

Lecture 8

Lemma 2.16. *Let $a \in A$ be a self-adjoint element. Then there is a state φ on A such that $|\varphi(a)| = \|a\|$.*

Proof. If A is non-unital, then we will work in A^+ . Consider the commutative C^* -algebra $C^*(a)$. Then there is a multiplicative linear functional φ_0 on $C^*(a)$ such that $|\varphi_0(a)| = \|a\|$ (we must take as φ_0 the mapping, which is the taking of the value of functions at that point of $\text{Sp}(a)$, where the function \hat{a} reaches its maximum). Then $\varphi_0(1) = 1 = \|\varphi_0\|$. Consider the extension of φ_0 by the Hahn-Banach theorem to a functional φ on A^+ . Then, since $\|\varphi\| = 1 = \varphi(1)$, then φ is a state by Lemma 2.15. \square

Corollary 2.17. *For any $a \in A$ there exists a representation π and a unit vector ξ in the space of representation such that $\|\pi(a)\xi\| = \|a\|$.*

Proof. By the previous lemma, we find a state φ such that $\varphi(a^*a) = \|a\|^2$. Let $\pi = \pi_\varphi$ and $\xi = \xi_\varphi$ were obtained for φ using the GNS construction. Then $\|\pi(a)\xi\|^2 = (\xi, \pi(a^*a)\xi) = \varphi(a^*a) = \|a\|^2$. \square

Theorem 2.18 (Gelfand-Naimark). *Any C^* -algebra is isometrically $*$ -isomorphic to a C^* -subalgebra of $\mathbb{B}(H)$ for some Hilbert space H . If A is separable, then H can be chosen to be separable.*

Proof. Let us set $\pi = \bigoplus_\varphi \pi_\varphi$, where the direct sum is taken over all states on A . More precisely, we consider the Hilbert direct sum $H := \bigoplus_\varphi H_\varphi$ (completion with respect to the ℓ_2 norm of the space of compactly supported mappings $\varphi \mapsto h_\varphi \in H_\varphi$, that is, the sets $h = \{h_\varphi\}$, $h_\varphi \in H_\varphi$, and only a finite number h_φ is nonzero, and the norm is defined as $\|h\|^2 = \sum_\varphi \|h_\varphi\|^2$) with diagonal action $\pi(a)(\{h_\varphi\}) = \{\pi_\varphi(a)(h_\varphi)\}$. Then, as can be seen from the proof of the previous consequences, $\|\pi(a)\| = \sup_\varphi \|\pi_\varphi(a)\| = \|a\|$. If A is separable, then it is sufficient to take the sum over a countable set $\{\varphi_n\}$, where $\|\pi_{\varphi_n}(a_n)\| = \|a_n\|$, for elements a_n forming a dense subset in A . Then $\pi = \bigoplus_{n \in \mathbb{N}} \pi_{\varphi_n}$, and the corresponding Hilbert space is separable, since each H_{φ_n} is separable (as a completion of a factor-space of a separable space). \square

Definition 2.19. The representation constructed in the theorem (in its first part) is called the *universal representation* of A . The von Neumann algebra $\pi(A)''$, where π is the universal representation, contains $\pi(A) \cong A$ as a dense subset and is called the *enveloping von Neumann algebra* for A .

2.5 Jordan decomposition

Lemma 2.20. *Let φ be a linear functional on A . Then $\varphi = \psi_1 + i\psi_2$, where ψ_1 and ψ_2 are self-adjoint.*

Proof. Let us take, in the same way as we did for elements of algebra, $\psi_1(a) = (\varphi(a) + \overline{\varphi(a^*)})/2$ and $\psi_2(a) = (\varphi(a) - \overline{\varphi(a^*)})/2i$. \square

Let A_{sa} denote the set of all self-adjoint elements of A . Then it is evident that A_{sa} is a real Banach space.

Problem 46. There is a natural bijection between self-adjoint linear functionals on A and (real) linear functionals on A_{sa} .

To prove the Jordan decomposition theorem, we need the following statement, which is of independent interest.

Theorem 2.21 (on extension of positive functionals). *Let $B \subset A$ be a C^* -subalgebra, and $\varphi : B \rightarrow \mathbb{C}$ be a positive functional. Then there exists a positive functional $\varphi' : A \rightarrow \mathbb{C}$ such that $\varphi'|_B = \varphi$ and $\|\varphi'\| = \|\varphi\|$.*

Proof. The following cases are possible:

- a) both algebras have a common unit,
- b) A has one, but B does not,
- c) both algebras do not have a unit,
- d) B has one, but A does not.
- e) both algebras with 1, but $1_A \neq 1_B$.

By Corollary 2.14, (c) and (b) can be reduced by adjoining 1 to (a) (for (b) it should be noted that $B^+ \cong B \oplus \mathbb{C}1_A$). In turn, (d) obviously reduces to (e).

In case (a) we extend φ (using the Hahn-Banach theorem) to some $\varphi' : A \rightarrow \mathbb{C}$ of the same norm. Then by Lemma 2.10, $\|\varphi'\| = \|\varphi\| = \varphi(1) = \varphi'(1)$ and φ' is positive by Lemma 2.15.

In case (e), consider the C^* -subalgebra $B_1 := B \oplus \mathbb{C}1_A = B \oplus \mathbb{C}(1_A - 1_B)$ and extend φ to $\varphi_1 : B_1 \rightarrow \mathbb{C}$, setting $\varphi_1(1_A - 1_B) = 0$. Then $\varphi_1(a) = \varphi(1_B \cdot a)$, where $a \in B_1$. Indeed, if $a \in B$, then $\varphi_1(a) = \varphi(1_B \cdot a) = \varphi(a)$, and if $a = 1_A - 1_B$, then $\varphi_1(a) = \varphi(1_B(1_A - 1_B)) = \varphi(0) = 0$. In this case, the unit of B_1 is 1_A . Moreover, $\|\varphi_1\| \leq \|\varphi\| \cdot \|1_B\| = \|\varphi\|$, and $\varphi_1(1_A) = \varphi(1_B) = \|\varphi\|$. This means that $\|\varphi_1\| = \|\varphi\| = \varphi_1(1_A) = \varphi_1(1_{B_1})$ and, by Lemma 2.15, φ_1 is positive. Thus, case (e) is also reduced to the proven case (a). \square

The Jordan theorem about decomposition of a measure in the sum of positive and negative ones [8, Ch. VI, §5, Theorem 1] in the functional language (in the sense of the Riesz-Markov-Kakutani theorem [5, Ch. I, §6, Theorem 4]) can be written as: for any bounded real linear functional $\tau : C(\Omega, \mathbb{R}) \rightarrow \mathbb{R}$ there are positive linear functionals τ_+ and τ_- such that $\tau = \tau_+ - \tau_-$ and $\|\tau\| = \|\tau_+\| + \|\tau_-\|$, where Ω is a compact Hausdorff space and $C(\Omega, \mathbb{R})$ is the real algebra of all real continuous functions on Ω .