Lecture 9

2.7 Finite-dimensional C^*-algebras

Consider the $*$-weak topology on A defined by the seminorm system $a \mapsto |\varphi(a)|$ for all linear functionals φ. From Lemma 2.20 and Theorem 2.22 it follows that the same topology can be obtained by using only seminorms, defined by states.

Note also that the corresponding LTS has the homothety property 2.27.

Lemma 2.32. *A finite-dimensional C^*-algebra is always unital.*

Proof. If A is finite-dimensional, then the topology of the norm coincides with the $*$-weak topology according to Theorem 2.31. Let u_n be an approximate unit of the algebra A. Then for any state φ the sequence $\varphi(u_n)$ is non-decreasing and bounded. Therefore u_n converges in $*$-weak topology, and therefore in norm. Thus, there is a limit $\lim_n u_n = a$. Then $ax = xa = x$ for any $x \in A$, so $a = 1$. $\square$

Lemma 2.33. *Let $I \subset A$ be an ideal in a finite-dimensional C^*-algebra A. Then $I = Ap$ for some central projector (=idempotent from the center) p.*

Proof. Since I is finite-dimensional, it is unital by Lemma 2.32. Let $p \in I$ be the unit of I. Then for every $x \in A$, one has $xp \in I$, so $p(xp) = xp$. Hence $px^*p = x^*p$ for any $x \in A$, whence $xp = pxp = px$ and p belongs to the center of A. Obviously, $p^2 = p$. $\square$

Lemma 2.34. *A simple finite-dimensional C^*-algebra A is isometrically $*$-isomorphic to the matrix algebra M_n for some n.*

Proof. First of all, note that $aAb \neq 0$ for any non-zero $a, b \in A$. Indeed, AaA is a non-zero ideal (since A is unital and $0 \neq a = 1 \cdot a \cdot 1 \in A$), so by simplicity, $AaA = A$. Therefore $1 = \sum_i x_i a y_i$ and $b = \sum_i x_i a y_i b$. Hence, if $ayb = 0$ for any $y \in A$, then $b = \sum_i x_i (a y_i b) = 0$. This contradicts the assumption.

Let B be some maximal commutative subalgebra of A. Then it can be identified with $C(X) = \mathbb{C}^n = \mathbb{C} \cdot e_1 \oplus \ldots \oplus \mathbb{C} \cdot e_n$ for some n, where X consists of n points, and $e_i \in B$ denotes the element corresponding to the characteristic functions at point i. Here e_i are projections with the relations $e_i e_j = 0$ for $i \neq j$ and $\sum_{i=1}^n e_i = 1$. Since $e_i A e_i \cdot e_j = e_j \cdot e_i A e_i = 0$ and B is maximal, then $e_i A e_i \subset B$. Therefore $e_i A e_i = \mathbb{C} \cdot e_i$ (since, obviously, $0 \neq e_i A e_i \ni e_i$, or you can use the statement from the beginning of the proof).

For any i, j there is $x = x_{ij} \in A$ such that $x = e_i x e_j \neq 0$, $\|x\| = 1$. Indeed, by virtue of the statement from the beginning of the proof, $e_i A e_j \neq 0$, so we have $x = e_i y e_j$ with $\|x\| = 1$. In this case $e_i x e_j = e_i e_i y e_j e_j = e_i y e_j = x$. Then $x^* x = e_j x^* e_i e_i x e_j \in e_j A e_j$, and therefore, according to what has been proven, this element has the form αe_j, $\alpha \in \mathbb{C}$. Since $x^* x$ is a positive element with norm equal to one, then $\alpha = 1$, so $x^* x = e_j$. Likewise, $x x^* = e_i$. Let us denote such $x = x_{ij}$ for $j = 1$ by u_i, so that $u_i = e_i x e_1 = e_i u_i e_1$. Then $u_i^* u_i = e_1$, $u_i u_i^* = e_i$, $i = 1, \ldots, n$. Let us set $u_{ij} := u_i u_j^*$. In this case, $u_i e_1 u_i^* = u_i u_i^* u_i u_i^* = e_i e_i = e_i$, So $u_{ij}u_{ji} = u_i u_j^* u_j u_i^* = u_i e_1 u_i^* = e_i$. Also $e_j u_{ji} = u_j u_j^* u_j u_i^* = u_j e_1 u_i^* = u_j u_i^* u_i u_i^* = u_{ji} e_i$, and $e_i u_{ij} = u_i u_i^* u_i u_j^* = u_i e_1 u_j^* = u_i u_j^* u_j u_j^*$.

