C^*-algebras and K-theory. Introduction

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Contents

Chapter 1

C^*-algebras

Lecture 1, September 14, 2026

1.1 Definition and first examples

Definition 1.1. An *algebra* A (over a field $\mathbb{K}$) is a ring that is a linear space over $\mathbb{K}$, and the addition in the definition of a ring and in the definition of a linear spaces are the same, and multiplications are connected by the relation $\lambda(ab) = (\lambda a)b$ for all $\lambda \in \mathbb{K}$, $a, b \in A$.

We will consider algebras only over the field of complex numbers $\mathbb{C}$.

Definition 1.2. An algebra A over $\mathbb{C}$ is called a *Banach algebra*, if the underlying linear space is a Banach space and $\|ab\| \leq \|a\|\|b\|$ for any $a, b \in A$.

Problem 1 (easy)**.** Show that in this case the multiplication is continuous (as a mapping $A \times A \to A$).

Definition 1.3. A mapping $* : A \to A$, $a \mapsto a^*$ is called an *involution* if

1. $(a^*)^* = a$;
2. $(a + b)^* = a^* + b^*$;
3. $(\lambda a)^* = \bar{\lambda} a^*$;
4. $(ab)^* = b^* a^*$;
5. $\|a^*\| = \|a\|$

for any $a, b \in A$, $\lambda \in \mathbb{C}$. A Banach algebra with involution is called *involutive Banach algebra.*

Definition 1.4. An involutive Banach algebra A is called a *C^*-algebra,* if the involution satisfies the equality $\|a^* a\| = \|a\|^2$ for all $a \in A$ (this equality is called the *C^*-property*).

Problem 2. Give an example of an involutive Banach algebra that is not a C^*-algebra.

Problem 3 (easy)**.** Show that property (5) of the definition 1.3 follows from properties (1 – 4) and the C^*-property.

Definition 1.5. An element $1 \in A$ is called a (left) *unit*, if $1a = a$ for any $a \in A$. A C^*-algebra A is called *unital* or *algebra with unit*, if it has a (left) unit (identity element).

Problem 4. Show that a left unit element is also a right one, which has $1^* = 1$; that the identity element is unique and that $\|1\| = 1$. It is called the *unit of algebra*.

Problem 5. Verify that the algebra $C(X)$ formed by all continuous complex-valued functions on a compact space X and the algebra $C_0(X)$ of all continuous complex-valued functions on a locally compact space X tending to 0 at infinity (that is, $f : X \to \mathbb{C}$ such that for any $\varepsilon > 0$ there exists a compact $K \subseteq X$ such that $\sup\{|f(x)| \mid x \in K\} < \varepsilon$) are commutative C^*-algebras if the *supremum-norm*: $\|f\| = \sup_{x\in X} |f(x)|$, is taken as the norm and the pointwise multiplication is taken as the multiplication. Moreover, the algebra $C(X)$ is unital.

Problem 6. Verify that the algebra $\mathbb{B}(H)$ of all bounded operators acting on a Hilbert space H is a C^*-algebra with identity. Here as a norm we take the *operator norm* $\|a\| = \sup_{h\in H,\, \|h\|\le 1} \|a(h)\|$, and the multiplication is the composition of operators.

These examples of C^*-algebras are the most important, as we will see later.

1.2 Unitalization, or attaching of a unit

If an involutive Banach algebra A does not have a unit, then it can be embedded into an involutive unital Banach algebra A as follows. Let $A^+ = A \oplus \mathbb{C}$ be a linear space (the direct sum of linear spaces). Let's define a structure of an involutive Banach algebra on A^+ by formulas

$$\begin{aligned} (a, \lambda)(b, \mu) &= (ab + \lambda b + \mu a, \lambda\mu), \\ (a, \lambda)^* &= (a^*, \bar{\lambda}), \\ \|(a, \lambda)\| &= \|a\| + |\lambda| \end{aligned} \tag{1.1}$$

for any $(a, \lambda), (b, \mu) \in A \oplus \mathbb{C}$.

We further assume that the initial algebra A is a C^*-algebra.

Problem 7. Show that this norm turns A^+ into an involutive Banach algebra, but not into a C^*-algebra. (To check completeness, use the completeness of the subspace $A \subset A^+$ of finite codimension (one), this follows from Problem 337(b) in [7].)

Definition 1.6. A subset $B \subseteq A$ is called a *subalgebra* of a (Banach) algebra A if B is an algebra with respect to the operations (and norm) of A.

A subset of $B \subseteq A$ is called (C^*-)*subalgebra* C^*-algebras A, if B is a C^*-algebra with respect to the operations and the norm of A. In particular, B is closed in A.

Definition 1.7. A subalgebra $I \subseteq A$ is called a (two-sided) *ideal*, if $aI \subseteq I$ and $Ia \subseteq I$ for any $a \in A$.

Problem 8. Prove that A is an ideal in A^+.

Lemma 1.8. *There is a norm on A^+ such that*

1) *it is equivalent to the one defined above;*

2) *on A it coincides with the original norm:*

3) *it is a C^*-norm.*

Proof. For $b = (b', \lambda) \in A^+$, consider the linear mapping $L_b : A \to A$ according to the formula $L_b(a) = ba$, $a \in A$. For L_b, we can thus define the operator norm: $\|L_b\| := \sup_{a\in A,\, \|a\|\le 1} \|L_b(a)\|$. If $b \in A$, then $\|L_b\| \le \|b\|$. Since $\|L_b(b^*)\| = \|bb^*\| = \|b\|^2$, we have $\|L_b\| = \|b\|$. For every $b = (b', \lambda) \in A^+$ we set $\|b\|_{new} = \|L_b\|$. Note that the above reasoning shows that $\|.\|_{new}$ satisfies condition 2) from the formulation of the lemma.

First of all, let's check that $\|\cdot\|_{new}$ is a norm. Obviously, the axioms of linearity and triangle are satisfied (that is, this is a *semi-norm*, or *pre-norm*), so it remains to check the nondegeneracy.

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Let $\|b\|_{new} = 0$ for some $b = (b', \lambda) \in A^+$. It means that $(b', \lambda) \cdot (a, 0) = (b'a + \lambda a, 0) = (0, 0)$ for any $a \in A$, so $-\frac{1}{\lambda} b' \cdot a = a$ and $-\frac{1}{\lambda} b'$ is the unit of A. But A does not have a unit. This means that $\|\cdot\|_{new}$ is the norm. Like any operator norm, the new norm is a norm of a Banach algebra, that is,

$$\|bc\|_{new} \le \|b\|_{new} \|c\|_{new} \tag{1.2}$$

for any $b, c \in A^+$ (Easy problem: check this). We do not claim yet that the involution is an isometry.

This new norm is equivalent to the previously defined norm on A^+, since $A \subset A^+$ is a subspace of codimension 1. Indeed, by the triangle inequality, we have $\|(a, \lambda)\|_{new} \le \|a\| + |\lambda| = \|(a + \lambda)\|$, so it is sufficient to show that there is a constant $c > 0$ such that $\|(a, \lambda)\|_{new} \ge c\|(a, \lambda)\|$. Let's assume the opposite: there are such pairs (a_n, λ_n) and numbers $c_n > 0$ such that $\lim_{n\to\infty} c_n = 0$ and

$$\|(a_n, \lambda_n)\|_{new} \le c_n \|(a_n, \lambda_n)\|.$$

Since none of λ_n is zero, we can assume (by dividing both sides by $\lambda_n \ne 0$ if necessary) that $\lambda_n = 1$. Applying the triangle inequality again, we get

$$\|a_n\| - 1 \le \|(a_n, 1)\|_{new} \le c_n(\|a_n\| + 1),$$

whence $\|a_n\| \le \frac{1+c_n}{1-c_n}$, and for sufficiently large n we have $\|a_n\| < 2$. But then for these n we have $\|(a_n, 1)\|_{new} \le 3c_n$, so

$$\lim_{n\to\infty} (a_n, 1) = 0. \tag{1.3}$$

Now recall that $A^+/A \cong \mathbb{C}$, and all norms on $\mathbb{C}$ are equivalent (moreover, they differ only by a constant multiple). Thus, the quotient norm on A^+/A given by $\|(a,\lambda)+A\|_{new} := \inf_{a\in A}\|(a,\lambda)\|_{new}$ is equivalent to the usual norm on $\mathbb{C}$. So $\inf_{a\in A}\|(a,1)\|_{new} = \alpha > 0$, what contradicts (1.3).

Now we need to check the C^* property. By definition, for any $b \in A^+$ and any $\varepsilon > 0$ there is an element $a \in A$ such that $\|a\| = 1$ and

$$\|L_b(a)\| \geq (1-\varepsilon)\|L_b\|, \qquad \text{that is} \qquad \|ba\|_{new} = \|ba\| \geq (1-\varepsilon)\|b\|_{new}.$$

We also have

$$\|b^*b\|_{new} \geq \|a^*(b^*b)a\|_{new} = \|a^*(b^*b)a\| = \|(ba)^*(ba)\| = \|ba\|^2 \geq (1-\varepsilon)^2\|b\|^2_{new}$$

(where the first inequality is satisfied by virtue of (1.2), the next equality is satisfied by virtue of that A is an ideal in A^+, and then we apply the C^*-property in A). Passing to the limit as ε tends to zero, we obtain $\|b^*b\|_{new} \geqslant \|b\|^2_{new}$. In particular, $\|b^*\|_{new}\cdot\|b\|_{new} \geqslant \|b^*b\|_{new} \geqslant \|b\|^2_{new}$, so $\|b^*\|_{new} \geqslant \|b\|_{new}$. Therefore $\|b\|_{new} = \|(b^*)^*\|_{new} \geqslant \|b^*\|_{new}$. Thus, the involution is an isometry and, by (1.2), $\|b^*b\|_{new} \leqslant \|b^*\|_{new}\|b\|_{new} = \|b\|^2_{new}$. This means that this is a C^*-norm. □

Definition 1.9. An element $a \in A$ is called *self-adjoint*, if $a^* = a$, *unitary*, if $a^*a = aa^* = 1$ (here the algebra A is assumed to be unital), *normal*, if $a^*a = aa^*$.

Definition 1.10. In a unital C^*-algebra, an element a is called *invertible*, if there is an element $a' \in A$ such that $aa' = a'a = 1$. There is only one a' with this property (check!), called the *inverse to a* and is denoted by a^{-1}. The set of invertible elements $G(A)$ is a group.

Problem 9. Check this.

Lemma 1.11. *If $\|1-a\| < 1$, then a^{-1} exists and is equal to $a' = \sum_{n=0}^{\infty}(1-a)^n$, and the series converges in norm.*

Proof. Convergence immediately follows from the domination by a geometric progression. Next we calculate: $a'a = aa' = -(1-a)a' + a' = -\sum_{n=1}^{\infty}(1-a)^n + \sum_{n=0}^{\infty}(1-a)^n = 1$. □

Problem 10. Prove in a similar way that if $a_0 \in A$ is invertible and $\|a - a_0\| < \frac{1}{\|a_0^{-1}\|}$, then a is also invertible, and $a^{-1} = \sum_{n=0}^{\infty}[a_0^{-1}(a_0-a)]^n a_0^{-1}$.

