

Lecture 8

2.3 GNS-construction (Gelfand-Naimark-Segal)

Definition 2.12. A vector $\xi \in H$ is called *cyclic* for $\pi : A \rightarrow \mathbb{B}(H)$, if $\pi(A)\xi$ is dense in H .

Theorem 2.13. Let φ be a positive linear functional on the C^* -algebra A . Then there exists a representation π_φ of the algebra A on the Hilbert space H and a cyclic vector $\xi_\varphi \in H$ such that $\|\xi_\varphi\|^2 = \|\varphi\|$ and $(\xi_\varphi, \pi_\varphi(a)\xi_\varphi) = \varphi(a)$ for all $a \in A$.

Proof. Let $N := \{a \in A : \varphi(a^*a) = 0\}$. Then $N = \{a \in A : \varphi(b^*a) = 0 \text{ for all } b \in A\}$ by the Cauchy-Schwartz-Bunyakovsky inequality. Therefore N is closed as an intersection of kernels of continuous functionals $a \mapsto \varphi(b^*a)$. Besides, N is a left ideal, since $\varphi(b^*an) = \varphi((a^*b)^*n) = 0$ for any $a, b \in A$ for $n \in N$, so $an \in N$.

Let us define an Hermitian inner product on the Banach quotient space A/N by the formula $(\dot{a}, \dot{b}) = \varphi(a^*b)$, where $\dot{a}$ denotes the coset class $a + N$. This product is well defined because if $n_1, n_2 \in N$, then $\varphi((a + n_1)^*(b + n_2)) = \varphi(a^*b) + \varphi((a + n_1)^*n_2) + \overline{\varphi(b^*n_1)} = \varphi(a^*b)$. Also $(\dot{a}, \dot{a}) > 0$ holds for $\dot{a} \neq 0$. Let H be the Hilbert space obtained from A/N by the completion w.r.t. the norm given by this inner product. Let us denote by π_0 the representation of A on A/N (here we slightly expand the concept of representation to a pre-Hilbert space) by the formula $\pi_0(a)\dot{x} = (ax)^\cdot$, where $\dot{x} \in A/N$. If $n \in N$ then $(a(x + n))^\cdot = (ax)^\cdot$, so π_0 is well defined. It is involutive, because $(\pi_0(a)\dot{x}, \dot{y}) = \varphi((ax)^*y) = \varphi(x^*(a^*y)) = (\dot{x}, \pi_0(a^*)\dot{y}) = (\pi_0(a^*)^*\dot{x}, \dot{y})$ and $\pi_0(a^*)^* = \pi_0(a)$. In this case, $\|\pi_0\| \leq 1$. Indeed,

$$\begin{aligned} \|\pi_0(a)\|^2 &= \sup_{\|\dot{x}\| \leq 1} \|\pi_0(a)\dot{x}\|^2 = \sup_{\|\dot{x}\| \leq 1} \varphi(x^*a^*ax) \leq \\ &\leq \sup_{\|\dot{x}\| \leq 1} \|a^*a\| \varphi(x^*x) \leq \|a\|^2. \end{aligned}$$

Therefore π_0 extends by continuity to a representation π_φ of the algebra A on H .

If the algebra A is unital, then we set $\xi_\varphi := \dot{1}$. Then $(\xi_\varphi, \pi_\varphi(a)\xi_\varphi) = \varphi(a)$ and ξ_φ is cyclic since $\pi_\varphi(A)\xi_\varphi = A/N$ is dense in H . Finally, $\|\varphi\| = \varphi(1) = \|\xi_\varphi\|^2$.

For a general algebra A , consider its approximate unit u_λ . Let us show that $\dot{u}_\lambda$ is a Cauchy directed net. Let us choose an $\varepsilon > 0$. Then there is an index $\alpha \in \Lambda$ such that $\varphi(u_\alpha) > \|\varphi\| - \varepsilon$ (since $\|\varphi\| = \lim_{\lambda \in \Lambda} \varphi(u_\lambda)$ by Lemma 2.10). Now let us find an index $\beta \in \Lambda$ such that $\beta \geq \alpha$ and $\|u_\lambda u_\alpha - u_\alpha\| < \varepsilon$ for any $\lambda \geq \beta$. Then

$$\operatorname{Re}(\varphi(u_\lambda u_\alpha)) = \varphi(u_\alpha) + \operatorname{Re}(\varphi(u_\lambda u_\alpha - u_\alpha)) > \|\varphi\| - 2\varepsilon.$$