If $x \in e_i A e_j$, that is, $x = e_i a e_j$, then $x u_{ji} = e_i a e_j u_{ji} = e_i a u_{ji} e_i \in e_i A e_i$, so $x u_{ji} = \lambda e_i$ for some $\lambda \in \mathbb{C}$. Then $x = x e_j = x u_{ji} u_{ij} = \lambda e_i u_{ij} = \lambda u_{ij}$, so for any $x \in A$ there is a number $\lambda_{ij}(x) \in \mathbb{C}$ such that $e_i x e_j = \lambda_{ij}(x) u_{ij}$. Thus, $x = \sum_{i,j} e_i x e_j = \sum_{ij} \lambda_{ij}(x) u_{ij}$. The correspondence $x \mapsto (\lambda_{ij}(x))$ defines an isomorphism $\kappa : A \to M_n$ (Problem 47). □

Problem 47. Check the bijectivity and necessary algebraic properties of κ.

Theorem 2.35. *If A is finite-dimensional, then $A = \oplus_k A p_k$, where p_k are central projections, and each $A p_k$ is a matrix algebra $M_{n(k)}$.*

Proof. For a simple algebra, the result follows from Lemma 2.34. If A is not simple, then $I = Ap$ by Lemma 2.33, where p is a central projection. Then $A = I \oplus J$, where $J := A(1-p)$. Then J is also an ideal, since $(1-p)$ is also a central projection, so $A(1-p)A = AA(1-p) \subseteq A(1-p)$. In this case, the center of A, being a finite-dimensional commutative algebra, is isomorphic to $\mathbb{C}^m$ (functions on finite set), and characteristic functions correspond to the projections. Next, we argue by induction, reducing the dimension, until we arrive to the sum of simple algebras. □

2.8 Non-degenerate representations

Definition 2.36. Let π be a representation of a C^*-algebra A on a Hilbert space H. We denote by $\pi(A)H$ the (possibly non-closed) linear space of finite linear combinations of the form $\sum_i \pi(a_i)\xi_i$, where $a_1, \dots, a_n \in A$, $\xi_1, \dots, \xi_n \in H$. A representation π is called *non-degenerate*, if $\pi(A)H$ is dense in H.

Problem 48. If A is unital, then π is non-degenerate if and only if $\pi(1) = 1$.

Lemma 2.37. *Let $I \subset A$ be an ideal and π a non-degenerate representation of I on a Hilbert space H. Then there is a unique extension π to a representation $\tilde{\pi}$ of the entire algebra A on H.*

Proof. Let us first define $\tilde{\pi}$ on vectors from the dense subspace $\pi(I)H \subset H$ by the formula

$$\tilde{\pi}(a)\left(\sum_i \pi(j_i)\xi_i\right) := \sum_i \pi(a j_i)\xi_i. \tag{2.4}$$

