Lecture 9

Lemma 2.16. *Let $a \in A$ be a self-adjoint element. Then there is a state φ on A such that $|\varphi(a)| = \|a\|$.*

Proof. If A is non-unital, then we will work in A^+. Consider the commutative C^*-algebra $C^*(a)$. Then there is a multiplicative linear functional φ_0 on $C^*(a)$ such that $|\varphi_0(a)| = \|a\|$ (we must take as φ_0 the mapping, which is the taking of the value of functions at that point of $\mathrm{Sp}(a)$, where the function $\hat{a}$ reaches its maximum). Then $\varphi_0(1) = 1 = \|\varphi_0\|$. Consider the extension of φ_0 by the Hahn-Banach theorem to a functional φ on A^+. Then, since $\|\varphi\| = 1 = \varphi(1)$, then φ is a state by Lemma 2.15. □

Corollary 2.17. *For any $a \in A$ there exists a representation π and a unit vector ξ in the space of representation such that $\|\pi(a)\xi\| = \|a\|$.*

Proof. By the previous lemma, we find a state φ such that $\varphi(a^*a) = \|a\|^2$. Let $\pi = \pi_\varphi$ and $\xi = \xi_\varphi$ were obtained for φ using the GNS construction. Then $\|\pi(a)\xi\|^2 = (\xi, \pi(a^*a)\xi) = \varphi(a^*a) = \|a\|^2$. □

Theorem 2.18 (Gelfand-Naimark)**.** *Any C^*-algebra is isometrically $*$-isomorphic to a C^*-subalgebra of $\mathbb{B}(H)$ for some Hilbert space H. If A is separable, then H can be chosen to be separable.*

Proof. Let us set $\pi = \oplus_\varphi \pi_\varphi$, where the direct sum is taken over all states on A. More precisely, we consider the Hilbert direct sum $H := \oplus_\varphi H_\varphi$ (completion with respect to the ℓ_2 norm of the space of compactly supported mappings $\varphi \mapsto h_\varphi \in H_\varphi$, that is, the sets $h = \{h_\varphi\}$, $h_\varphi \in H_\varphi$, and only a finite number h_φ is nonzero, and the norm is defined as $\|h\|^2 = \sum_\varphi \|h_\varphi\|^2$) with diagonal action $\pi(a)(\{h_\varphi\}) = \{\pi_\varphi(a)(h_\varphi)\}$. Then, as can be seen from the proof of the previous consequences, $\|\pi(a)\| = \sup_\varphi \|\pi_\varphi(a)\| = \|a\|$. If A is separable, then it is sufficient to take the sum over a countable set $\{\varphi_n\}$, where $\|\pi_{\varphi_n}(a_n)\| = \|a_n\|$, for elements a_n forming a dense subset in A. Then $\pi = \oplus_{n\in\mathbb{N}} \pi_{\varphi_n}$, and the corresponding Hilbert space is separable, since each H_{φ_n} is separable (as a completion of a factor-space of a separable space). □

Definition 2.19. The representation constructed in the theorem (in its first part) is called the *universal representation* of A. The von Neumann algebra $\pi(A)''$, where π is the universal representation, contains $\pi(A) \cong A$ as a dense subset and is called the *enveloping von Neumann algebra* for A.

2.5 Jordan decomposition

Lemma 2.20. *Let φ be a linear functional on A. Then $\varphi = \psi_1 + i\psi_2$, where ψ_1 and ψ_2 are self-adjoint.*

Proof. Let us take, in the same way as we did for elements of algebra, $\psi_1(a) = (\varphi(a) + \overline{\varphi(a^*)})/2$ and $\psi_2(a) = (\varphi(a) - \overline{\varphi(a^*)})/2i$. □

Let A_{sa} denote the set of all self-adjoint elements of A. Then it is evident that A_{sa} is a real Banach space.

Problem 46. There is a natural bijection between self-adjoint linear functionals on A and (real) linear functionals on A_{sa}.

To prove the Jordan decomposition theorem, we need the following statement, which is of independent interest.

Theorem 2.21 (on extension of positive functionals)**.** *Let $B \subset A$ be a C^*-subalgebra, and $\varphi : B \to \mathbb{C}$ be a positive functional. Then there exists a positive functional $\varphi' : A \to \mathbb{C}$ such that that $\varphi'|_B = \varphi$ and $\|\varphi'\| = \|\varphi\|$.*

Proof. The following cases are possible:

a) both algebras have a common unit,

b) A has one, but B does not,

c) both algebras do not have a unit,

d) B has one, but A does not.

e) both algebras with 1, but $1_A \neq 1_B$.

