Lecture 9

Theorem 2.22 (Jordan decomposition)**.** *Let ψ be a self-adjoint linear functional on A. Then $\psi = \varphi_+ - \varphi_-$, where φ_+ and φ_- are positive linear functionals on A and $\|\psi\| = \|\varphi_+\| + \|\varphi_-\|$.*

Proof. Denote by K the set of all self-adjoint linear functionals of norm $\leqslant 1$, i.e., $K \subset (A^*)_{sa}$. Then K is a $*$-weak closed subset of the unit ball and hence it is $*$-weak compact. Define an $\mathbb{R}$-linear map

$$\theta : A_{sa} \to C(K, \mathbb{R}), \qquad \theta(a)(\tau) = \tau(a),$$

so, if $a \in A$, $a \geqslant 0$, then $\theta(a) \geqslant 0$ in K. By Lemma 2.16 the mapping θ is an isometry onto its image.

There is a natural isometry $\tau \mapsto \tau'$ of real spaces $(A^*)_{sa}$ and $(A_{sa})^*_{\mathbb{R}}$ (real functionals) (see Problem46). By the Hahn-Banach theorem there is a functional $\rho \in (C(K, \mathbb{R}))^*_{\mathbb{R}}$ such that $\rho \circ \theta = \psi'$ and $\|\rho\| = \|\psi'\|$ (an extension of a functional from the closed subspace $\theta(A_{sa})$). Then by the Jordan theorem for measures (as it is explained above before the formulation) there are positive functionals ρ_+ and ρ_- such that $\rho = \rho_+ - \rho_-$ and $\|\rho\| = \|\rho_+\| + \|\rho_-\|$. Consider $\varphi'_+ := \rho_+ \circ \theta$ and $\varphi'_- := \rho_- \circ \theta$. These are functionals from $(A_{sa})^*_{\mathbb{R}}$. Let φ_+ and φ_- correspond to them under the identification with $(A^*)_{sa}$. Evidently they satisfy all the conditions, except maybe the norm property. Let us verify it:

$$\|\psi\| = \|\psi'\| = \|\rho\| = \|\rho_+\| + \|\rho_-\| \geqslant \|\varphi'_+\| + \|\varphi'_-\| = \|\varphi_+\| + \|\varphi_-\| \geqslant \|\psi\|.$$

□

2.6 Linear topological spaces

Definition 2.23. A subset M of a linear space is called *balanced*, if for any $v \in M$ the vector λv belongs to M for any $|\lambda| \leqslant 1$. In particular, M is a star set relative to the zero of space.

Definition 2.24. A subset M of a linear space is called *absorbing*, if for any vector v of the space there is a number $\alpha > 0$ such that $v \in \beta M$ for $|\beta| \geqslant \alpha$.

Definition 2.25. A linear space equipped with a topology is called *linear topological space* (LTS), if the operations of linear space are continuous.

In the basic course of functional analysis, the following simple statements are proved: (see [8, Chapter III, §5]):

Proposition 2.26. *1). A base of LTS consists of shifts of neighborhoods of zero.*

2). Any vector of LTS and a closed set not containing it have disjoint neighborhoods.

Definition 2.27. An LTS L satisfies the *homothety condition*, if for any neighborhood of zero W its homothety λW is also a neighborhood of zero for any $\lambda \neq 0$ from the main field.

Remark 2.28. Obviously, the topology of a normed space satisfies the homothety condition.

Proposition 2.29. *For any neighborhood of zero U of an LTS L with the homothety condition, there is a balanced neighborhood contained in it.*

Proof. Consider the continuous mapping $\mathbb{C} \times L \to L$ (multiplication) mapping $(0, 0_L) \mapsto 0_L$. Then, by virtue of continuity, there are $\delta > 0$ and a neighborhood of zero W such that $\lambda W \subseteq U$ for $|\lambda| \leqslant \delta$ (a non-strict inequality can be achieved by reducing δ from the standard definition). Let $W' := \cup_{0<|\lambda|\leqslant 1} \lambda W$. By virtue of 2.27, this W' is what we are looking for. □

Remark 2.30. In fact, it can be proven that the base of neighborhoods of zero of an LTS L can be chosen from absorbing balanced sets, and also that the homothety condition is in fact not a condition, but we will not need this (see [9, Chapter II, §4]).