Corollary 1.12. *The subset $G(A) \subset A$ is an open set. Taking the inverse element $a \mapsto a^{-1}$ is a continuous map $G(A)$ into itself.*

Problem 11. Prove the corollary using the formulas established above.

1.3 Spectrum and functional calculus

Definition 1.13. Let A be a unital Banach algebra. For any $a \in A$ we call the set $\mathrm{Sp}(a) = \{\lambda \in \mathbb{C} : (a - \lambda 1) \text{ is not invertible}\}$ the *spectrum* of the element a. The function $R_a(\lambda) = (a - \lambda 1)^{-1}$ is called the *resolvent* of a. If A is a non-unital C^*-algebra, then *quasi-spectrum* $\mathrm{Sp}'(a)$ of an element $a \in A$ is assumed to be equal to the spectrum of a as an element of the algebra A^+.

Problem 12. Show that the quasi-spectrum always contains zero.

Theorem 1.14. *The spectrum of any element is a compact non-empty set. The resolvent is an analytic function outside the spectrum (that is, for any point of the completed complex plane from the complement of the spectrum, it can be represented by a power series with coefficients from the algebra, uniformly convergent on some closed disk).*

Proof. If $|\lambda| > \|a\|$, then the series $\sum_{n=0}^{\infty} \lambda^{-(n+1)} a^n$ converges in norm, and converges to $-(a - \lambda 1)^{-1}$, since $-(a - \lambda 1) \sum_{n=0}^{k} \lambda^{-(n+1)} a^n = 1 - \lambda^{-(k+2)} a^{k+1}$ converges to 1 . Thus, $R_a(\lambda)$ is analytic in the neighborhood of infinity, defined by $|\lambda| > \|a\|$, and

$$\lim_{\lambda \to \infty} \|R_a(\lambda)\| \leq \lim_{\lambda \to \infty} \frac{1}{|\lambda|} \frac{1}{1 - \frac{\|a\|}{|\lambda|}} = 0.$$

In particular, $\mathrm{Sp}(a)$ is contained in a closed disk of radius $\|a\|$ and thus, is a bounded set.

If $a - \lambda_0 1$ is invertible (that is, $\lambda_0 \notin \mathrm{Sp}(a)$) and $|\lambda - \lambda_0| < \frac{1}{\|(a - \lambda_0 1)^{-1}\|}$ then, the same way,

$$(a - \lambda 1)^{-1} = \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n (a - \lambda_0 1)^{-n-1}$$

is the Taylor series R_a in a neighborhood of λ_0, so $R_a(\lambda)$ is analytic outside $\mathrm{Sp}(a)$. The same reasoning shows, in particular, that the complement to $\mathrm{Sp}(a)$ is open, that is, $\mathrm{Sp}(a)$ is closed. Since the spectrum is bounded, it is compact.

Since $R_a(\lambda)$ is analytic, then the complex-valued function $\lambda \mapsto \varphi(R_a(\lambda))$ is also analytic for φ being an arbitrary bounded linear functional on A. If $\mathrm{Sp}(a)$ is empty, then $\varphi(R_a(\lambda))$ is an analytic function on the entire $\mathbb{C}$. However, it is bounded because $|\varphi(R_a(\lambda))| \leqslant \|\varphi\| \cdot \|R_a(\lambda)\|$ and $\|R_a(\lambda)\|$ is bounded: we estimate separately on a compact disk of radius $\lambda_0 > \|a\|$ and outside it using the estimation obtained above

$$\|R_a(\lambda)\| \leq \frac{1}{|\lambda| - \|a\|} \leq \frac{1}{|\lambda_0| - \|a\|}.$$

This means $\varphi(R_a(\lambda)) = 0$ for any φ, so $R_a(\lambda) = 0$ (see problem 13 below) for all $\lambda \in \mathbb{C}$. This is impossible (it turns out that $\lambda_1 1 - \lambda_2 1 = 0$ for $\lambda_1 \neq \lambda_2$). $\square$

Problem 13. For an element $x \neq 0$ of a normed space X, there is a continuous linear functional φ that does not vanish on x. (This is a theorem from the standard course, a corollary of the Hahn-Banach theorem, see, for example, Corollary 2 on page 189 in [8]).

Problem 14. Let a and b be commuting elements of a Banach algebra. Then the product ab is invertible if and only if each of the elements a and b are invertible.

Theorem 1.15 (Spectral mapping theorem)**.** *If $p(z)$ is a polynomial, then* $\mathrm{Sp}(p(a)) = p(\mathrm{Sp}(a))$.

Proof. For $\alpha \in \mathbb{C}$ we decompose $p(z)-\alpha$ into linear factors $p(z)-\alpha = c\prod_i(z-\beta_i)$. Then $p(a)-\alpha 1 = c\prod_i(a-\beta_i 1)$ and $p(a)-\alpha 1$ are invertible if and only if all $a-\beta_i 1$ are (applying inductively Problem 14). Then $\alpha \in \mathrm{Sp}(p(a))$ if and only if at least one of β_i belongs to $\mathrm{Sp}(a)$. On the other hand, substituting this β_{i_0} into the above representation of p, we obtain $p(\beta_{i_0}) - \alpha = c\prod_i(\beta_{i_0} - \beta_i) = 0$, that is, $\alpha = p(\beta_{i_0})$. So $\mathrm{Sp}(p(a)) \subseteq p(\mathrm{Sp}(a))$. The inverse statement is similar. □

Lemma 1.16. *Let A be a C^*-algebra (unital). Then* $\mathrm{Sp}(a^*) = \overline{\mathrm{Sp}(a)}$. *If a is unitary, then* $\mathrm{Sp}(a) \subset \mathbb{S}^1$, *where $\mathbb{S}^1 \subset \mathbb{C}$ is the unit circle.*

Proof. Since $a^*(a^{-1})^* = (a^{-1}a)^* = 1^* = 1$, and similarly, in a different order, then the element is invertible if and only if we invert its conjugate. This gives the first statement.

Now let a be unitary, in particular, invertible. Then $\|a\|^2 = \|a^*a\| = 1$, so for $|\lambda| > 1$ we have $\lambda \notin \mathrm{Sp}(a)$. If $|\lambda| < 1$, then $1 - \lambda a^*$ is invertible, which means that $a(1-\lambda a^*) = a - \lambda 1$ is also invertible. Thus, $\mathrm{Sp}(a) \subset \mathbb{S}^1$. □

Definition 1.17. The number $r(a) = \sup\{|\lambda| \,:\, \lambda \in \mathrm{Sp}(a)\}$ is called the *spectral radius* of a.

Problem 15. Show that $r(a) \leq \|a\|$ for any $a \in A$.

Lemma 1.18. $r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$.

Proof. Let us expand the resolvent R_a in a neighborhood of infinity: $R_a(\lambda) = \sum_{n=0}^{\infty} \frac{a^n}{\lambda^{n+1}}$. This function is analytic for $|\lambda| > r(a)$, so for any $\rho > r(a)$, terms tends to zero: $\lim_{n\to\infty} \|\frac{a^n}{\rho^{n+1}}\| = 0$. That's why $\overline{\lim}_{n\to\infty}\|a^n\|^{1/n} \leq \rho$ for any $\rho > r(a)$, so

$$\overline{\lim}_{n\to\infty}\|a^n\|^{1/n} \leq r(a). \tag{1.4}$$

On the other hand, due to compactness of the spectrum, there is $\alpha \in \mathrm{Sp}(a)$ such that $|\alpha| = r(a)$. According to the spectral mapping theorem, $\alpha^n \in \mathrm{Sp}(a^n)$, therefore $|\alpha^n| \leq r(a^n)$. By Problem 15, $r(a^n) \leq \|a^n\|$. Combining, we get

$$r(a) = |\alpha| = (|\alpha^n|)^{1/n} \leq (r(a^n))^{1/n} \leq \|a^n\|^{1/n}$$

for all n, so

$$r(a) \leq \inf_n \|a^n\|^{1/n}. \tag{1.5}$$

Comparing (1.4) and (1.5), we conclude that $\lim_{n\to\infty} \|a^n\|^{1/n}$ exists and is equal to $r(a)$. □

1.4 Multiplicative functionals and maximal ideals of commutative Banach algebras

Recall that an algebra A is called *simple*, if it doesn't have proper (i.e., distinct from $\{0\}$ and A) ideals.

Lemma 1.19. *The algebra of complex numbers $\mathbb{C}$ is the only simple commutative algebra.*

Proof. If $A \neq \mathbb{C}$, then there is a non-scalar element $a \in A$ (that is, $a \neq \lambda 1$ for any λ). Let's take some $\alpha \in \mathrm{Sp}(a)$ and set $I = \overline{(a - \alpha 1)A}$, so that $I \neq \{0\}$ is closed (two-sided) ideal in A. For any $b \in A$, the element $(a - \alpha 1)b$ is not invertible (see Problem 14), so by Lemma 1.11 $\|(a - \alpha 1)b - 1\| \geq 1$. Therefore $1 \notin I$ and we arrive to a contradiction with the simplicity of A. □

Definition 1.20. A *multiplicative functional* on A is a nontrivial homomorphism $\varphi : A \to \mathbb{C}$. The set of all multiplicative functionals is denoted by M_A.

Problem 16. Prove that $\varphi(1) = 1$. You can use a simple general observation: image of an *idempotent* ($p^2 = p$) under homomorphism it is always idempotent.

Lemma 1.21. *A multiplicative functional on a commutative unital Banach algebra A has norm 1. The mapping that associates to each multiplicative functional its kernel is a bijection onto the set of* maximal ideals *of A, that is, such proper ideals that are not contained in any other ideal except the entire algebra A*

Proof. Since, according to Problem 16, $\varphi(1) = 1$, we have $\|\varphi\| \geq 1$. Let $\|\varphi\| > 1$, so there is an element $a \in A$ such that $\|a\| < 1$ and $\varphi(a) = 1$. Then the series $b = \sum_{n=1}^{\infty} a^n$ converges and $a + ab = b$. This means that $\varphi(b) = \varphi(a)(1 + \varphi(b)) = 1 + \varphi(b)$. A contradiction. We obtained $\|\varphi\| = 1$.

Since $\mathrm{Ker}\,\varphi$ has codimension 1, it is a maximal ideal.

Any functional φ is completely determined by its kernel and the condition $\varphi(1) = 1$ (see Problem 17 below), so the indicated correspondence is a bijection onto the image.

Let us show that it is an epimorphism. If $M \subset A$ is a maximal ideal, then $\mathrm{dist}(M, 1) = 1$, since the unit open ball with center 1 consists of invertible elements (Lemma 1.11), and M cannot contain invertible elements (if $a \in M$ is invertible, then $1 \in M$, and hence $M = A$). Then the closed ideal $\overline{M}$ (the closure of M) also does not contain 1, so by maximality $\overline{M} = M$. Consider the factor algebra A/M being simple (since otherwise M would not be maximal) commutative unital Banach algebra. Therefore, according to Lemma 1.19, $A/M \cong \mathbb{C}$. The corresponding factorization map is a non-zero homomorphism, that is, a multiplicative functional (with kernel M). □

Problem 17. Prove that any functional φ is completely determined by its kernel and the condition $\varphi(1) = 1$ (factor by kernel and consider the induced functional on $\mathbb{C}$).

