0} for which $\varphi(u_{\lambda_0}^2) > m + \varepsilon$, we come to the equality $\|\varphi\| = m$. Since $m \leqslant \lim_{\lambda\in\Lambda} \varphi(u_\lambda) \leqslant \|\varphi\| = m$, we have $\lim_{\lambda\in\Lambda} \varphi(u_\lambda) = \|\varphi\|$. ☐

Corollary 2.11. *If φ is a state on a unital C^*-algebra, then $\varphi(1) = 1$.*

Proof. By the previous lemma, $1 = \|\varphi\| = \varphi(1)$. ☐

2.3 GNS-construction (Gelfand-Naimark-Segal)

Definition 2.12. A vector $\xi \in H$ is called *cyclic* for $\pi : A \to \mathbb{B}(H)$, if $\pi(A)\xi$ is dense in H.

Theorem 2.13. *Let φ be a positive linear functional on the C^*-algebra A. Then there exists a representation π_φ of the algebra A on the Hilbert space H and a cyclic vector $\xi_\varphi \in H$ such that $\|\xi_\varphi\|^2 = \|\varphi\|$ and $(\xi_\varphi, \pi_\varphi(a)\xi_\varphi) = \varphi(a)$ for all $a \in A$.*

Proof. Let $N := \{a \in A : \varphi(a^*a) = 0\}$. Then $N = \{a \in A : \varphi(b^*a) = 0 \text{ for all } b \in A\}$ by the Cauchy-Schwartz-Bunyakovsky inequality. Therefore N is closed as an intersection kernels of continuous functionals $a \mapsto \varphi(b^*a)$). Besides, N is a left ideal, since $\varphi(b^*an) = \varphi((a^*b)^*n) = 0$ for any $a, b \in A$ for $n \in N$, so $an \in N$.

Let us define an Hermitian inner product on the Banach quotient space A/N by the formula $(\dot a, \dot b) = \varphi(a^*b)$, where $\dot a$ denotes the coset class $a+N$. This product is well defined because if $n_1, n_2 \in N$, then $\varphi((a+n_1)^*(b+n_2)) = \varphi(a^*b) + \varphi((a+n_1)^*n_2) + \overline{\varphi(b^*n_1)} = \varphi(a^*b)$. Also $(\dot a, \dot a) > 0$ holds for $\dot a \neq 0$. Let H be the Hilbert space obtained from A/N by the completion w.r.t. the norm given by this inner product. Let us denote by π_0 the representation of A on A/N (here we slightly expand the concept of representation to a pre-Hilbert space) by the formula $\pi_0(a)\dot x = (ax)^{\cdot}$, where $\dot x \in A/N$. If $n \in N$ then $(a(x+n))^{\cdot} = (ax)^{\cdot}$, so π_0 is well defined. It is involutive, because $(\pi_0(a)\dot x, \dot y) = \varphi((ax)^*y) = \varphi(x^*(a^*y)) = (\dot x, \pi_0(a^*)\dot y) = (\pi_0(a^*)^*\dot x, \dot y)$ and $\pi_0(a^*)^* = \pi_0(a)$. In this case, $\|\pi_0\| \leqslant 1$. Really,

$$\begin{aligned}\|\pi_0(a)\|^2 &= \sup_{\|\dot x\|\leqslant 1} \|\pi_0(a)\cdot x\|^2 = \sup_{\|\dot x\|\leqslant 1} \varphi(x^*a^*ax) \leqslant \\ &\leqslant \sup_{\|\dot x\|\leqslant 1} \|a^*a\|\varphi(x^*x) \leqslant \|a\|^2.\end{aligned}$$

Therefore π_0 extends by continuity to a representation π_φ of the algebra A on H.

If the algebra A is unital, then we set $\xi_\varphi := \dot 1$. Then $(\xi_\varphi, \pi_\varphi(a)\xi_\varphi) = \varphi(a)$ and ξ_φ is cyclic since $\pi_\varphi(A)\xi_\varphi = A/N$ is dense in H. Finally, $\|\varphi\| = \varphi(1) = \|\xi_\varphi\|^2$.

For a general algebra A, consider its approximate unit u_λ. Let us show that $\dot u_\lambda$ is a Cauchy directed net. Let us choose an $\varepsilon > 0$. Then there is an index $\alpha \in \Lambda$ such that $\varphi(u_\alpha) > \|\varphi\| - \varepsilon$ (since $\|\varphi\| = \lim_{\lambda\in\Lambda} \varphi(u_\lambda)$ by Lemma 2.10). Now let us find an index $\beta \in \Lambda$ such that $\beta \geqslant \alpha$ and $\|u_\lambda u_\alpha - u_\alpha\| < \varepsilon$ for any $\lambda \geqslant \beta$. Then

$$\mathrm{Re}(\varphi(u_\lambda u_\alpha)) = \varphi(u_\alpha) + \mathrm{Re}(\varphi(u_\lambda u_\alpha - u_\alpha)) > \|\varphi\| - 2\varepsilon.$$