We will need the following important result.

Theorem 2.31. *Let L be a finite-dimensional space,* $\dim L = n$. *Then any Hausdorff topology τ making L a linear topological space L_τ with the homothety condition coincides with the topology of the Euclidean norm $\|v\|^2 = \sum_{i=1}^n |v^i|^2$, where $e_1, \dots, e_n$ is some base of L, and $v = v^1 e_1 + \cdots v^n e_n$.*

Proof. The space L with Euclidean (or unitary) topology will be denoted by L_u, and neighborhoods of zero of two topologies (τ and Euclidean) will be denoted by T and U, respectively.

Consider an arbitrary T. Then there is a neighborhood T_0 such that $T_0 + \cdots + T_0 \subset T$ (n terms) due to the continuity of the addition operation. For every k there is $\varepsilon_k > 0$ such that $v^k e_k \in T_0$ for $|v_k| < \varepsilon_k$ ($k = 1, \dots, n$). Let $\varepsilon := \min_k \varepsilon_k$, and $U := \{v \in L \mid \|v\| < \varepsilon\}$. Then $v^k e_k \in T_0$ for any $v \in U$ and any $k = 1, \dots, n$. Thus, $U \subset T$. From what has been proved, in particular, it follows that the identity mapping $\iota : L_u \to L_\tau$ is continuous.

Conversely, let U be an arbitrary neighborhood, we can assume that $U = B(0, \varepsilon)$ is an open ball of radius ε with boundary (sphere) S, which is a compact set. Then $S = \iota(S)$ is compact in L_τ. This means that it is closed, since the topology is Hausdorff. Then there is a stellar neighborhood of zero T (for example, balanced) that does not intersect S by virtue of propositions 2.26 and 2.29. Moreover, $T \subseteq U$, since otherwise there exists a vector $v \in T$ such that $\|v\| \geqslant \varepsilon$, and if we put $\alpha := \varepsilon / \|v\|$, $w := \alpha v$, then $\alpha \leqslant 1$, so $w \in T$ by the star property. But $\|w\| = \varepsilon$, so $w \in T \cap S = \varnothing$. A contradiction. □

2.7 Finite-dimensional C^*-algebras

Consider the $*$-weak topology on A defined by the seminorm system $a \mapsto |\varphi(a)|$ for all linear functionals φ. From Lemma 2.20 and Theorem 2.22 it follows that the same topology can be obtained by using only seminorms, defined by states.

Note also that the corresponding LTS has the homothety property 2.27.

Lemma 2.32. *A finite-dimensional C^*-algebra is always unital.*

Proof. If A is finite-dimensional, then the topology of the norm coincides with the $*$-weak topology according to Theorem 2.31. Let u_n be an approximate unit of the algebra A. Then for any state φ the sequence $\varphi(u_n)$ is non-decreasing and bounded. Therefore u_n converges in $*$-weak topology, and therefore in norm. Thus, there is a limit $\lim_n u_n = a$. Then $ax = xa = x$ for any $x \in A$, so $a = 1$. $\square$

Lemma 2.33. *Let $I \subset A$ be an ideal in a finite-dimensional C^*-algebra A. Then $I = Ap$ for some central projector (=idempotent from the center) p.*

Proof. Since I is finite-dimensional, it is unital by Lemma 2.32. Let $p \in I$ be the unit of I. Then for every $x \in A$, one has $xp \in I$, so $p(xp) = xp$. Hence $px^*p = x^*p$ for any $x \in A$, whence $xp = pxp = px$ and p belongs to the center of A. Obviously, $p^2 = p$. $\square$

Lemma 2.34. *A simple finite-dimensional C^*-algebra A is isometrically $*$-isomorphic to the matrix algebra M_n for some n.*

Proof. First of all, note that $aAb \neq 0$ for any non-zero $a, b \in A$. Indeed, AaA is a non-zero ideal (since A is unital and $0 \neq a = 1 \cdot a \cdot 1 \in A$), so by simplicity, $AaA = A$. Therefore $1 = \sum_i x_i a y_i$ and $b = \sum_i x_i a y_i b$. Hence, if $ayb = 0$ for any $y \in A$, then $b = \sum_i x_i(ay_ib) = 0$. This contradicts the assumption.