Since multiplicative functionals are bounded by 1, M_A is a subset of the unit ball in the dual Banach space A' (the space of all bounded linear functionals on A). The space A' can be equipped with the $*$-*weak topology*, determined by the prebase of neighborhoods of the form $U_{\varphi_0,\varepsilon,a} = \{\varphi \in A' : |(\varphi - \varphi_0)(a)| < \varepsilon\}$, $\varphi_0 \in A'$, $a \in A$, $\varepsilon > 0$. In terms of directed nets: a directed net φ_α converges to φ if $\varphi_\alpha(a)$ converges to $\varphi(a)$ for each $a \in A$. The unit ball of space A' is compact and Hausdorff with respect to the $*$-weak topology (Banach-Alaoglu theorem, see [5, Theorem 5, p. 325]).

Problem 18. Verify that M_A is $*$-weakly closed.

Therefore M_A is also $*$-weakly compact.

Problem 19. The following example describes a typical situation, as it will become clear a little further. Let $A = C[0,1]$, so the multiplicative functionals correspond to points of $[0,1]$. Namely, $\varphi(g) = g(t)$. Show that for the "ordinary" norm of the dual space for any two points $t \neq s$, the distance between φ_t and φ_s is 1. That is the set of multiplicative functionals is discrete, non-compact. There is no limit points, unlike the situation with the $*$-weak topology.

If A does not have unit, consider the Banach algebra $A^+ = A \oplus \mathbb{C}$ (with the original norm (1.2), not the C^*-norm from Lemma 1.8). In it, A is itself a maximal ideal corresponding to the multiplicative functional $\varphi_0((a,\lambda)) = \lambda$. Any other maximal ideal I of the algebra A^+ must have a proper intersection with A. Then $I \cap A$ is an ideal of codimension 1, since I had codimension 1 in A^+. Thus, $I \cap A$ is a maximal ideal in A. Since its codimension is 1, then the quotient mapping is defined being a nonzero homomorphism to $\mathbb{C}$. Conversely, if $\varphi : A \to \mathbb{C}$ is a (nonzero) multiplicative functional on A, then the formula $\widetilde{\varphi}((a,\lambda)) = \varphi(a) + \lambda$ defines a unique extension of φ to a multiplicative functional on A^+ (Problem 21). We obtain a bijective correspondence between M_A and $M_{A^+} \setminus \{\varphi_0\}$.

Problem 20. Prove that M_A is a locally compact Hausdorff space, and M_{A^+} is its one-point compactification.

Problem 21. Check that if $\varphi : A \to \mathbb{C}$ is a (non-zero) multiplicative functional on A, then the formula $\widetilde{\varphi}((a,\lambda)) = \varphi(a) + \lambda$ defines a unique extension of φ to a multiplicative functional on A^+.

Lecture 3

1.5 Gelfand transform

Definition 1.22. For a commutative Banach algebra A we define *Gelfand transform* $\Gamma : A \to C_0(M_A)$ by the relation $\Gamma(a) = \hat{a}$, where $\hat{a}(\varphi) := \varphi(a)$ (the necessary properties will be verified in the next lemma).

Lemma 1.23. *The Gelfand transform is a (non-strict) contraction homomorphism of algebras and the image A separates the points M_A, that is, for any two points M_A there is a function from the image with distinct values at these points.*

Proof. Functions $\hat{a}$ are continuous by the definition of the $*$-weak topology. The mapping is contractive because

$$\|\Gamma(a)\| = \sup_{\varphi\in M_A} |\varphi(a)| \le \sup_{\varphi\in M_A} \|\varphi\| \cdot \|a\| = \sup_{\varphi\in M_A} \|a\| = \|a\|.$$

The separation of points is obvious since two (multiplicative) functionals are distinct if and only if their values on some element a are distinct. If A is not unital, then we note that $\hat{a}(\tilde{0}) = 0(a) = 0$, where $M_{A^+} = M_A \cup \tilde{0}$. Therefore $\hat{a} \in C_0(M_A)$. □

Problem 22. If an algebra is unital, then $\Gamma(1) = 1$.

Corollary 1.24. *Let A be a commutative unital Banach algebra. Then $a \in A$ is invertible if and only if $\hat{a}$ is invertible, and if and only if $\hat{a}(\varphi) \neq 0$ for any $\varphi \in M_A$. Therefore $\mathrm{Sp}(a) = \mathrm{Sp}(\hat{a}) = \{\varphi(a) : \varphi \in M_A\}$ and $\|\hat{a}\| = r(a)$.*

Proof. If a is invertible, then $\hat{a}$ is invertible by Problem 22. If a is not invertible, then consider the ideal $I = \overline{aA}$. As discussed above (see the proof of Lemma 1.19), this ideal cannot contain 1, so it is proper. Let I_M be the maximal ideal containing I (see problem 23 below), and φ be the multiplicative functional corresponding to I_M . Then $\varphi(a) = 0$ and $\hat{a}$ is not invertible in $C(M_A)$.

The remaining statements are immediately obtained from what has been proven. □

Problem 23. Any ideal I of a commutative unital Banach algebra is contained in some maximal ideal. Hint: consider the union of J of all proper ideals I_α containing I partially ordered by inclusion. For each chain (a completely ordered subsystem) I_{α_τ} using, as above, by the fact that 1 does not belong to every I_{α_τ}, make sure that it does not belong to $\cup_\tau I_{\alpha_\tau}$, so this is its proper ideal. Then apply Zorn's lemma.

Theorem 1.25. *Let A be a commutative C^*-algebra. Then the Gelfand transformation is an isometric $*$-isomorphism of A onto $C_0(M_A)$.*

Proof. Let us prove the theorem for a unital algebra. The necessary adaptation for the case without a unit is left to the reader as Problem 24.

Let $\varphi \in M_A$. Let us first consider a self-adjoint element $a^* = a \in A$. Let us set $u_t = \sum_{n=0}^{\infty} \frac{(ita)^n}{n!}$, $t \in \mathbb{R}$. It is easy to check by considering partial sums and passing to the limit that $u_t^* = u_t^{-1}$, so $u_t \in A$ is unitary. Then

$$1 \geq |\varphi(u_t)| = \left| \sum_{n=0}^{\infty} \frac{(it\varphi(a))^n}{n!} \right| = |e^{it\varphi(a)}| = e^{-t \operatorname{Im} \varphi(a)}.$$

Due to the arbitrariness of $t \in \mathbb{R}$ in this estimate, we conclude that $\operatorname{Im} \varphi(a) = 0$, that is, $\varphi(a) \in \mathbb{R}$.

We write an arbitrary element $c \in A$ in the form $c = a + ib$, where $a = (c + c^*)/2$ and $b = (c - c^*)/2i$ are self-adjoint. According to what has been proven, $\varphi(a), \varphi(b) \in \mathbb{R}$, that means $\varphi(c^*) = \varphi(a) - i\varphi(b) = \overline{\varphi(c)}$, so the Gelfand transformation preserves involution and is thus a $*$-homomorphism.

For a self-adjoint element ($a = a^*$) we have $\|a\|^2 = \|a^*a\| = \|a^2\|$, therefore

$$\|\hat{a}\| = r(a) = \lim_{n\to\infty} \|a^{2^n}\|^{1/2^n} = \lim_{n\to\infty} (\|a\|^{2^n})^{1/2^n} = \|a\|.$$

For an element $b \in A$ of general form we have $\|b\|^2 = \|b^*b\| = \|\widehat{b^*b}\| = \|\hat{b}^*\hat{b}\| = \|\hat{b}\|^2$, so the Gelfand transformation is an isometry (onto its image).

Therefore $\Gamma(A)$ is norm closed and involutive subalgebra with identity in $C(M_A)$ separating the points. By theorem Stone-Weierstrass[1], $\Gamma(A) = C(M_A)$. $\square$

Problem 24. Prove the theorem for a non-unital algebra.

So in particular there is an inverse mapping for the Gelfand transform, which is also an isometric $*$-isomorphism.

Let $a \in A$ be a normal element. Let us denote by $C^*(1, a)$ (respectively, $C^*(a)$) C^*-algebra generated by 1 and a (resp., only a). Due to normality, these algebras are commutative, with the first definition supposing that A is unital.

Let us clarify that by a *C^*-algebra generated by a set* we call the minimal C^*-subalgebra of A containing the set, that is, the intersection all C^*-subalgebras of A containing this set.

Problem 25. Verify that the C^*-algebra generated by a set is indeed a C^*-algebra.

Problem 26. If a is an invertible element, then the algebra $C^*(a)$ is unital. In this case $C^*(a) = C^*(1, a)$.

Corollary 1.26. *If $a^* = a$, then* $\operatorname{Sp}(a) \subset \mathbb{R}$.

Proof. As we know, $\operatorname{Sp}(a) = \operatorname{Sp}(\widehat{a})$ and $\widehat{a}^* = \widehat{a}$. But a self-adjoint function is exactly a function with real values, while the spectrum of a function is the set of all its values. $\square$

Corollary 1.27. *The algebra $C^*(1, a)$ is isometrically $*$-isomorphic to the algebra $C(\operatorname{Sp}(a))$ under a mapping that takes a to the function $z(t) = t$, $z : \operatorname{Sp}(a) \subset \mathbb{C} \to \mathbb{C}$. The algebra $C^*(a)$ is mapped onto $C_0(\operatorname{Sp}(a) \setminus \{0\})$.*

[1]The theorem is not always included in the standard course on functional analysis, so we present its proof in Section 1.6.

Proof. For a commutative C^*-algebra $C^*(1,a)$ we find $X = M_{C^*(1,a)}$. Any multiplicative functional $\varphi \in X$ is determined by its value $\varphi(a) = \lambda$ on a. Moreover, due to multiplicativity $\varphi(p(a,a^*)) = p(\lambda,\bar{\lambda})$ for any polynomial p. Thus, X is identified with the set of all possible values λ that $\varphi(a) = \hat{a}(\varphi)$ takes for $\varphi \in X$. According to Corollary 1.24, we have $\hat{a}(X) = \mathrm{Sp}(a)$. We obtain the identification $\mathrm{Sp}(a) \cong X$ using the correspondence $\mathrm{Sp}(a) \ni \lambda \mapsto \varphi_\lambda \in X$, where φ_λ is determined by the condition $\varphi_\lambda(a) = \lambda$.

This identification carries over to functions: every continuous function on X is identified with a continuous function on $\mathrm{Sp}(a)$, namely, the function $\hat{b} = \hat{b}(\varphi)$ is associated with the function argument $\lambda \in \mathrm{Sp}(a)$, specified as $\lambda \mapsto \varphi_\lambda(b)$. For example, if we take the polynomial $p(a,a^*) = b$, then the corresponding function will be $\lambda \mapsto \varphi_\lambda(p(a,a^*)) = p(\lambda,\bar{\lambda})$. In particular, the function $\hat{a}$ is mapped to $\lambda \mapsto \varphi_\lambda(a) = \lambda$, so the Gelfand transform identifies $\hat{a}$ with the identity mapping of $X \subset \mathbb{C}$. By Theorem 1.25, this mapping is an isometric $*$-isomorphism.

