Chapter 2

Representations of C^*-algebras

Lecture 7

2.1 Definition and basic properties

Definition 2.1. A *representation* of a C^*-algebra A on a Hilbert space H is a $*$-homomorphism from A to $\mathbb{B}(H)$.

Definition 2.2. A representation of a C^*-algebra A is called *algebraically irreducible*, if there is no proper invariant linear subspace in H (when operated by operators from the image of the representation). A representation is *topologically irreducible*, if there is no proper closed invariant subspaces.

We will see soon that for C^*-algebras these two concepts coincide.

Lemma 2.3. *A representation π is topologically irreducible if and only if $\pi(A)' = \mathbb{C}1$.*

Proof. If $\pi(A)'$ contains something other than scalars, then it also contains a self-adjoint non-scalar operator (this immediately follows from the expansion of a non-scalar operator into a linear combination of two self-adjoint ones $a = \frac{a+a^*}{2} + i \cdot \frac{a-a^*}{2i}$). Using Borel functional calculus (see note 1.64) for this self-adjoint operator b, we can obtain a proper projection p in $\pi(A)'$. Namely, if an operator is nonscalar, then it has at least two distinct points in the spectrum, say, t_0 and t_1, and we need to consider a Borel function f, taking values 0 and 1, and $f(t_0) = 0$, $f(t_1) = 1$ (task 43). (You can also not use calculus, but simply take suitable spectral projections from the standard spectral theorem, that by construction have the necessary commutation properties). Then pH is a closed invariant subspace, since $p \in \pi(A)'$.

Conversely, let $L \subset H$ be a closed $\pi(A)$-invariant subspace, and $p \in \mathbb{B}(H)$ is a projection onto this subspace. Then $\pi(a)p = p\pi(a)p$ for any $a \in A$. Therefore $p\pi(a) = (\pi(a^*)p)^* = (p\pi(a^*)p)^* = p\pi(a)p = \pi(a)p$ and $p \in \pi(A)'$. Moreover, p is not a scalar. $\square$

Problem 43. Verify in the proof above that $f(b)$ is a proper projection, since $f^2 = f$ and $\mathrm{Sp}(f(b)) = \{0, 1\}$.

Problem 44. Prove a more general fact: if a self-adjoint element a in a unital C^*-algebra has $\mathrm{Sp}(a) = \{0, 1\}$, then a is a nonscalar idempotent.

Lemma 2.4. *Let π be a topologically irreducible representation of a C^*-algebra A in a Hilbert space H. Then for any $t \in \mathbb{B}(H)$, a finite-dimensional subspace $L \subset H$ and $\varepsilon > 0$, there is an element $a \in A$ such that $\|a\| \leq \|t|_L\|$ and $\|(\pi(a) - t)|_L\| < \varepsilon$.*

Proof. Since π is topologically irreducible, then by Lemma 2.3 $\pi(A)'$ coincides with scalars, hence $\pi(A)'' = \mathbb{B}(H)$. That is why $\pi(A)$ is dense in $\mathbb{B}(H)$ in the weak (strong) topology. Without loss of generality, we can assume that $\|t|_L\| = 1$. Let us put $s = tp_L$, where p_L is the projection onto L. Since L is finite-dimensional, then, by Kaplansky's density theorem, there is $b \in A$ such that $\|\pi(b)\| \leqslant 1$ and $\|(\pi(b) - s)|_L\| < \varepsilon/2$. Then there is an element $c \in A$ such that $\pi(c) = \pi(b)$ and $\|c\| < \|\pi(b)\|(1 + \varepsilon/2)$ (see Theorem 1.50). Let us put $a := \frac{c}{1+\varepsilon/2}$. Then $\|a\| \leqslant 1$ and

$$\|(\pi(a) - t)|_L\| \leqslant \|(\pi(c) - t)|_L\| + \|\pi(a) - \pi(c)\| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