That is why

$$\begin{aligned}\|\dot u_\lambda - \dot u_\alpha\|^2 &= \varphi((u_\lambda - u_\alpha)^2) = \varphi(u_\lambda^2) + \varphi(u_\alpha^2) - 2\mathrm{Re}(\varphi(u_\lambda u_\alpha)) \leqslant \\ &\leqslant \varphi(u_\lambda^2) + \varphi(u_\alpha^2) - 2(\|\varphi\| - 2\varepsilon) \leqslant 4\varepsilon.\end{aligned}$$

This means that for $\lambda, \mu \geqslant \beta$, we have

$$\|\dot u_\lambda - \dot u_\mu\| \leqslant \|\dot u_\lambda - \dot u_\alpha\| + \|\dot u_\alpha - \dot u_\mu\| \leqslant 4\varepsilon^{1/2}.$$

Thus, $\dot u_\lambda$ is a Cauchy net. Let $\xi_\varphi := \lim_{\lambda\in\Lambda} \dot u_\lambda \in H$. Then $(\xi_\varphi, \pi_\varphi(a)\xi_\varphi) = \lim_{\lambda\in\Lambda} \varphi(u_\lambda a u_\lambda) = \varphi(a)$. Since $\pi_\varphi(A)\xi_\varphi = A/N$, then ξ_φ is cyclic. From $\dot a = \pi_\varphi(a)\xi_\varphi$ it follows that

$$\lim_{\lambda\in\Lambda} \pi_\varphi(u_\lambda)\dot a = \lim_{\lambda\in\Lambda} \pi_\varphi(u_\lambda)\pi_\varphi(a)\xi_\varphi = \pi_\varphi(a)\xi_\varphi = \dot a$$

for any $\dot a \in A/N$, so the directed net $\pi_\varphi(u_\lambda)$ strongly converges to 1. Therefore $\|\varphi\| = \lim_{\lambda\in\Lambda} \varphi(u_\lambda) = \lim_{\lambda\in\Lambda}(\xi_\varphi, \pi_\varphi(u_\lambda)\xi_\varphi) = \|\xi_\varphi\|^2$. □

2.4 Realization of C^*-algebras as operator algebras on Hilbert space

Corollary 2.14. *Any state φ on a non-unital C^*-algebra A admits a unique extension to a state on A^+.*

Proof. Let π_φ be the representation of A given by the GNS construction. Let us set $\pi_\varphi(1) = 1$. Then π_φ can be extended to a representation of A^+ and $\tilde{\varphi}(a) := (\xi_\varphi, \pi(a)\xi_\varphi)$ is a state. It is unique, since $\tilde{\varphi}(1) = 1$ must hold (Corollary 2.11). □

Problem 45. Let u_λ, $\lambda \in \Lambda$, be some approximate unit in a unital algebra. Prove that $1 = \lim_{\lambda\in\Lambda} u_\lambda$.

Lemma 2.15. *Let $\varphi : A \to \mathbb{C}$ be a continuous linear functional such that $\|\varphi\| = 1 = \lim_{\lambda\in\Lambda} \varphi(u_\lambda)$ for some approximate unit u_λ. Then φ is a state.*

Proof. Let us first reduce the proof to the unital case. Let $\tilde{\varphi}$ be some extension (by the Hahn-Banach theorem) functional φ to a continuous functional on A^+. Let $\tilde{\varphi}(1) =: \alpha$. Because the $\|\tilde{\varphi}\| = 1$, then $|\alpha| \leqslant 1$. From inequality $\|2u_\lambda - 1\| \leqslant 1$ it follows that $|2 - \alpha| = \lim_{\lambda\in\Lambda} |\varphi(2u_\lambda - 1)| \leqslant 1$. Thus, $\alpha = 1$. This means that we can assume that A is unital and $\varphi(1) = 1$ (if A was unital from the very beginning, then we use the problem 45).

Let us now show that $\varphi(a) \in \mathbb{R}$ if $a = a^*$ (and therefore contained in $[-\|a\|, \|a\|]$). Let a — self-adjoint element of norm 1. Then $\|a \pm in1\|^2 = \|a^2 + n^2 1\| = n^2 + 1$, so $|\varphi(a) \pm in| \leqslant \sqrt{n^2 + 1}$ for any $n \in \mathbb{N}$. This means that $\varphi(a)$ is contained in the intersection of all disks with centers at $\pm in$ and radii $\sqrt{n^2 + 1}$. This intersection is equal to the real interval $[-1, 1]$.

If $0 \leq a \leq 1$, then $\|2a - 1\| \leq 1$. Applying the previous reasoning to the self-adjoint element $2a - 1$, we obtain that $-1 \leqslant 2\varphi(a) - 1 \leqslant 1$, so $\varphi(a) \geqslant 0$ and φ is positive. □