If a is invertible, then by Problem 26, $C^*(a)$ isometrically $*$-isomorphic to $C(\mathrm{Sp}(a))$. If a is not is invertible, then $C^*(a)$ does not have unity (see Problem 27). It corresponds under the constructed mapping for $C^*(1,a) \cong C^*(a)^+$ to the ideal $C(\mathrm{Sp}'(a))$ consisting of functions, tending to 0. □

Problem 27. Prove that if a is not invertible, then $C^*(a)$ does not have a unit. Hint: if there is a unit, then it has to be approximated by a polynomial in a and a^*, which cannot be an invertible element.

Corollary 1.28 (continuous functional calculus)**.** *Let a be a normal element of a unital C^*-algebra A, and f is a continuous function on* $\mathrm{Sp}(a)$*. Then the element $f(a) \in A$ is defined as the inverse image of f under the Gelfand transformation: $\Gamma = \Gamma_a : C^*(1,a) \to C(\mathrm{Sp}(a))$, $f(a) := (\Gamma_a)^{-1}(f)$. If $0 \in \mathrm{Sp}(a)$ and $f(0) = 0$, then $f(a) \in C^*(a)$. Moreover, $f(\mathrm{Sp}(a)) = \mathrm{Sp}(f(a))$ and if g is a continuous function on $f(\mathrm{Sp}(a))$, then $g(f(a)) = (g \circ f)(a)$.*

Proof. Everything has already been proven except the last statement. Let's first consider the polynomial $p(\lambda,\bar{\lambda})$ as $f = f(\lambda)$. Then $\Gamma(p(a,a^*))$ is a function $\lambda \mapsto p(\lambda,\bar{\lambda})$, so $\mathrm{Sp}(p(a,a^*))$ coincides with the set of values of this function, $\{\mu : \mu = p(\lambda,\bar{\lambda}), \lambda \in \mathrm{Sp}(a)\}$. Approximating f by polynomials, we get $f(\mathrm{Sp}(a)) = \mathrm{Sp}(f(a))$ (Problem 28).

Similarly, consider a polynomial $q(\lambda,\bar{\lambda})$ as g. It's easy to see that

$$q(f(a)) = \lim_\alpha (q(p_\alpha(a)) = \lim_\alpha (q \circ p_\alpha)(a) = (q \circ f)(a).$$

Now we approximate g by polynomials and use the isometricity of the inverse Gelfand transform. □

Problem 28. Prove that $f(\mathrm{Sp}(a)) = \mathrm{Sp}(f(a))$ in the proof above, approximating f by polynomials, and correctly stating what it means that the image is continuous under a uniform approximation, and using the isometricity of the Gelfand transform.

Corollary 1.29. *If a is a normal element, then $\|a\| = r(a)$.*

Proof. $\|a\| = \|\hat{a}\| = \sup_{\varphi \in M_A} |\hat{a}(\varphi)| = \sup_{\lambda \in \mathrm{Sp}(a)} |\lambda| = r(a)$. □

1.6 Addition: Stone-Weierstrass theorem

Let us first consider the algebra $C_{\mathbb{R}}(X)$ over $\mathbb{R}$ formed by all continuous real-valued functions on a compact Hausdorff space X.

Theorem 1.30. *Let $A \subseteq C_{\mathbb{R}}(X)$, where X is a compact Hausdorff space, is a closed subalgebra*[2] *such that A separates the points X and contains $1 \in C_{\mathbb{R}}(X)$ (and hence all constant functions). Then $A = C_{\mathbb{R}}(X)$.*

Proof. First of all, we note that the condition for separating points can be strengthened, namely: for any x and y from X and any u and v from $\mathbb{R}$ there is a function $g \in A$ such that $g(x) = u$ and $g(y) = v$. Indeed, since there is $f \in A$ with the property $u' = f(x) \neq f(y) = v'$, then g can be taken equal

$$g = \frac{u - v}{u' - v'} \cdot f + \frac{u'v - v'u}{u' - v'} \cdot 1.$$

For $f, g \in A$ we define continuous functions $f \vee g$, $f \wedge g$, $\gamma(g)$ as

$$(f \vee g)(s) = \max\{f(s), g(s)\}, \qquad (f \wedge g)(s) = \min\{f(s), g(s)\}, \qquad \gamma(g)(s) = |g(s)|.$$

According to Weierstrass's theorem on the approximation of continuous functions by polynomials, there is a sequence of polynomials p_n such that

$$|\,|\lambda| - p_n(\lambda)| \leqslant \frac{1}{n} \qquad \text{with } -n \leqslant \lambda \leqslant n.$$

Then

$$|\,|g(s)| - p_n(g)(s)| = |\,|g(s)| - p_n(g(s))| \leqslant \frac{1}{n} \qquad \text{for } -n \leqslant g(s) \leqslant n.$$

So $\gamma(g) \in A$. Therefore, $f \vee g \in A$, $f \wedge g \in A$, since

$$f \vee g = \frac{f + g}{2} + \frac{\gamma(f - g)}{2}, \qquad f \wedge g = \frac{f + g}{2} - \frac{\gamma(f - g)}{2}.$$

Let us now consider an arbitrary $F \in C_{\mathbb{R}}(X)$ and, by the remark from the beginning of the proof, find for arbitrary $x, y \in X$ a function $f_{x,y} \in A$ such that $f_{x,y}(x) = F(x)$ and $f_{x,y}(y) = F(y)$. Having temporarily fixed y, we find for each $x \in X$ a neighborhood U_x such that $f_{x,y}(u) > F(u) - \varepsilon$ for $u \in U_x$. Let us choose a finite subcover $U_{x_1}, \dots U_{x_p}$ and define $f_y = f_{x_1,y} \vee \dots \vee f_{x_p,y}$. Then $f_y(u) > F(u) - \varepsilon$ for any $u \in X$. Since $f_{x_i,y}(y) = F(y)$ for any $i = 1, \dots, p$, then $f_y(y) = F(y)$. This means that there is a neighborhood V_y of a point y such that $f_y(u) < F(u) + \varepsilon$ for $u \in V_y$. Let's choose a finite subcover $V_{y_1}, \dots, V_{y_q}$ and define $f := f_{y_1} \wedge \dots \wedge f_{y_q}$. Since every $f_{y_i}(u) > F(u) - \varepsilon$ for any $u \in X$, then $f(u) > F(u) - \varepsilon$ for any $u \in X$. On the other hand, for any $u \in X$ there is $V_{y_i} \ni u$, so $f(u) < f_{y_i}(u) < F(u) + \varepsilon$. Combining the inequalities, we obtain that $|f(u) - F(u)| < \varepsilon$ for any $u \in X$. Due to arbitrariness ε we obtain the required result. $\square$

[2] this condition can be weakened

Theorem 1.31. *Let $A \subseteq C(X)$ (where X is a compact Hausdorff space) be a closed involutive subalgebra such that A separates the points X and contains $1 \in C(X)$ (and hence all constant functions). Then $A = C(X)$.*

Proof. The involution has the form $f^*(x) = \overline{f(x)}$. Let A_R consist of real-valued functions belonging to A. Note that this is a unital subalgebra of the algebra $C_{\mathbb{R}}(X)$. Since A_R coincides with the kernel of a continuous $\mathbb{R}$-linear mapping $f \mapsto f - f^*$, then it is closed in A, and hence in $C(X)$. Therefore $A_R = A \cap C_{\mathbb{R}}(X)$ is closed in $C_{\mathbb{R}}(X)$. Finally, A_R separates the points X. Indeed, if $f(x) \neq f(y)$, where $f \in A$, then $f = f_1 + if_2$ for $f_1 = (f + f^*)/2 \in A_R$, $f_1 = (f - f^*)/2i \in A_R$, so at least one of f_1, f_2 separates x and y.

Therefore, by the previous theorem, $C_{\mathbb{R}}(X) = A_R \subset A$. Using the representation $f = f_1 + if_2$ again, but for the entire $C(X)$, we see that $\mathbb{C}$-linear combinations of elements of $C_{\mathbb{R}}(X)$ give $C(X)$ and, at the same time, give A by virtue of what has been proven. So $A = C(X)$. $\square$

Lecture 4

1.7 Positive elements

Definition 1.32. A self-adjoint element a in a unital C^*-algebra A is called *positive,* if $\mathrm{Sp}(a) \subset [0, \infty)$. If A is not unital, then a is called *positive,* if it is positive in A^+.

Positivity is written as $a \geqslant 0$. For two self-adjoint elements $a, b \in A$ we say that $a \geqslant b$ if $a - b \geqslant 0$.

Problem 29. Show that if $a \geqslant 0$ and $0 \geqslant a$, then $a = 0$; and also that $-\|a\|1 \leqslant a \leqslant \|a\|1$ for every self-adjoint a.

Now let's look at applications of continuous functional calculus to positivity.

Corollary 1.33. *Let $a \in A$ be a positive element. Then there exists a unique positive square root b of a, that is, $b \geqslant 0$ such that $b^2 = a$.*

Proof. The function $f(z) = \sqrt{z}$ is defined and continuous on $[0, \infty)$, so $b = f(a)$ is defined. It is self-adjoint and even positive (since f maps $[0, \infty)$ to itself) and $b^2 = f(a)^2 = a$ (by corollary 1.28). If c is another positive square root of a, then $c = f(c^2) = f(a) = b$. $\square$

Corollary 1.34. *Let $a \in A$ be a self-adjoint element. Then there are positive elements $a_+, a_- \in A$, such that $a = a_+ - a_-$ and $a_+ a_- = 0$.*

Proof. Let us define a continuous function $f : \mathbb{R} \to [0, +\infty)$, putting $f(x) = x$ for $x \geq 0$ and $f(x) = 0$ for $x < 0$. Let's denote $g(x) = f(-x)$. These functions satisfy $f(x) - g(x) = x$ and $f(x)g(x) = 0$. It remains to put $a_+ = f(a)$, $a_- = g(a)$. $\square$

Corollary 1.35. *For a self-adjoint element $a \in A$ the following conditions are equivalent:*

(i) $a \geqslant 0$;

(ii) $a = b^2$ *for some self-adjoint* b;

(iii) $\|\mu 1 - a\| \leqslant \mu$ *for every* $\mu \geqslant \|a\|$;

(iv) $\|\mu 1 - a\| \leqslant \mu$ *for some* $\mu \geqslant \|a\|$.

Proof. By Corollary 1.33, from (i) it follows (ii). Moreover, (iii) implies (iv) by evident reasons.

Let us show that (ii) implies (iii). By assumption, $a = f(b)$, where $f(x) = x^2$. Moreover, the norm of f on $\mathrm{Sp}(b)$ is equal to $\|a\|$, so $0 \leqslant \mu - x^2 \leq \mu$ for any $\mu \geqslant \|a\|$ and $x \in \mathrm{Sp}(b)$. Since $\mu 1 - a = (\mu - f)(b)$, then $\|(\mu - f)(b)\|$ equals the norm of $\mu - f$ on $\mathrm{Sp}(b)$, which does not exceed μ.