Let B be some maximal commutative subalgebra of A. Then it can be identified with $C(X) = \mathbb{C}^n = \mathbb{C} \cdot e_1 \oplus \ldots \oplus \mathbb{C} \cdot e_n$ for some n, where X consists of n points, and $e_i \in B$ denotes the element corresponding to the characteristic functions at point i. Here e_i are projections with the relations $e_ie_j = 0$ for $i \neq j$ and $\sum_{i=1}^n e_i = 1$. Since $e_iAe_i \cdot e_j = e_j \cdot e_iAe_i = 0$ and B is maximal, then $e_iAe_i \subset B$. Therefore $e_iAe_i = \mathbb{C} \cdot e_i$ (since, obviously, $0 \neq e_iAe_i \ni e_i$, or you can use the statement from the beginning of the proof).

For any i, j there is $x = x_{ij} \in A$ such that $x = e_ixe_j \neq 0$, $\|x\| = 1$. Indeed, by virtue of the statement from the beginning of the proof, $e_iAe_j \neq 0$, so we have $x = e_iye_j$ with $\|x\| = 1$. In this case $e_ixe_j = e_ie_iye_je_j = e_iye_j = x$. Then $x^*x = e_jx^*e_ie_ixe_j \in e_jAe_j$, and therefore, according to what has been proven, this element has the form αe_j, $\alpha \in \mathbb{C}$. Since x^*x is a positive element with norm equal to one, then $\alpha = 1$, so $x^*x = e_j$. Likewise, $xx^* = e_i$. Let us denote such $x = x_{ij}$ for $j = 1$ by u_i, so that $u_i = e_ixe_1 = e_iu_ie_1$. Then $u_i^*u_i = e_1$, $u_iu_i^* = e_i$, $i = 1, \ldots, n$. Let us set $u_{ij} := u_iu_j^*$. In this case, $u_ie_1u_i^* = u_iu_i^*u_iu_i^* = e_ie_i = e_i$, So $u_{ij}u_{ji} = u_iu_j^*u_ju_i^* = u_ie_1u_i^* = e_i$. Also $e_ju_{ji} = u_ju_j^*u_ju_i^* = u_je_1u_i^* = u_ju_i^*u_iu_i^* = u_{ji}e_i$, and $e_iu_{ij} = u_iu_i^*u_iu_j^* = u_ie_1u_j^* = u_iu_j^*u_ju_j^*$.

If $x \in e_iAe_j$, that is, $x = e_iae_j$, then $xu_{ji} = e_iae_ju_{ji} = e_iau_{ji}e_i \in e_iAe_i$, so $xu_{ji} = \lambda e_i$ for some $\lambda \in \mathbb{C}$. Then $x = xe_j = xu_{ji}u_{ij} = \lambda e_iu_{ij} = \lambda u_{ij}$, so for any $x \in A$ there is a number $\lambda_{ij}(x) \in \mathbb{C}$ such that $e_ixe_j = \lambda_{ij}(x)u_{ij}$. Thus, $x = \sum_{i,j} e_ixe_j = \sum_{ij} \lambda_{ij}(x)u_{ij}$. The correspondence $x \mapsto (\lambda_{ij}(x))$ defines an isomorphism $\kappa : A \to M_n$ (Problem 47). $\square$

Problem 47. Check the bijectivity and necessary algebraic properties of κ.

Theorem 2.35. *If A is finite-dimensional, then $A = \oplus_k Ap_k$, where p_k are central projections, and each Ap_k is a matrix algebra $M_{n(k)}$.*

Proof. For a simple algebra, the result follows from Lemma 2.34. If A is not simple, then $I = Ap$ by Lemma 2.33, where p is a central projection. Then $A = I \oplus J$, where $J := A(1-p)$. Then J is also an ideal, since $(1-p)$ is also a central projection, so $A(1-p)A = AA(1-p) \subseteq A(1-p)$. In this case, the center of A, being a finite-dimensional commutative algebra, is isomorphic to $\mathbb{C}^m$ (functions on finite set), and characteristic functions correspond to the projections. Next, we argue by induction, reducing the dimension, until we arrive to the sum of simple algebras. □