□

Lemma 2.5. *Let π be a topologically irreducible representation of the C^*-algebra A in the Hilbert space H. Then for any $t \in \mathbb{B}(H)$, finite-dimensional subspace $L \subset H$ and $\varepsilon > 0$, there is an element $a \in A$ such that $\pi(a)|L = t|_L$ and $\|a\| \leqslant \|t\| + \varepsilon$.*

Proof. By the previous lemma, there is an element $a_0 \in A$ such that $\|a_0\| \leqslant \|t\|$ and $\|(\pi(a_0) - t)|_L\| < \varepsilon/2$. By induction we can find for each n, an element $a_n \in A$ such that $\|a_n\| \leqslant 2^{-n}\varepsilon$ and $\|(\sum_{k=0}^{n} \pi(a_n) - t)p_L|\| < 2^{-n-1}\varepsilon$. Indeed, suppose that the elements are found for some n and all the smaller ones. Applying the previous lemma to $s = -\sum_{k=0}^{n} \pi(a_k) + t$, the same subspace L and $2^{-n-2}\varepsilon$, we find an element a_{n+1} such that $\|a_{n+1}\| \leqslant 2^{-n-1}\varepsilon$ and $\|(\sum_{k=1}^{n+1} \pi(a_k) - t)p_L\| < 2^{-n-2}\varepsilon$. Now let's put $a = \sum_{k=0}^{\infty} a_k$. Then $a \in A$ and it is evident that $\|a\| \leqslant \|t\| + \varepsilon$ and $a|_L = t|_L$. □

Theorem 2.6. *Every topologically irreducible representation of a C^*-algebra is algebraically irreducible.*

Proof. Let's assume the opposite Let $V \subset H$ be a non-closed invariant space, and $\overline{V}$ is its closure. It is also an invariant subspace (since the action is continuous), so $\overline{V} = H$. Let us take $\eta \in H \setminus V$, of norm 1 for example. Let $\xi \in V$ is a nonzero vector, and t is an operator on H such that $t\xi = \eta$. Then, by the previous lemma, there is an $a \in A$ such that $\pi(a)\xi = \eta$. Contradiction with the invariance of V. □

2.2 Positive linear functionals

Definition 2.7. Linear functional (we do not require continuity, see Lemma 2.10 below) φ on the C^*-algebra A is called *positive*, if $\varphi(a) \geqslant 0$ for any $a \geqslant 0$. If a positive linear functional is continuous and has norm 1, then it is called a *state*.

Example 2.8. If π is a representation of A in the Hilbert space H and $\xi \in H$, then the functional $\varphi(a) := (\xi, \pi(a)\xi)$ is positive. If A is unital and $\|\xi\| = 1$, then such φ is a state.

With every positive linear functional φ we can associate a sesquilinear form on A given by the formula $\langle a, b\rangle := \varphi(a^*b)$, that is, the form $\langle\cdot,\cdot\rangle$ is linear in the second argument is conjugate linear in the first argument. By definition of positivity of the functional $\langle a, a\rangle = \varphi(a^*a) \geqslant 0$ for any $a \in A$. Therefore, by the following lemma it is Hermitian symmetric: $\langle b, a\rangle = \overline{\langle a, b\rangle}$.

Lemma 2.9 (from linear algebra course)**.** *If a sesquilinear form has $\langle a, a\rangle \in \mathbb{R}$ for any a, then it is Hermitian symmetric.*

Proof. Let us write down the polarization identities

$$\langle a+b, a+b\rangle = \langle a, a\rangle + \langle a, b\rangle + \langle b, a\rangle + \langle b, b\rangle, \tag{2.1}$$

$$\langle a+ib, a+ib\rangle = \langle a, a\rangle + \langle a, ib\rangle + \langle ib, a\rangle + \langle ib, ib\rangle = \langle a, a\rangle + i(\langle a, b\rangle - \langle b, a\rangle) + \langle b, b\rangle. \tag{2.2}$$

From the first we obtain that $\langle a, b\rangle + \langle b, a\rangle$ is real, and from the second — that $\langle a, b\rangle - \langle b, a\rangle$ is imaginary. So $\overline{\langle a, b\rangle} = \langle b, a\rangle$. □

Thus, $\langle a, b\rangle$ is a positive Hermitian form and, therefore, the Cauchy-(Schwartz-Bunyakovsky) inequality holds for it: $|\langle a, b\rangle|^2 \leq \langle a, a\rangle\langle b, b\rangle$, that is, $|\varphi(a^*b)|^2 \leqslant \varphi(a^*a)\varphi(b^*b)$.