Let us finally show that (iv) implies (i). Since, for some $\mu \geqslant \|a\|$, we have $\|\mu - x\| = \sup_{x \in \mathrm{Sp}(a)} |\mu - x| \leqslant \mu$, then no $x \in \mathrm{Sp}(a)$ can be negative. $\square$

Corollary 1.36. *If a and b are positive, then $a+b$ is positive.*

Proof. Let us choose some $\mu \geqslant \|a\|$, $\nu \geqslant \|b\|$. Then $\|a+b\| \leqslant \mu+\nu$ and the item (iii) of Corollary 1.35 implies

$$\|(\mu+\nu)1-(a+b)\| = \|(\mu 1-a)+(\nu 1-b)\| \leqslant \|\mu 1-a\|+\|\nu 1-b\| \leqslant \mu+\nu.$$

Therefore, by item (iv) of Corollary 1.35, $a+b$ is positive. □

Problem 30. If $0 \leqslant a \leqslant b$, then $\|a\| \leqslant \|b\|$. *Hint:* As we know, $b \leqslant \|b\| \cdot 1_A$. Thus $a \leqslant \|b\| \cdot 1_A$, i.e. $\mathrm{Sp}(\|b\| \cdot 1_A - a) \subset [0,+\infty)]$. From $\mathrm{Sp}(\mu 1_A - a) = \mu - \mathrm{Sp}(a)$ deduce that $\|a\| = \max\{\lambda \in \mathrm{Sp}(a)\}$ equals $\min\{\mu\colon \mathrm{Sp}(\mu \cdot 1_A - a) \subset [0,+\infty)\}$. This implies the statement.

The following Proposition 1.38 is almost obvious for operators in a Hilbert space, but is very nontrivial for elements of a C^*-algebra. We will need the following result about spectra of products.

Lemma 1.37. *If $a,b \in A$, then $\mathrm{Sp}(ab)\cup\{0\} = \mathrm{Sp}(ba)\cup\{0\}$.*

Proof. Let $0 \neq \lambda \notin \mathrm{Sp}(ab)$. This means that $(ab-\lambda 1) = -\lambda(1-\lambda^{-1}ab)$ is invertible, so there exists an element $u \in A$ such that $(1-\lambda^{-1}ab)u = 1$. Let us set $v = 1+\lambda^{-1}bua$. Then

$$(1-\lambda^{-1}ba)v = (1-\lambda^{-1}ba)(1+\lambda^{-1}bua) = 1-\lambda^{-1}ba+\lambda^{-1}bua-\lambda^{-2}babua =$$

$$= 1-\lambda^{-1}ba+\lambda^{-1}b(1-\lambda^{-1}ab)ua = 1-\lambda^{-1}ba-\lambda^{-1}ba = 1,$$

so $\lambda \notin \mathrm{Sp}(ba)$. □

Proposition 1.38. *The element a^*a is positive for every $a \in A$.*

Proof. Since a^*a is self-adjoint, we can write $a^*a = b_+ - b_-$ by Corollary 1.34. Let $c := \sqrt{b_-}$, $t := ac$. Note that $f(0) = 0$ for $f(x) = \sqrt{x}$, so c is approximated by polynomials in b_- without a free term, this means that $cb_+ = 0$. We have:

$$-t^*t = -c(b_+ - b_-)c = b_-^2. \tag{1.6}$$

Therefore, $-t^*t$ is positive.

Let us write t in the form $t = x+iy$, where x and y are self-adjoint elements of A (that is $x = (t+t^*)/2$, $y = (t-t^*)/2i$). Then $t^*t+tt^* = 2(x^2+y^2)$ is positive by Corollary 1.36. By the same Corollary, we see that the element

$$tt^* = (t^*t+tt^*) - t^*t = (t^*t+tt^*) + b_-^2$$

is also positive, that is, $\mathrm{Sp}(tt^*) \subset [0,\infty)$. By Lemma 1.37, $\mathrm{Sp}(t^*t) \subset [0,\infty)$, so t^*t is positive. But it is also negative by (1.6), so $t^*t = 0$ for problem 29. This means that $b_- = 0$, since positive square root is unique. □

Corollary 1.39. *If $b \leqslant c$, then $a^*ba \leqslant a^*ca$ for any $a \in A$.*

Proof. Since $c - b$ is positive, then $c - b = d^2$ for some self-adjoint d. That is why $a^*(c - b)a = a^*d^2a = (da)^*(da) \geqslant 0$. □

Corollary 1.40. *If a and b are invertible and $0 \leqslant a \leqslant b$, then $b^{-1} \leqslant a^{-1}$.*

Proof. Let us first consider the special case $b = 1$. Then $\mathrm{Sp}(a) \subseteq [0, 1]$. According to the spectral mapping theorem, we see that $\mathrm{Sp}(a^{-1}) \subseteq [1, \infty)$, so $a^{-1} \geqslant 1$. Let's pass to the general case. Since, by Corollary 1.39, $b^{-1/2}ab^{-1/2} \leqslant 1$, then $b^{1/2}a^{-1}b^{1/2} \geqslant 1$ by the first part of the proof. Multiplying this inequality by $b^{-1/2}$ on both sides and again applying corollary of 1.39, we obtain $a^{-1} \geqslant b^{-1}$. □

1.8 Approximate identity

Let Λ be a directed set. A family $(u_\lambda)_{\lambda\in\Lambda}$ elements of the C^*-algebra A is called an *approximate identity (unit),* if $\lim_\Lambda \|xu_\lambda - x\| = 0$ for $x \in A$ (and therefore $\lim_\Lambda \|u_\lambda x - x\| = 0$). If A is unital, then we can take $u_\lambda = 1$ for any λ, so this concept is of interest only for non-unital algebras. In the definition of an approximate unit also include the conditions $0 \leqslant u_\lambda \leqslant 1$ and $u_\lambda \leqslant u_\mu$ for all $\lambda \leqslant \mu$ from Λ.

Theorem 1.41. *Every C^*-algebra has an approximate unit.*

Proof. Let $\Lambda = \{a \in A \mid a \geqslant 0; \|a\| < 1\}$. An order on Λ is given by $\leqslant$. Let us show that the set of elements Λ, indexed tautologically (a has index a), is an approximate unit.

First of all, we need to check that Λ is a directed set, that is, for any two elements $a, b \in \Lambda$ there is a $c \in \Lambda$ such that $a \leqslant c$ and $b \leqslant c$. Let $f(t) := \frac{t}{1-t}$ and $g(t) := \frac{t}{1+t}$. Moreover, the function f is defined on $[0, 1)$, the function g is defined on $[0, \infty)$ and $g(f(t)) = t$. Let us put $x := f(a)$, $y := f(a) + f(b)$, $c := g(y)$. Since $0 \leqslant g(t) < 1$, then $c \in \Lambda$. The inequality $x \leqslant y$ implies $1 + x \leqslant 1 + y$, so $(1 + x)^{-1} \geqslant (1 + y)^{-1}$ and

$$a = 1 - (1 + x)^{-1} \leqslant 1 - (1 + y)^{-1} = c.$$

The inequality $b \leqslant c$ can be obtained similarly, so we have verified that the set Λ is directed.

Now let's check that $\lim_\Lambda \|x - ax\| = 0$ for every $x \in A$. Since by Corollary 1.34 each element can be decomposed into a linear combination of four positive elements:

$$x = \frac{x + x^*}{2} + i\frac{x - x^*}{2i} = \left(\frac{x + x^*}{2}\right)_+ - \left(\frac{x + x^*}{2}\right)_- + i\left(\frac{x - x^*}{2i}\right)_+ - i\left(\frac{x - x^*}{2i}\right)_-, \quad (1.7)$$

then it is sufficient to verify the statement for $x \geqslant 0$. Since for $a \in \Lambda$ we have $0 \leqslant 1 - a \leqslant 1$, then by Corollary 1.39, $(1 - a)^{1/2}(1 - a)(1 - a)^{1/2} \leqslant (1 - a)^{1/2}(1 - a)^{1/2} = 1 - a$. That's why (see Problem 30)

$$\|(1 - a)x\|^2 = \|x^*(1 - a)^2x\| \leqslant \|x^*(1 - a)x\|,$$

and it is sufficient to verify that $\lim_\Lambda \|x(1 - a)x\| = 0$ for any $x \geqslant 0$ from A, and without loss of generality we can assume that $\|x\| = 1$.

Similar to the previous reasoning, if $a, b \in \Lambda$ and $a \leqslant b$ then $\|x^*(1-b)x\| \leq \|x^*(1-a)x\|$, so, for $x \geqslant 0$,

$$\sup_{b\in\Lambda,\, b\geqslant a} \|x(1-b)x\| = \|x(1-a)x\|.$$

Therefore we need to show that, for any positive $x \in A$ of norm one and for any $\varepsilon > 0$, there is an element $a \in \Lambda$ such that $\|x^*(1-a)x\| < \varepsilon$. Let us put $a_n := g(nx)$, $n \in \mathbb{N}$ (see the beginning of the proof). Then $\|x(1-a_n)x\| = \|h(x)\|$, where $h(t) := t^2(1-g(nt)) = \frac{t^2}{1+nt}$. For any $t \in [0,1]$, we have $0 \leqslant h(t) \leqslant \frac{1}{n}$, so $\|h(x)\| \leqslant \frac{1}{n}$. Hence,

$$\sup_{b\in\Lambda,\, b\geqslant a_n} \|x(1-b)x\| = \|x(1-a_n)x\| \leqslant \frac{1}{n}$$

for any n, so $\lim_{b\in\Lambda} \|x(1-b)x\| = 0$. □

Definition 1.42. An approximate unit is called *countable,* if the set Λ is countable.

Corollary 1.43. *A separable C^*-algebra has a countable approximate unit.*

Proof. Let us choose a dense sequence $(x_n)_{n\in\mathbb{N}}$ in A. Then there is an element $a_1 \in A$, $0 \leqslant a_1$, $\|a_1\| < 1$ such that $\|x_1a_1 - x_1\| \leqslant 1$ (see the proof of the previous theorem). Suppose by induction that we have already found such $a_2, \dots, a_n$, $a_1 \leqslant a_2 \leqslant \dots \leqslant a_n$, such that, for all $k = 1, 2, \dots, n$, the inequalities $\|a_k\| < 1$ and $\|x_ia_k - x_i\| \leqslant \frac{1}{k}$ are satisfied for $i = 1, 2, \dots, k$. Now let's find an element $a_{n+1} \geqslant a_n$ with norm $\|a_{n+1}\| < 1$, such that $\|x_ia_{n+1} - x_i\| \leqslant \frac{1}{n+1}$ for $i = 1, 2, \dots, n+1$. Because the sequence (x_n) is dense in A, then by induction we obtain a non-decreasing sequence $(a_n)_{n\in\mathbb{N}}$ positive elements of the unit ball with $\lim_{n\to\infty} \|xa_n - x\| = 0$ for each $x \in A$. Indeed, for any $\varepsilon > 0$ we can find an element x_i such that $\|x - x_i\| < \varepsilon/3$, and for x_i we can find a number j such that $\|x_ia_k - x_i\| < \varepsilon/3$ for all $k > j$. Then for these k

$$\|xa_k - x\| = \|x_ia_k - x_i + (x - x_i)a_k + x - x_i\| < \varepsilon/3 + \varepsilon/3\|a_k\| + \varepsilon/3 \leqslant \varepsilon.$$

□

Problem 31. Prove that the converse is not true: an algebra with countable approximate unit does not have to be separable.

Lecture 5

1.9 Ideals, factors and homomorphisms

Under an *ideal of C^*-algebra* we will always mean norm-closed two-sided ideal (for maximal ideals in in the commutative case this happens automatically).