This is well-defined because if $\sum_i \pi(j_i)\xi_i = \sum_i \pi(j_i')\xi_i'$, then

$$\tilde{\pi}(a)\left(\sum_i \pi(j_i)\xi_i\right) = \lim_{\lambda\in\Lambda} \tilde{\pi}(a)\left(\sum_i \pi(u_\lambda j_i)\xi_i\right) = \lim_{\lambda\in\Lambda} \pi(a u_\lambda)\left(\sum_i \pi(j_i)\xi_i\right)$$

and, similarly, $\tilde{\pi}(a)(\sum_i \pi(j_i')\xi_i') = \lim_{\lambda\in\Lambda} \pi(a u_\lambda)(\sum_i \pi(j_i')\xi_i')$, where $u_\lambda \in I$ is an approximate unit of I. Note that the existence of the last limit in the chain follows from the existence of the penultimate limit. Hence, for each of the two cases it should be proved separately. Since

$$\begin{aligned}\left\|\tilde{\pi}(a)\left(\sum_i \pi(j_i)\xi_i\right)\right\| &= \lim_{\lambda\in\Lambda}\left\|\pi(au_\lambda)\left(\sum_i \pi(j_i)\xi_i\right)\right\| \leqslant \sup_{\lambda\in\Lambda}\|\pi(au_\lambda)\|\cdot\left\|\sum_i \pi(j_i)\xi_i\right\| \leqslant \\ &\leqslant \|a\|\cdot\sup_{\lambda\in\Lambda}\|u_\lambda\|\cdot\left\|\sum_i \pi(j_i)\xi_i\right\| = \|a\|\cdot\left\|\sum_i \pi(j_i)\xi_i\right\|,\end{aligned}$$

$\tilde{\pi}$ is bounded, i.e., $\tilde{\pi}(a)$ extends to a bounded operator in H.

At the same time, it is easy to check $\tilde{\pi}(ab) = \tilde{\pi}(a)\tilde{\pi}(b)$ and $\tilde{\pi}(a^*) = \tilde{\pi}(a)^*$ for any $a, b \in A$, so $\tilde{\pi}$ is a representation of A. Uniqueness follows from the fact that any extension of π has to satisfy (2.4). □

Lemma 2.38. *Under the conditions of Lemma 2.37 the representation π is irreducible if and only if $\tilde{\pi}$ is irreducible.*

Proof. Let π be reduced by a proper invariant subspace $L \subset H$. Then, due to non-degeneracy, $H = \overline{\pi(I)(L + L^\perp)} \subseteq \overline{\pi(I)L} + \overline{\pi(I)L^\perp}$. Since $L^\perp$ is also invariant, then $\pi(I)L^\perp \subset L^\perp$, so $\overline{\pi(I)L} = L$. Then $\overline{\tilde{\pi}(A)L} = \overline{\tilde{\pi}(A)\pi(I)L} = \overline{\pi(I)L} = L$ and L reduces $\tilde{\pi}$. The opposite statement is trivial. □

Lemma 2.39. *Let π be a representation of A on a Hilbert space H, and $I \subset A$ is an ideal. Then the orthogonal projection p onto $\overline{\pi(I)H}$ lies in the center of $\pi(A)''$. If π is irreducible and $\pi(I) \neq 0$, then $\pi|_I$ is also irreducible.*

Proof. Since $\pi(A)\pi(I)H = \pi(I)H$, then $\overline{\pi(I)H}$ is an invariant space for $\pi(A)$, hence $p \in \pi(A)'$ (see the end of proof of Lemma 2.3). If $x \in \pi(I)'$, then $x\pi(j)\xi = \pi(j)x\xi \in \pi(I)H$ for any $j \in I$, $\xi \in H$, so pH is an invariant subspace of $\pi(I)'$ and, therefore, $p \in \pi(I)''$. So,

$$p \in \pi(I)'' \cap \pi(A)' \subset \pi(A)'' \cap \pi(A)',$$

that is the center of $\pi(A)''$.