That is why

$$\begin{aligned} \|\dot{u}_\lambda - \dot{u}_\alpha\|^2 &= \varphi((u_\lambda - u_\alpha)^2) = \varphi(u_\lambda^2) + \varphi(u_\alpha^2) - 2\operatorname{Re}(\varphi(u_\lambda u_\alpha)) \leq \\ &\leq \varphi(u_\lambda^2) + \varphi(u_\alpha^2) - 2(\|\varphi\| - 2\varepsilon) \leq 4\varepsilon. \end{aligned}$$

This means that for $\lambda, \mu \geq \beta$, we have

$$\|\dot{u}_\lambda - \dot{u}_\mu\| \leq \|\dot{u}_\lambda - \dot{u}_\alpha\| + \|\dot{u}_\alpha - \dot{u}_\mu\| \leq 4\varepsilon^{1/2}.$$

Thus, $\dot{u}_\lambda$ is a Cauchy net. Let $\xi_\varphi := \lim_{\lambda \in \Lambda} \dot{u}_\lambda \in H$. Then $(\xi_\varphi, \pi_\varphi(a)\xi_\varphi) = \lim_{\lambda \in \Lambda} \varphi(u_\lambda a u_\lambda) = \varphi(a)$. Since $\pi_\varphi(A)\xi_\varphi = A/N$, then ξ_φ is cyclic. From $\dot{a} = \pi_\varphi(a)\xi_\varphi$ it follows that

$$\lim_{\lambda \in \Lambda} \pi_\varphi(u_\lambda)\dot{a} = \lim_{\lambda \in \Lambda} \pi_\varphi(u_\lambda)\pi_\varphi(a)\xi_\varphi = \pi_\varphi(a)\xi_\varphi = \dot{a}$$

for any $\dot{a} \in A/N$, so the directed net $\pi_\varphi(u_\lambda)$ strongly converges to 1. Therefore $\|\varphi\| = \lim_{\lambda \in \Lambda} \varphi(u_\lambda) = \lim_{\lambda \in \Lambda} (\xi_\varphi, \pi_\varphi(u_\lambda)\xi_\varphi) = \|\xi_\varphi\|^2$. $\square$

2.4 Realization of C^* -algebras as operator algebras on Hilbert space

Corollary 2.14. *Any state φ on a non-unital C^* -algebra A admits a unique extension to a state on A^+ .*

Proof. Let π_φ be the representation of A given by the GNS construction. Let us set $\pi_\varphi(1) = 1$. Then π_φ can be extended to a representation of A^+ and $\tilde{\varphi}(a) := (\xi_\varphi, \pi(a)\xi_\varphi)$ is a state. It is unique, since $\tilde{\varphi}(1) = 1$ must hold (Corollary 2.11). $\square$

Problem 45. Let u_λ , $\lambda \in \Lambda$, be some approximate unit in a unital algebra. Prove that $1 = \lim_{\lambda \in \Lambda} u_\lambda$.

Lemma 2.15. *Let $\varphi : A \rightarrow \mathbb{C}$ be a continuous linear functional such that $\|\varphi\| = 1 = \lim_{\lambda \in \Lambda} \varphi(u_\lambda)$ for some approximate unit u_λ . Then φ is a state.*

Proof. Let us first reduce the proof to the unital case. Let $\tilde{\varphi}$ be some extension (by the Hahn-Banach theorem) functional φ to a continuous functional on A^+ . Let $\tilde{\varphi}(1) =: \alpha$. Because the $\|\tilde{\varphi}\| = 1$, then $|\alpha| \leq 1$. From inequality $\|2u_\lambda - 1\| \leq 1$ it follows that $|2 - \alpha| = \lim_{\lambda \in \Lambda} |\varphi(2u_\lambda - 1)| \leq 1$. Thus, $\alpha = 1$. This means that we can assume that A is unital and $\varphi(1) = 1$ (if A was unital from the very beginning, then we use the problem 45).

Let us now show that $\varphi(a) \in \mathbb{R}$ if $a = a^*$ (and therefore contained in $[-\|a\|, \|a\|]$). Let a — self-adjoint element of norm 1. Then $\|a \pm in1\|^2 = \|a^2 + n^2 1\| = n^2 + 1$, so $|\varphi(a) \pm in| \leq \sqrt{n^2 + 1}$ for any $n \in \mathbb{N}$. This means that $\varphi(a)$ is contained in the intersection of all disks with centers at $\pm in$ and radii $\sqrt{n^2 + 1}$. This intersection is equal to the real interval $[-1, 1]$.