Lemma 1.44. *Every ideal I in a C^*-algebra is self-adjoint: $I = I^*$.*

Proof. If $I \subset A$ is an ideal, then $B := I \cap I^* \subseteq A$ is a C^*-subalgebra. In this case, $B \supset I \cdot I^*$. Let (u_λ) is an approximate unit in B, and $j \in I$. Then

$$\lim_{\lambda\in\Lambda} \|j^* u_\lambda - j^*\|^2 = \lim_{\lambda\in\Lambda} \|u_\lambda(jj^* u_\lambda - jj^*) - (jj^* u_\lambda - jj^*)\| \leqslant 2 \lim_{\lambda\in\Lambda} \|jj^* u_\lambda - jj^*\| = 0.$$

Since $u_\lambda \in I$, then $j^* u_\lambda \in I$, so $j^* \in I$, since I is closed. $\square$

The following technical lemma is often used.

Lemma 1.45. *If $x^*x \leqslant a$ is in A, then there is an element $b \in A$ such that $\|b\| \leqslant \|a\|^{1/4}$ and $x = ba^{1/4}$.*

Proof. Let us put $b_n := x(a + \frac{1}{n}1)^{-1/2} a^{1/4}$ (this element lies in A, even if A does not have a unit, but in this case it is convenient for us to carry out calculations in A^+). Let also

$$d_{nm} := \left(a + \frac{1}{n}1\right)^{-1/2} - \left(a + \frac{1}{m}1\right)^{-1/2}, \qquad f_n(t) := t^{3/4}\left(t + \frac{1}{n}\right)^{-1/2}.$$

Then the sequence of functions $\{f_n(t)\}$ converges to $f(t) := t^{1/4}$ uniformly on $[0, \|a\|]$, since due to $u^2 + v^2 \geqslant 2uv$ we have

$$\left(t^{1/4}\left(1 - \frac{t^{1/2}}{(t+1/n)^{1/2}}\right)\right)^2 = t^{1/2} \frac{t + 1/n + t - 2t^{1/2}(t+1/n)^{1/2}}{t + 1/n} <$$

$$< t^{1/2} \frac{t + 1/n + t - 2t}{t + 1/n} = t^{1/2} \frac{1/n}{t + 1/n} = \frac{2\sqrt{t/n}}{t + 1/n} \cdot \frac{1}{2\sqrt{n}} \leqslant \frac{1}{2\sqrt{n}}.$$

We have

$$\begin{aligned} \|b_n - b_m\|^2 &= \|x d_{nm} a^{1/4}\|^2 = \|a^{1/4} d_{nm} x^* x d_{nm} a^{1/4}\| \leqslant \|a^{1/4} d_{nm} a d_{nm} a^{1/4}\| = \\ &= \|d_{nm} a^{3/4}\|^2 = \|f_n(a) - f_m(a)\|^2 = \sup_{t\in[0,\|a\|]} |f_n(t) - f_m(t)|. \end{aligned}$$

Thus, since f_n is a Cauchy sequence, so is b_n. Let us put $b := \lim_{n\to\infty} b_n$. Then $ba^{1/4} = \lim_{n\to\infty} b_n a^{1/4} = \lim_{n\to\infty} x(a + \frac{1}{n}1)^{-1/2} a^{1/2} = x$. $\square$

Problem 32. Verify that the last limit is indeed x. This can be done in a similar way to the calculation for f_n in the proof, using $x^*x \leqslant a$.

Definition 1.46. A subalgebra $B \subset A$ is called *hereditary* if for any positive $b \in B$ and $a \in A$, from the condition $0 \leqslant a \leqslant b$ it follows that $a \in B$.

Problem 33. Prove that a positive element of an arbitrary C^*-subalgebra is a positive element of the entire algebra.

Lemma 1.47. *Let $I \subset A$ be an ideal and $j \in I$ a positive element. If $a^*a \leqslant j$, then $a \in I$. In particular, any ideal is a hereditary subalgebra.*

Proof. Let us represent $a = bj^{1/4}$ in accordance with Lemma 1.45. Moreover, $j^{1/4} \in C^*(j) \subset I$, and therefore $a \in I$. □

If $I \subset A$ is an ideal, then we can define Banach factor algebra A/I with norm $\|a+I\| := \inf_{j \in I} \|a + j\|$. This is an involutive algebra: since I is self-adjoint, then $\|(a + I)^*\| = \|a^* + I\| = \|a + I\|$. To be short we will denote $a + I$ by $\dot{a} \in A/I$.

Theorem 1.48. *The involutive algebra A/I is a C^*-algebra.*

Proof. Only the C^* property needs to be verified. Let $(u_\lambda)_{\lambda \in \Lambda}$ be an approximate unit of I (note that ideals typically do not have a unit, and in any case, a proper ideal does not contain the unit of A, even if the latter exists). Let us first show that

$$\|\dot{a}\| = \lim_{\lambda \in \Lambda} \|a - au_\lambda\|. \tag{1.8}$$

Indeed, since $u_\lambda \in I$, then $\|\dot{a}\| \leq \|a - au_\lambda\|$. To prove the reverse inequality, we choose arbitrarily $\varepsilon > 0$. Then there is an element $j \in I$ such that $\|\dot{a}\| \geqslant \|a - j\| - \varepsilon$. We have

$$\lim_{\lambda \in \Lambda} \|a - au_\lambda\| \leqslant \lim_{\lambda \in \Lambda} (\|a - au_\lambda - (j - ju_\lambda)\| + \|j - ju_\lambda\|) = \lim_{\lambda \in \Lambda} \|a - au_\lambda - (j - ju_\lambda)\|.$$

Writing in A^+, where the equality $a - au_\lambda - (j - ju_\lambda) = (a - j)(1 - u_\lambda)$ holds, we obtain the estimation $\|(a - j)(1 - u_\lambda)\| \leqslant \|a - j\| < \|\dot{a}\| + \varepsilon$. Due to arbitrariness of $\varepsilon > 0$ we obtain (1.8).

Now, calculating in A^+, we find the estimation

$$\begin{aligned} \|\dot{a}^*\dot{a}\| &= \lim_{\lambda \in \Lambda} \|a^*a(1 - u_\lambda)\| \geqslant \lim_{\lambda \in \Lambda} \|(1 - u_\lambda)a^*a(1 - u_\lambda)\| = \\ &= \lim_{\lambda \in \Lambda} \|(a(1 - u_\lambda)\|^2 = \|\dot{a}\|^2. \end{aligned}$$

The inverse inequality $\|\dot{a}^*\dot{a}\| \leqslant \|\dot{a}\|^2$ is true in any involutive Banach algebra. □

Definition 1.49. Let A and B be C^*-algebras. A $*$-*homomorphism* from A to B is any homomorphism φ preserving the involution: $\varphi(a^*) = \varphi(a)^*$. If both algebras are unital, φ is called *unital*, if $\varphi(1_A) = 1_B$.

Problem 34. Let $\varphi : A \to B$ be a $*$-homomorphism of non-unital algebras. Prove that there is a unique unital $*$-homomorphism $\varphi^+ : A^+ \to B^+$, extending φ. Note: The only way to determine φ^+ is the requirement to be unital: $\varphi^+(1) = 1$.

Problem 35. Let $\varphi : A \to B$ be a $*$-homomorphism of algebras, with A non-unital, and B unital. Prove that there is a unique unital $*$-homomorphism $\varphi^{(+)} : A^+ \to B$, extending φ. *Hint:* the same as above.

Theorem 1.50. *Let $\varphi : A \to B$ be a nonzero $*$-homomorphism. Then $\|\varphi\| = 1$ (in particular, it is continuous) and $\varphi(A)$ is a C^*-subalgebra of B. If φ is injective, then it is isometric (on the image).*

Proof. If the algebra A is non-unital, then we will consider φ^+ from Problem 34 or $\varphi^{(+)}$ from problem 35. If the algebra A is unital, then we can assume that B is unital too (if not — then we attach a unity without requiring the homomorphism to be unital). Then $\varphi(1_A) = p$ is a self-adjoint idempotent ($p^2 = p$), the space $B_p := pBp$ is a subalgebra of B (see problem 36) with identity $p = p \cdot 1_B \cdot p$, and φ, considered as a homomorphism in B_p, is unital.

Thus, in the proof we can restrict ourselves to the case of a unital homomorphism $\varphi : A \to B$ of unital algebras.

To distinguish the spectrum of an element in A and B, we will write Sp_A (resp., Sp_B) for the spectrum of elements in A (resp., in B).

Let $a = a^* \in A$. Then $\mathrm{Sp}_B(\varphi(a)) \subset \mathrm{Sp}_A(a)$, since φ is a unital $*$-homomorphism of algebras and $\|\varphi(a)\| = r(\varphi(a)) \leqslant r(a) = \|a\|$. For an element $a \in A$ of general form, we have $\|\varphi(a)\|^2 = \|\varphi(a^*a)\| \leqslant \|a^*a\| = \|a\|^2$, so $\|\varphi\| \leqslant 1$, that is, φ is continuous and does not increase the norm.

Suppose now that φ is injective but not isometric. Then there is an element $a \in A$ such that $\|\varphi(a)\| < \|a\|$. This means $\|\varphi(b)\| < \|b\|$ for $b := a^*a$. Let us denote $\|\varphi(b)\| =: r$ and $\|b\| =: s$. Let h be a continuous real function that satisfies the conditions $q(t) = 0$ for $t \in [0, r]$ and $h(s) = 1$. Then $\|\varphi(h(b))\| = \|h(\varphi(b))\| = \sup_{\lambda \in \mathrm{Sp}_B(\varphi(b))} |h(\lambda)| = 0$, while $\|h(b)\| = \sup_{\lambda \in \mathrm{Sp}_A(b)} |h(\lambda)| \geqslant 1$. A contradiction with injectivity. (The commutation condition is obvious for polynomials, h_n, uniformly approximating h, and in the limit we obtain it for h.)

In the case of a general (not necessarily injective) $*$-homomorphism, note that that $I = \operatorname{Ker} \varphi$ is closed since φ is continuous, so I is an ideal in A. Therefore φ induces an injective $*$-homomorphism $\dot\varphi : A/I \to B$ by the rule $\dot\varphi(\dot a) = \varphi(a)$. Then, by what has been proven, $\dot\varphi$ is isometric, and $\varphi(A) = \dot\varphi(A/I)$ is closed in B, so it is a C^*-subalgebra. Since φ is non-zero, then there is $a \in A$ with $\varphi(a) \neq 0$. Since $\dot\varphi$ is isometric, we have the equality $\|\dot a\| = \|\dot\varphi(\dot a)\| = \|\varphi(a)\|$. Moreover, for any $\varepsilon > 0$ there is an element $c \in A$ such that $\dot c = \dot a$ and $\|c\| < \|\dot a\| + \varepsilon$. Thus, $\|\varphi(c)\| > \|c\| - \varepsilon$. Since ε is arbitrary, we obtain $\|\varphi\| \geqslant 1$, so $\|\varphi\| = 1$. $\square$

Problem 36. Prove that the algebra B_p is closed, first obtaining the equality $pBp = \operatorname{Ker}(L_{1-p}) \cap \operatorname{Ker}(R_{1-p})$, where L_{1-p} and R_{1-p} are the linear operators of left and right multiplication by $1 - p$ in B, given by $L_{1-p} : b \mapsto (1-p)b$ and $R_{1-p} : b \mapsto b(1-p)$.