If π is irreducible, then p is a scalar operator (that is, 0 or 1) (cf. Lemma 2.3), and since $\pi(I) \neq 0$, then $p = 1$. Thus, $\pi|_I$ is non-degenerate. So by Lemma 2.38 it is irreducible. □

Chapter 3

Special classes of C^*-algebras

3.1 C^*-algebra of compact operators

In this section we will consider C^*-subalgebras of C^*-algebra $\mathbb{K}(H)$ of compact operators on the Hilbert space H. We will say that C^*-subalgebra of the algebra $\mathbb{B}(H)$ *irreducible*, if its identical representation is irreducible.

Definition 3.1. The projection p is called *minimal*, if there is no projection $q \neq 0$, $q \neq p$ such that $qp = q$. In other words, p does not *dominate* any non-trivial projection.

Lemma 3.2. *Any nonzero C^*-algebra A consisting of compact operators contains a minimal projection e and $eAe = \mathbb{C} \cdot e$. If A is irreducible, then e is a rank 1 projection (as a projection in Hilbert space).*

Proof. Since A is nonzero, it contains a nonzero positive operator (see (1.7)), which (as is known from the basic course of functional analysis, see [5, Theorem 1, p. 360]), has a discrete spectrum (except of 0) with eigenvalues of finite multiplicities. Let us consider the spectral projection for a non-zero point of the spectrum. Since the characteristic function of this isolated point is continuous on the spectrum, then this projection belongs to A. Then among the nonzero projections dominated by it there is some projection $e \in A$ of minimal rank among the dominated (since they have finite ranks). Then e is minimal (the uniqueness of the minimal and even the equality of ranks of different minimal projections is not supposed). If eAe consists not only of $\mathbb{C} \cdot e$, then in the same way we can construct a projection dominated by e and arrive to a contradiction.

Now suppose that A is irreducible, but the rank of e is greater than 1. Let us choose a pair of nonzero orthogonal vectors ξ, η in the image e. Since for any a there is a number $\lambda \in \mathbb{C}$ such that $eae = \lambda e$, we have $(\xi, a\eta) = (e\xi, ae\eta) = (\xi, eae\eta) = \lambda(\xi, \eta)$, that is $a\eta \perp \xi$ for any $a \in A$. Considering all ξ from the image e being orthogonal to η, we see that the subspace $\overline{A\eta}$ is a proper invariant subspace. A contradiction. □

Lemma 3.3. *The only irreducible C^*-subalgebra of $\mathbb{K}(H)$ is itself.*

Proof. Let A be an irreducible C^*-subalgebra of $\mathbb{K}(H)$, and $e \in A$ a minimal projection of rank 1. Then there is a unit vector $\xi \in H$ such that $e\eta = \xi(\xi, \eta)$ for any η (we take ξ from the image of e). Due to irreducibility, for any $\eta, \zeta \in H$ there are elements $a, b \in A$ such that $a\xi = \eta$, $b\xi = \zeta$ (see Lemma 2.5). Moreover, $A \ni aeb^*$ and $aeb^*(\kappa) = a\xi(\xi, b^*\kappa) = \eta(\zeta, \kappa)$, $\kappa \in H$. Thus, A contains all operators of rank 1. Such operators generate $\mathbb{K}(H)$ (any compact operator is approximated by finite-dimensional), so $A = \mathbb{K}(H)$. □

Corollary 3.4. *The algebra* $\mathbb{K}(H)$ *is simple.*

Proof. Since $\mathbb{K}(H)$ is irreducible, then any non-zero ideal is also irreducible (by Lemma 2.39), so it coincides with $\mathbb{K}(H)$ (by Lemma 3.3). □

Corollary 3.5. *Let A be an irreducible C^*-subalgebra of $\mathbb{B}(H)$ containing a nonzero compact operator. Then $\mathbb{K}(H) \subseteq A$.*

Proof. Since $A \cap \mathbb{K}(H)$ is a nonzero ideal of A, it is irreducible by Lemma 2.39. By Lemma 3.3 this subalgebra of $\mathbb{K}(H)$ should coincide with the entire $\mathbb{K}(H)$. □