Lemma 2.10. *Positive linear functionals are continuous. If u_λ is an approximate unit for A, then $\|\varphi\| = \lim_{\lambda\in\Lambda} \varphi(u_\lambda)$. In particular, if A is unital, then $\|\varphi\| = \varphi(1)$.*

Proof. Let us first consider the unital case. If $0 \leqslant a \leqslant 1$, then since φ is positive, we obtain that $0 \leqslant \varphi(a) \leqslant \varphi(1)$. For $x \in A$ with $\|x\| \leqslant 1$ we have $0 \leqslant x^*x \leqslant 1$, so $|\varphi(x)|^2 = |\varphi(1 \cdot x)|^2 \leqslant \varphi(1) \cdot \varphi(x^*x) \leqslant \varphi(1)^2$ by the Cauchy-Schwartz-Bunyakovsky inequality. Thus, $\|\varphi\| \leqslant \varphi(1) \leqslant \|\varphi\|$.

Now consider the non-unital case. Suppose that φ is not bounded on the unit ball A. Then it is not bounded on the subset of the unit ball consisting of positive elements (since any element a is decomposable into a linear combination of four positive elements with norms not exceeding $\|a\|$, see (1.7)). Thus, for every $k \in \mathbb{N}$ there is a positive element $a_k \in A$ such that $\|a_k\| \leq 1$ and $\varphi(a_k) > 2^k$. Let us put $a := \sum_{k=1}^\infty \frac{a_k}{2^k} \in A$. Then for any $n \in \mathbb{N}$, we have $a \geqslant \sum_{k=1}^n \frac{a_k}{2^k}$ and

$$\varphi(a) \geqslant \varphi\left(\sum_{k=1}^n \frac{a_k}{2^k}\right) = \sum_{k=1}^n \frac{\varphi(a_k)}{2^k} > n,$$

that is impossible. Thus, φ is bounded in the non-unital case as well.

Let $m := \lim_{\lambda\in\Lambda} \varphi(u_\lambda)$, and the limit exists since the directed net is increasing and bounded by $\|\varphi\|$ from above. Then, for any $x \in A$ with $\|x\| \leqslant 1$ we have $|\varphi(x)| = \lim_{\lambda\in\Lambda} |\varphi(u_\lambda x)|$ due to the continuity of φ. Therefore, by the Cauchy-Schwartz-Bunyakovsky inequality we have

$$|\varphi(x)|^2 = \lim_{\lambda\in\Lambda} |\varphi(u_\lambda x)|^2 \leqslant \sup_\lambda \varphi(u_\lambda^2)\varphi(x^*x) \leqslant \sup_\lambda \varphi(u_\lambda)\varphi(x^*x) \leqslant m\|\varphi\|.$$

For any $\varepsilon > 0$ we choose an element $x \in A$ such that $\|\varphi\|^2 < |\varphi(x)|^2 + \varepsilon$. Then $\|\varphi\|^2 < m\|\varphi\| + \varepsilon$. Hence, $\|\varphi\|^2 \leqslant m\|\varphi\|$ and $\|\varphi\| \leqslant m$. Since for any $\varepsilon > 0$ there is u_{λ_0} for which $\varphi(u_{\lambda_0}) > m - \varepsilon$, we come to the equality $\|\varphi\| = m$. □

Corollary 2.11. *If φ is a state on a unital C^*-algebra, then $\varphi(1) = 1$.*

Proof. By the previous lemma, $1 = \|\varphi\| = \varphi(1)$. □