Problem 37. Develop the result of the previous problem by verifying the decomposition into a direct sum of closed subspaces $B = pBp \oplus pB(1-p) \oplus (1-p)Bp \oplus (1-p)B(1-p)$. Moreover, if we write down the quadruple (a, b, c, d), representing an element of a given direct sum in the form of a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then the multiplication in B passes under this isomorphism to the matrix multiplication according to the standard rule.

Problem 38. Derive from Theorem 1.50 the statement $\varphi(f(a)) = f(\varphi(a))$ for any normal a and f, which is continuous on the appropriate set (not only for a polynomial).

Problem 39. Obtain a proof of Theorem 1.50 via a reduction to a map of commutative subalgebras.

Corollary 1.51. *Let $I \subset A$ be an ideal, and $B \subset A$ be a C^*-subalgebra. Then $I + B$ coincides with the C^*-subalgebra $C^*(I, B)$ generated by I and B.*

Proof. It is obvious that $I + B \subset C^*(I, B)$ is an involutive subalgebra. Let $q : A \to A/I$ be the $*$-homomorphism of factorization. We know from the previous theorem that $q(B)$ is closed in A/I, so $I + B = q^{-1}(q(B))$ is closed in A. This means that $I + B$ is a C^*-algebra, contained in $C^*(I, B)$. $\square$

So far we were very careful when considering spectrum of an element in a C^*-algebra and its C^*-subalgebra. The next lemma shows that this is not so important.

Lemma 1.52. *Let $B \subset A$ be a unital C^*-subalgebra of a unital C^*-algebra, $1_A = 1_B$, and $a \in B$. Then $\mathrm{Sp}_B(a) = \mathrm{Sp}_A(a)$.*

Proof. Obviously, if an element has an inverse in B, then so does A, whence $\mathrm{Sp}_A(a) \subset \mathrm{Sp}_B(a)$. The reverse inclusion follows from the statement: if a is invertible into A, then its inverse belongs to B. To prove this, consider first the case $a = a^*$. Then the C^*-algebra $C = C^*(a, a^{-1})$ generated by a and a^{-1}, is a commutative unital C^*-subalgebra of A, and therefore it is isomorphic to some algebra of functions $C(X)$. Let $\hat{a}$ denote the image of a under this isomorphism. Then $0 \notin \mathrm{Sp}_{C(X)}(\hat{a}) \subset \mathbb{R}$. Let us choose polynomials p_n such that $p_n(t)$ converges uniformly to t^{-1} on $\mathrm{Sp}_{C(X)}(\hat{a})$. Then $\widehat{a^{-1}} = \lim_{n\to\infty} p_n(\hat{a})$, so $a^{-1} = \lim_{n\to\infty} p_n(a) \in C^*(a) \subset B$.

For a general element a, if a^{-1} exists in A, then $a^{-1}(a^*)^{-1} = (a^*a)^{-1} \in B$ as proven. That is why $a^{-1} = (a^*a)^{-1}a^* \in B$. $\square$

Problem 40. Show with an example that, without the condition $1_A = 1_B$, the previous proposition does not hold.

Lecture 6

1.10 Topologies on $\mathbb{B}(H)$ and von Neumann algebras

Besides the norm topology, there are other useful topologies on the C^*-algebra $\mathbb{B}(H)$.

Definition 1.53. *Strong topology* is defined by a system of seminorms $a \to \|a\xi\|$, $\xi \in H$.
Weak topology is defined by a seminorm system $a \to (a\xi, \eta)$, $\xi, \eta \in H$.

Theorem 1.54. *For a linear functional $\varphi : \mathbb{B}(H) \to \mathbb{C}$ the following conditions are equivalent:*

(i) There exist $\xi_k, \eta_k \in H$, $k = 1, \dots, n$, such that $\varphi(a) = \sum_{k=1}^n (a\xi_k, \eta_k)$ for any $a \in \mathbb{B}(H)$;

(ii) φ is weakly continuous;

(iii) φ is strongly continuous.

Proof. It is obvious that (i)$\Longrightarrow$(ii) $\Longrightarrow$(iii). Let us show that (iii) implies (i).

The strong continuity of φ means that the preimage $\{a : |\varphi(a)| < 1\}$ of the open unit disk is an open set in the strong topology, that is, there are positive constants $\varepsilon_1, \dots, \varepsilon_n$ and vectors $\xi_1, \dots, \xi_n$, such that for any $a \in \mathbb{B}(H)$ the condition $\|a\xi_k\| < \varepsilon_k$ (for all $k = 1, \dots, n$) implies $|\varphi(a)| < 1$. Changing the length of these vectors if necessary, we see that in an equivalent way we can say, that there exist vectors $\xi_1, \dots, \xi_n$ such that, for any $a \in \mathbb{B}(H)$, from $\max_k \|a\xi_k\| \leqslant 1$ it follows that $|\varphi(a)| \leqslant 1$. Then

$$|\varphi(a)| \leqslant \left(\sum_{k=1}^n \|a\xi_k\|^2 \right)^{1/2}. \tag{1.9}$$

Indeed, if $|\varphi(a)|^2 > \sum_{k=1}^n \|a\xi_k\|^2$ for some a, then $|\varphi(a)| > \|a\xi_k\|$ for all k, so since the number of them is finite, one can find $\alpha \in \mathbb{R}$ such that $|\varphi(\alpha a)| > 1$, and $\max_k \|\alpha a\xi_k\| < 1$ (for example, $\alpha^{-1} := (|\varphi(a)| + \max \|a\xi_k\|)/2$). A contradiction.

Let $K := \oplus_{k=1}^n H$. The algebra $\mathbb{B}(K)$ can be identified with the algebra of $n \times n$-matrices with elements from $\mathbb{B}(H)$. Let $\rho : \mathbb{B}(H) \to \mathbb{B}(K)$ map $a \in \mathbb{B}(H)$ to a diagonal matrix with all diagonal elements equal to a.

Let us denote $\xi := \xi_1 \oplus \dots \oplus \xi_n \in K$ and note that, putting $\psi(\rho(a)\xi) = \varphi(a)$, we obtain a linear functional on the closed subspace $L \subset K$, where L is the closure of the space $L_0 := \{\rho(a)\xi \mid a \in \mathbb{B}(H)\}$. Indeed, first we need to verify that ψ is well defined on L_0: if $\rho(a)(\xi) = \rho(b)(\xi)$, then $(a - b)\xi_k = 0$ for $k = 1, \dots, n$. In particular for arbitrarily large $R > 0$ we have $\|R(a - b)\xi_k\| \leqslant 1$, and therefore $|\varphi(R(a - b))| \leqslant 1$. Therefore $|\varphi(a - b)| \leqslant 1/R$. Due to the arbitrariness of R, we obtain that $\varphi(a - b) = 0$ and thus ψ is well defined on L_0. From (1.9) we see that $|\psi(\rho(a))| \leqslant \|\rho(a)\xi\|$, so $|\psi(\zeta)| \leq \|\zeta\|$ for any $\zeta \in L_0$, and hence L. So, ψ is a bounded functional on L. By the Riesz theorem on representation of functionals, there is a vector $\eta \in L \subset K$ such that $\psi(\zeta) = (\zeta, \eta)$

for all $\zeta \in L$ (we assume the Hermitian product to be linear in the first argument), so $\varphi(a) = (\rho(a)\xi, \eta)$. Decomposing it into components $\eta = \eta_1 \oplus \ldots \oplus \eta_n$, we obtain $(\rho(a)\xi, \eta) = \sum_{k=1}^{n} (a\xi_k, \eta_k)$. □

Corollary 1.55. *In $\mathbb{B}(H)$ a convex set is closed for the weak topology if and only if it is closed for the strong one.*

Proof. This immediately follows from the previous theorem, since, according to the Hahn-Banach theorem, closed convex sets are obtained as the intersection of closed half-spaces, corresponding to linear functionals. □

Definition 1.56. A *von Neumann algebra* is a C^*-subalgebra $\mathbb{B}(H)$ containing unity (identity operator) and closed in the weak topology.

The simplest examples are $\mathbb{C}$ and $\mathbb{B}(H)$ (in fact, the first algebra is a special case of the second).

Definition 1.57. For the set $S \subseteq \mathbb{B}(H)$ we denote by S' its *commutant*, that is, the set of all operators $a \in \mathbb{B}(H)$ such that $as = sa$ for every $s \in S$.

Problem 41. Verify that

- If S is self-adjoint, then so is S'.
- The commutant of any set is a unital algebra.
- The commutant of any set is weakly closed.
- Thus, S' is the von Neumann algebra for any self-adjoint set S.
- If $S_1 \subset S_2$, then $S_1' \supset S_2'$.
- Always $S \subset S''$.
- Therefore $S' = S'''$, $S'' = S''''$, etc.

Theorem 1.58 (von Neumann bicommutant theorem)**.** *Let A be a C^*-subalgebra of $\mathbb{B}(H)$ containing the identity operator. Then the following conditions are equivalent.*

(i) $A = A''$;

(ii) A *is weakly closed;*

(iii) A *is strongly closed.*

Proof. Since A is a convex subset, then (ii) and (iii) are equivalent as a consequence 1.55. Since A'' is weakly closed, then (ii) follows from (i). It remains to show that (iii) implies (i).

For a vector $\xi \in H$ we denote by p the projection onto the closure V of the linear subspace formed by vectors $a\xi$, $a \in A$.

Thus, $p\eta = \eta$ for $\eta \in V$. Since $1 \in A$, then $\xi \in V$, so $p\xi = \xi$. Therefore $pap\zeta = pa\eta = a\eta = ap\zeta$ for any $\zeta \in H$, where we denote $\eta = p\zeta \in V$. So $pap = ap$ for any $a \in A$. From here $pa = (a^*p)^* = (pa^*p)^* = pap$ and we get $ap = pa$, that is $p \in A'$. Let $b \in A''$. Then $pb = bp$, so $pb\xi = bp\xi = b\xi$ and $b\xi \in V$. Thus, for every $\varepsilon > 0$ there is an element $a \in A$, for which $\|(b-a)\xi\| < \varepsilon$.

Now consider some $\xi_1, \dots, \xi_n \in H$ and define $\xi := \xi_1 \oplus \dots \oplus \xi_n \in K := H \oplus \dots \oplus H$. Let $\rho : \mathbb{B}(H) \to \mathbb{B}(K)$ be the diagonal embedding. It is easy to see that $\rho(A)'$ consists of all $n\times n$-matrices with elements from A', and $\rho(A'') = \rho(A)''$ (Problem 42). Applying the first part of the proof to this situation, we obtain that for any $b \in A''$ and every $\varepsilon > 0$ there is an element $a \in A$ such that $\|(\rho(b) - \rho(a))\xi\| < \varepsilon$. Then $\sum_{k=1}^n \|(b-a)\xi_k\|^2 = \|(\rho(b) - \rho(a))\xi\|^2 < \varepsilon^2$, so we can strongly approximate $b \in A''$ by some $a \in A$. $\square$

Problem 42. Check that $\rho(A)'$ consists of all $n\times n$-matrices with elements from A', and $\rho(A'') = \rho(A)''$.