If $0 \leq a \leq 1$, then $\|2a - 1\| \leq 1$. Applying the previous reasoning to the self-adjoint element $2a - 1$, we obtain that $-1 \leq 2\varphi(a) - 1 \leq 1$, so $\varphi(a) \geq 0$ and φ is positive. $\square$

Lemma 2.16. *Let $a \in A$ be a self-adjoint element. Then there is a state φ on A such that $|\varphi(a)| = \|a\|$.*

Proof. If A is non-unital, then we will work in A^+ . Consider the commutative C^* -algebra $C^*(a)$. Then there is a multiplicative linear functional φ_0 on $C^*(a)$ such that $|\varphi_0(a)| = \|a\|$ (we must take as φ_0 the mapping, which is the taking of the value of functions at that point of $\text{Sp}(a)$, where the function $\hat{a}$ reaches its maximum). Then $\varphi_0(1) = 1 = \|\varphi_0\|$. Consider the extension of φ_0 by the Hahn-Banach theorem to a functional φ on A^+ . Then, since $\|\varphi\| = 1 = \varphi(1)$, then φ is a state by Lemma 2.15. $\square$

Corollary 2.17. *For any $a \in A$ there exists a representation π and a unit vector ξ in the space of representation such that $\|\pi(a)\xi\| = \|a\|$.*

Proof. By the previous lemma, we find a state φ such that $\varphi(a^*a) = \|a\|^2$. Let $\pi = \pi_\varphi$ and $\xi = \xi_\varphi$ were obtained for φ using the GNS construction. Then $\|\pi(a)\xi\|^2 = (\xi, \pi(a^*a)\xi) = \varphi(a^*a) = \|a\|^2$. $\square$

Theorem 2.18 (Gelfand-Naimark). *Any C^* -algebra is isometrically $*$ -isomorphic to a C^* -subalgebra of $\mathbb{B}(H)$ for some Hilbert space H . If A is separable, then H can be chosen to be separable.*

Proof. Let us set $\pi = \oplus_\varphi \pi_\varphi$, where the direct sum is taken over all states on A . More precisely, we consider the Hilbert direct sum $H := \oplus_\varphi H_\varphi$ (completion with respect to the ℓ_2 norm of the space of compactly supported mappings $\varphi \mapsto h_\varphi \in H_\varphi$, that is, the sets $h = \{h_\varphi\}$, $h_\varphi \in H_\varphi$, and only a finite number h_φ is nonzero, and the norm is defined as $\|h\|^2 = \sum_\varphi \|h_\varphi\|^2$) with diagonal action $\pi(a)(\{h_\varphi\}) = \{\pi_\varphi(a)(h_\varphi)\}$. Then, as can be seen from the proof of the previous consequences, $\|\pi(a)\| = \sup_\varphi \|\pi_\varphi(a)\| = \|a\|$. If A is separable, then it is sufficient to take the sum over a countable set $\{\varphi_n\}$, where $\|\pi_{\varphi_n}(a_n)\| = \|a_n\|$, for elements a_n forming a dense subset in A . Then $\pi = \oplus_{n \in \mathbb{N}} \pi_{\varphi_n}$, and the corresponding Hilbert space is separable, since each H_{φ_n} is separable (as a completion of a factor-space of a separable space). $\square$

Definition 2.19. The representation constructed in the theorem (in its first part) is called the *universal representation* of A . The von Neumann algebra $\pi(A)''$, where π is the universal representation, contains $\pi(A) \cong A$ as a dense subset and is called the *enveloping von Neumann algebra* for A .

2.5 Jordan decomposition

Lemma 2.20. *Let φ be a linear functional on A . Then $\varphi = \psi_1 + i\psi_2$, where ψ_1 and ψ_2 are self-adjoint.*

Proof. Let us take, in the same way as we did for elements of algebra, $\psi_1(a) = (\varphi(a) + \overline{\varphi(a^*)})/2$ and $\psi_2(a) = (\varphi(a) - \overline{\varphi(a^*)})/2i$. $\square$

Let A_{sa} denote the set of all self-adjoint elements of A . Then it is evident that A_{sa} is a real Banach space.

Problem 46. There is a natural bijection between self-adjoint linear functionals on A and (real) linear functionals on A_{sa} .

To prove the Jordan decomposition theorem, we need the following statement, which is of independent interest.

Theorem 2.21 (on extension of positive functionals). *Let $B \subset A$ be a C^* -subalgebra, and $\varphi : B \rightarrow \mathbb{C}$ be a positive functional. Then there exists a positive functional $\varphi' : A \rightarrow \mathbb{C}$ such that $\varphi'|_B = \varphi$ and $\|\varphi'\| = \|\varphi\|$.*

Proof. The following cases are possible:

- a) both algebras have a common unit,
- b) A has one, but B does not,
- c) both algebras do not have a unit,
- d) B has one, but A does not.
- e) both algebras with 1, but $1_A \neq 1_B$.