By Corollary 2.14, (c) and (b) can be reduced by adjoining 1 to (a) (for (b) it should be noted that $B^+ \cong B \oplus \mathbb{C}\, 1_A$). In turn, (d) obviously reduces to (e).

In case (a) we extend φ (using the Hahn-Banach theorem) to some $\varphi' : A \to \mathbb{C}$ of the same norm. Then by Lemma 2.10, $\|\varphi'\| = \|\varphi\| = \varphi(1) = \varphi'(1)$ and φ' is positive by Lemma 2.15.

In case (e), consider the C^*-subalgebra $B_1 := B \oplus \mathbb{C}\, 1_A = B \oplus \mathbb{C}\,(1_A - 1_B)$ and extend φ to $\varphi_1 : B_1 \to \mathbb{C}$, setting $\varphi_1(1_A - 1_B) = 0$. Then $\varphi_1(a) = \varphi(1_B \cdot a)$, where $a \in B_1$. Indeed, if $a \in B$, then $\varphi_1(a) = \varphi(1_B \cdot a) = \varphi(a)$, and if $a = 1_A - 1_B$, then $\varphi_1(a) = \varphi(1_B(1_A - 1_B)) = \varphi(0) = 0$. In this case, the unit of B_1 is 1_A. Moreover, $\|\varphi_1\| \leqslant \|\varphi\| \cdot \|1_B\| = \|\varphi\|$, and $\varphi_1(1_A) = \varphi(1_B) = \|\varphi\|$. This means that $\|\varphi_1\| = \|\varphi\| = \varphi_1(1_A) = \varphi_1(1_{B_1})$ and, by Lemma 2.15, φ_1 is positive. Thus, case (e) is also reduced to the proven case (a). □

The Jordan theorem about decomposition of a measure in the sum of positive and negative ones [8, Ch. VI, §5, Theorem 1] in the functional language (in the sense of the Riesz-Markov-Kakutani theorem [5, Ch. I, §6, Theorem 4]) can be written as: for any bounded real linear functional $\tau : C(\Omega, \mathbb{R}) \to \mathbb{R}$ there are positive linear functionals τ_+ and τ_- such that $\tau = \tau_+ - \tau_-$ and $\|\tau\| = \|\tau_+\| + \|\tau_-\|$, where Ω is a compact Haudorff space and $C(\Omega, \mathbb{R})$ is the real algebra of all real continuous functions on Ω.

Theorem 2.22 (Jordan decomposition)**.** *Let ψ be a self-adjoint linear functional on A. Then $\psi = \varphi_+ - \varphi_-$, where φ_+ and φ_- are positive linear functionals on A and $\|\psi\| = \|\varphi_+\| + \|\varphi_-\|$.*

Proof. Denote by K the set of all self-adjoint linear functionals of norm $\leqslant 1$, i.e., $K \subset (A^*)_{sa}$. Then K is a $*$-weak closed subset of the unit ball and hence it is $*$-weak compact. Define an $\mathbb{R}$-linear map

$$\theta : A_{sa} \to C(K, \mathbb{R}), \qquad \theta(a)(\tau) = \tau(a),$$

so, if $a \in A$, $a \geqslant 0$, then $\theta(a) \geqslant 0$ in K. By Lemma 2.16 the mapping θ is an isometry onto its image.

There is a natural isometry $\tau \mapsto \tau'$ of real spaces $(A^*)_{sa}$ and $(A_{sa})^*_{\mathbb{R}}$ (real functionals) (see Problem46). By the Hahn-Banach theorem there is a functional $\rho \in (C(K,\mathbb{R}))^*_{\mathbb{R}}$ such that $\rho \circ \theta = \psi'$ and $\|\rho\| = \|\psi'\|$ (an extension of a functional from the closed subspace $\theta(A_{sa})$). Then by the Jordan theorem for measures (as it is explained above before the formulation) there are positive functionals ρ_+ and ρ_- such that $\rho = \rho_+ - \rho_-$ and $\|\rho\| = \|\rho_+\| + \|\rho_-\|$. Consider $\varphi'_+ := \rho_+ \circ \theta$ and $\varphi'_- := \rho_- \circ \theta$. These are functionals from $(A_{sa})^*_{\mathbb{R}}$. Let φ_+ and φ_- correspond to them under the identification with $(A^*)_{sa}$. Evidently they satisfy all the conditions, except maybe the norm property. Let us verify it:

$$\|\psi\| = \|\psi'\| = \|\rho\| = \|\rho_+\| + \|\rho_-\| \geqslant \|\varphi'_+\| + \|\varphi'_-\| = \|\varphi_+\| + \|\varphi_-\| \geqslant \|\psi\|.$$

□

2.6 Linear topological spaces

Definition 2.23. A subset M of a linear space is called *balanced*, if for any $v \in M$ the vector λv belongs to M for any $|\lambda| \leqslant 1$. In particular, M is a star set relative to the zero of space.