By Corollary 2.14, (c) and (b) can be reduced by adjoining 1 to (a) (for (b) it should be noted that $B^+ \cong B \oplus \mathbb{C}1_A$). In turn, (d) obviously reduces to (e).

In case (a) we extend φ (using the Hahn-Banach theorem) to some $\varphi' : A \rightarrow \mathbb{C}$ of the same norm. Then by Lemma 2.10, $\|\varphi'\| = \|\varphi\| = \varphi(1) = \varphi'(1)$ and φ' is positive by Lemma 2.15.

In case (e), consider the C^* -subalgebra $B_1 := B \oplus \mathbb{C}1_A = B \oplus \mathbb{C}(1_A - 1_B)$ and extend φ to $\varphi_1 : B_1 \rightarrow \mathbb{C}$, setting $\varphi_1(1_A - 1_B) = 0$. Then $\varphi_1(a) = \varphi(1_B \cdot a)$, where $a \in B_1$. Indeed, if $a \in B$, then $\varphi_1(a) = \varphi(1_B \cdot a) = \varphi(a)$, and if $a = 1_A - 1_B$, then $\varphi_1(a) = \varphi(1_B(1_A - 1_B)) = \varphi(0) = 0$. In this case, the unit of B_1 is 1_A . Moreover, $\|\varphi_1\| \leq \|\varphi\| \cdot \|1_B\| = \|\varphi\|$, and $\varphi_1(1_A) = \varphi(1_B) = \|\varphi\|$. This means that $\|\varphi_1\| = \|\varphi\| = \varphi_1(1_A) = \varphi_1(1_{B_1})$ and, by Lemma 2.15, φ_1 is positive. Thus, case (e) is also reduced to the proven case (a). $\square$

The Jordan theorem about decomposition of a measure in the sum of positive and negative ones [8, Ch. VI, §5, Theorem 1] in the functional language (in the sense of the Riesz-Markov-Kakutani theorem [5, Ch. I, §6, Theorem 4]) can be written as: for any bounded real linear functional $\tau : C(\Omega, \mathbb{R}) \rightarrow \mathbb{R}$ there are positive linear functionals τ_+ and τ_- such that $\tau = \tau_+ - \tau_-$ and $\|\tau\| = \|\tau_+\| + \|\tau_-\|$, where Ω is a compact Hausdorff space and $C(\Omega, \mathbb{R})$ is the real algebra of all real continuous functions on Ω .

Theorem 2.22 (Jordan decomposition). *Let ψ be a self-adjoint linear functional on A . Then $\psi = \varphi_+ - \varphi_-$, where φ_+ and φ_- are positive linear functionals on A and $\|\psi\| = \|\varphi_+\| + \|\varphi_-\|$.*

Proof. Denote by K the set of all self-adjoint linear functionals of norm ≤ 1 , i.e., $K \subset (A^*)_{sa}$. Then K is a $*$ -weak closed subset of the unit ball and hence it is $*$ -weak compact. Define an $\mathbb{R}$ -linear map

$$\theta : A_{sa} \rightarrow C(K, \mathbb{R}), \quad \theta(a)(\tau) = \tau(a),$$

so, if $a \in A$, $a \geq 0$, then $\theta(a) \geq 0$ in K . By Lemma 2.16 the mapping θ is an isometry onto its image.

There is a natural isometry $\tau \mapsto \tau'$ of real spaces $(A^*)_{sa}$ and $(A_{sa})_{\mathbb{R}}^*$ (real functionals) (see Problem 46). By the Hahn-Banach theorem there is a functional $\rho \in (C(K, \mathbb{R}))_{\mathbb{R}}^*$ such that $\rho \circ \theta = \psi'$ and $\|\rho\| = \|\psi'\|$ (an extension of a functional from the closed subspace $\theta(A_{sa})$). Then by the Jordan theorem for measures (as it is explained above before the formulation) there are positive functionals ρ_+ and ρ_- such that $\rho = \rho_+ - \rho_-$ and $\|\rho\| = \|\rho_+\| + \|\rho_-\|$. Consider $\varphi'_+ := \rho_+ \circ \theta$ and $\varphi'_- := \rho_- \circ \theta$. These are functionals from $(A_{sa})_{\mathbb{R}}^*$. Let φ_+ and φ_- correspond to them under the identification with $(A^*)_{sa}$. Evidently they satisfy all the conditions, except maybe the norm property. Let us verify it:

$$\|\psi\| = \|\psi'\| = \|\rho\| = \|\rho_+\| + \|\rho_-\| \geq \|\varphi'_+\| + \|\varphi'_-\| = \|\varphi_+\| + \|\varphi_-\| \geq \|\psi\|.$$

□