Definition 2.24. A subset M of a linear space is called *absorbing*, if for any vector v of the space there is a number $\alpha > 0$ such that $v \in \beta M$ for $|\beta| \geqslant \alpha$.

Definition 2.25. A linear space equipped with a topology is called *linear topological space* (LTS), if the operations of linear space are continuous.

In the basic course of functional analysis, the following simple statements are proved: (see [8, Chapter III, §5]):

Proposition 2.26. *1). A base of LTS consists of shifts of neighborhoods of zero.*

2). Any vector of LTS and a closed set not containing it have disjoint neighborhoods.

Definition 2.27. An LTS L satisfies the *homothety condition*, if for any neighborhood of zero W its homothety λW is also a neighborhood of zero for any $\lambda \neq 0$ from the main field.

Remark 2.28. Obviously, the topology of a normed space satisfies the homothety condition.

Proposition 2.29. *For any neighborhood of zero U of an LTS L with the homothety condition, there is a balanced neighborhood contained in it.*

Proof. Consider the continuous mapping $\mathbb{C} \times L \to L$ (multiplication) mapping $(0, 0_L) \mapsto 0_L$. Then, by virtue of continuity, there are $\delta > 0$ and a neighborhood of zero W such that $\lambda W \subseteq U$ for $|\lambda| \leqslant \delta$ (a non-strict inequality can be achieved by reducing δ from the standard definition). Let $W' := \cup_{0<|\lambda|\leqslant 1} \lambda W$. By virtue of 2.27, this W' is what we are looking for. □

Remark 2.30. In fact, it can be proven that the base of neighborhoods of zero of an LTS L can be chosen from absorbing balanced sets, and also that the homothety condition is in fact not a condition, but we will not need this (see [9, Chapter II, §4]).

We will need the following important result.

Theorem 2.31. *Let L be a finite-dimensional space, $\dim L = n$. Then any Hausdorff topology τ making L a linear topological space L_τ with the homothety condition coincides with the topology of the Euclidean norm $\|v\|^2 = \sum_{i=1}^n |v^i|^2$, where $e_1, \dots, e_n$ is some base of L, and $v = v^1 e_1 + \cdots v^n e_n$.*

Proof. The space L with Euclidean (or unitary) topology will be denoted by L_u, and neighborhoods of zero of two topologies (τ and Euclidean) will be denoted by T and U, respectively.

Consider an arbitrary T. Then there is a neighborhood T_0 such that $T_0 + \cdots + T_0 \subset T$ (n terms) due to the continuity of the addition operation. For every k there is $\varepsilon_k > 0$ such that $v^k e_k \in T_0$ for $|v_k| < \varepsilon_k$ ($k = 1, \dots, n$). Let $\varepsilon := \min_k \varepsilon_k$, and $U := \{v \in L \mid \|v\| < \varepsilon\}$. Then $v^k e_k \in T_0$ for any $v \in U$ and any $k = 1, \dots, n$. Thus, $U \subset T$. From what has been proved, in particular, it follows that the identity mapping $\iota : L_u \to L_\tau$ is continuous.

Conversely, let U be an arbitrary neighborhood, we can assume that $U = B(0, \varepsilon)$ is an open ball of radius ε with boundary (sphere) S, which is a compact set. Then $S = \iota(S)$ is compact in L_τ. This means that it is closed, since the topology is Hausdorff. Then there is a stellar neighborhood of zero T (for example, balanced) that does not intersect S by virtue of propositions 2.26 and 2.29. Moreover, $T \subseteq U$, since otherwise there exists a vector $v \in T$ such that $\|v\| \geqslant \varepsilon$, and if we put $\alpha := \varepsilon/\|v\|$, $w := \alpha v$, then $\alpha \leqslant 1$, so $w \in T$ by the star property. But $\|w\| = \varepsilon$, so $w \in T \cap S = \varnothing$. A contradiction. □