Corollary 1.59. *If A is a von Neumann algebra, then A' is a von Neumann algebra.*

Definition 1.60. The *center* of an algebra is the set of its elements that commute with all its elements.

Corollary 1.61. *If A is a von Neumann algebra, then its center Z is also a von Neumann algebra.*

Proof. For a subalgebra $A \subseteq \mathbb{B}(H)$ we have $Z = A \cap A'$. $\square$

Let $A \subset \mathbb{B}(H)$ be a C^*-algebra containing the identity operator. Then the bicommutant theorem states that A is weakly (strongly) dense in A''. This result has the disadvantage that the approximation is done by elements with, generally speaking, an uncontrollable norm. This is overcome by the following theorem, which we present without the proof, which can be found in [10, § 4.3].

Theorem 1.62 (Kaplansky density theorem)**.** *The unit ball of A is weakly (strongly) dense in the unit ball of A''. The same is true for the sets of positive elements in these unit balls and for sets of unitary elements.*

Definition 1.63. If the center Z of the von Neumann algebra A consists only of scalar operators (that is, $Z = \mathbb{C}1$), then A is called a *factor.*

Remark 1.64. It should be noted that besides the continuous functional calculus for self-adjoint operators, there is a Borel functional calculus: instead of norm approximation of continuous functions by polynomials here Borel functions are approximated by polynomials, and the corresponding operators will converge in the weak topology. More precisely, let the polynomials p_i converge monotonically and pointwise to a Borel function f on the spectrum of a self-adjoint operator $a \in \mathbb{B}(H)$. Then $\{p_i(a)\}$ is a strongly convergent sequence of operators (this is a statement from the standard course, see for example [15, §§ 7 and 11]). Since all polynomials commute with the commutator of the self-adjoint operator, then for any Borel function f on the spectrum of a self-adjoint operator a, the operator $f(a)$ lies in $\{a\}''$.

Chapter 2

Representations of C^*-algebras

Lecture 7

2.1 Definition and basic properties

Definition 2.1. A *representation* of a C^*-algebra A on a Hilbert space H is a $*$-homomorphism from A to $\mathbb{B}(H)$.

Definition 2.2. A representation of a C^*-algebra A is called *algebraically irreducible*, if there is no proper invariant linear subspace in H (when operated by operators from the image of the representation). A representation is *topologically irreducible*, if there is no proper closed invariant subspaces.

We will see soon that for C^*-algebras these two concepts coincide.

Lemma 2.3. *A representation π is topologically irreducible if and only if $\pi(A)' = \mathbb{C}1$.*

Proof. If $\pi(A)'$ contains something other than scalars, then it also contains a self-adjoint non-scalar operator (this immediately follows from the expansion of a non-scalar operator into a linear combination of two self-adjoint ones $a = \frac{a+a^*}{2} + i \cdot \frac{a-a^*}{2i}$). Using Borel functional calculus (see note 1.64) for this self-adjoint operator b, we can obtain a proper projection p in $\pi(A)'$. Namely, if an operator is nonscalar, then it has at least two distinct points in the spectrum, say, t_0 and t_1, and we need to consider a Borel function f, taking values 0 and 1, and $f(t_0) = 0$, $f(t_1) = 1$ (task 43). (You can also not use calculus, but simply take suitable spectral projections from the standard spectral theorem, that by construction have the necessary commutation properties). Then pH is a closed invariant subspace, since $p \in \pi(A)'$.

Conversely, let $L \subset H$ be a closed $\pi(A)$-invariant subspace, and $p \in \mathbb{B}(H)$ is a projection onto this subspace. Then $\pi(a)p = p\pi(a)p$ for any $a \in A$. Therefore $p\pi(a) = (\pi(a^*)p)^* = (p\pi(a^*)p)^* = p\pi(a)p = \pi(a)p$ and $p \in \pi(A)'$. Moreover, p is not a scalar. $\square$

Problem 43. Verify in the proof above that $f(b)$ is a proper projection, since $f^2 = f$ and $\mathrm{Sp}(f(b)) = \{0, 1\}$.

Problem 44. Prove a more general fact: if a self-adjoint element a in a unital C^*-algebra has $\mathrm{Sp}(a) = \{0, 1\}$, then a is a nonscalar idempotent.

Lemma 2.4. *Let π be a topologically irreducible representation of a C^*-algebra A in a Hilbert space H. Then for any $t \in \mathbb{B}(H)$, a finite-dimensional subspace $L \subset H$ and $\varepsilon > 0$, there is an element $a \in A$ such that $\|a\| \leq \|t|_L\|$ and $\|(\pi(a) - t)|_L\| < \varepsilon$.*

Proof. Since π is topologically irreducible, then by Lemma 2.3 $\pi(A)'$ coincides with scalars, hence $\pi(A)'' = \mathbb{B}(H)$. That is why $\pi(A)$ is dense in $\mathbb{B}(H)$ in the weak (strong) topology. Without loss of generality, we can assume that $\|t|_L\| = 1$. Let us put $s = tp_L$, where p_L is the projection onto L. Since L is finite-dimensional, then, by Kaplansky's density theorem, there is $b \in A$ such that $\|\pi(b)\| \leqslant 1$ and $\|(\pi(b) - s)|_L\| < \varepsilon/2$. Then there is an element $c \in A$ such that $\pi(c) = \pi(b)$ and $\|c\| < \|\pi(b)\|(1 + \varepsilon/2)$ (see Theorem 1.50). Let us put $a := \frac{c}{1+\varepsilon/2}$. Then $\|a\| \leqslant 1$ and

$$\|(\pi(a) - t)|_L\| \leqslant \|(\pi(c) - t)|_L\| + \|\pi(a) - \pi(c)\| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

□

Lemma 2.5. *Let π be a topologically irreducible representation of the C^*-algebra A in the Hilbert space H. Then for any $t \in \mathbb{B}(H)$, finite-dimensional subspace $L \subset H$ and $\varepsilon > 0$, there is an element $a \in A$ such that $\pi(a)|L = t|_L$ and $\|a\| \leqslant \|t\| + \varepsilon$.*

Proof. By the previous lemma, there is an element $a_0 \in A$ such that $\|a_0\| \leqslant \|t\|$ and $\|(\pi(a_0) - t)|_L\| < \varepsilon/2$. By induction we can find for each n, an element $a_n \in A$ such that $\|a_n\| \leqslant 2^{-n}\varepsilon$ and $\|(\sum_{k=0}^{n} \pi(a_n) - t)p_L|\| < 2^{-n-1}\varepsilon$. Indeed, suppose that the elements are found for some n and all the smaller ones. Applying the previous lemma to $s = -\sum_{k=0}^{n} \pi(a_k) + t$, the same subspace L and $2^{-n-2}\varepsilon$, we find an element a_{n+1} such that $\|a_{n+1}\| \leqslant 2^{-n-1}\varepsilon$ and $\|(\sum_{k=1}^{n+1} \pi(a_k) - t)p_L\| < 2^{-n-2}\varepsilon$. Now let's put $a = \sum_{k=0}^{\infty} a_k$. Then $a \in A$ and it is evident that $\|a\| \leqslant \|t\| + \varepsilon$ and $a|_L = t|_L$. □

Theorem 2.6. *Every topologically irreducible representation of a C^*-algebra is algebraically irreducible.*

Proof. Let's assume the opposite Let $V \subset H$ be a non-closed invariant space, and $\overline{V}$ is its closure. It is also an invariant subspace (since the action is continuous), so $\overline{V} = H$. Let us take $\eta \in H \setminus V$, of norm 1 for example. Let $\xi \in V$ is a nonzero vector, and t is an operator on H such that $t\xi = \eta$. Then, by the previous lemma, there is an $a \in A$ such that $\pi(a)\xi = \eta$. Contradiction with the invariance of V. □

2.2 Positive linear functionals

Definition 2.7. Linear functional (we do not require continuity, see Lemma 2.10 below) φ on the C^*-algebra A is called *positive*, if $\varphi(a) \geqslant 0$ for any $a \geqslant 0$. If a positive linear functional is continuous and has norm 1, then it is called a *state*.

Example 2.8. If π is a representation of A in the Hilbert space H and $\xi \in H$, then the functional $\varphi(a) := (\xi, \pi(a)\xi)$ is positive. If A is unital and $\|\xi\| = 1$, then such φ is a state.

With every positive linear functional φ we can associate a sesquilinear form on A given by the formula $\langle a, b\rangle := \varphi(a^*b)$, that is, the form $\langle\cdot,\cdot\rangle$ is linear in the second argument is conjugate linear in the first argument. By definition of positivity of the functional $\langle a, a\rangle = \varphi(a^*a) \geqslant 0$ for any $a \in A$. Therefore, by the following lemma it is Hermitian symmetric: $\langle b, a\rangle = \overline{\langle a, b\rangle}$.

Lemma 2.9 (from linear algebra course)**.** *If a sesquilinear form has $\langle a, a\rangle \in \mathbb{R}$ for any a, then it is Hermitian symmetric.*

Proof. Let us write down the polarization identities

$$\langle a+b, a+b\rangle = \langle a, a\rangle + \langle a, b\rangle + \langle b, a\rangle + \langle b, b\rangle, \tag{2.1}$$

$$\langle a+ib, a+ib\rangle = \langle a, a\rangle + \langle a, ib\rangle + \langle ib, a\rangle + \langle ib, ib\rangle = \langle a, a\rangle + i(\langle a, b\rangle - \langle b, a\rangle) + \langle b, b\rangle. \tag{2.2}$$

From the first we obtain that $\langle a, b\rangle + \langle b, a\rangle$ is real, and from the second — that $\langle a, b\rangle - \langle b, a\rangle$ is imaginary. So $\overline{\langle a, b\rangle} = \langle b, a\rangle$. □

Thus, $\langle a, b\rangle$ is a positive Hermitian form and, therefore, the Cauchy-(Schwartz-Bunyakovsky) inequality holds for it: $|\langle a, b\rangle|^2 \le \langle a, a\rangle\langle b, b\rangle$, that is, $|\varphi(a^*b)|^2 \leqslant \varphi(a^*a)\varphi(b^*b)$.

Lemma 2.10. *Positive linear functionals are continuous. If u_λ is an approximate unit for A, then $\|\varphi\| = \lim_{\lambda\in\Lambda} \varphi(u_\lambda)$. In particular, if A is unital, then $\|\varphi\| = \varphi(1)$.*

Proof. Let us first consider the unital case. If $0 \leqslant a \leqslant 1$, then since φ is positive, we obtain that $0 \leqslant \varphi(a) \leqslant \varphi(1)$. For $x \in A$ with $\|x\| \leqslant 1$ we have $0 \leqslant x^*x \leqslant 1$, so $|\varphi(x)|^2 = |\varphi(1 \cdot x)|^2 \leqslant \varphi(1) \cdot \varphi(x^*x) \leqslant \varphi(1)^2$ by the Cauchy-Schwartz-Bunyakovsky inequality. Thus, $\|\varphi\| \leqslant \varphi(1) \leqslant \|\varphi\|$.

Now consider the non-unital case. Suppose that φ is not bounded on the unit ball A. Then it is not bounded on the subset of the unit ball consisting of positive elements (since any element a is decomposable into a linear combination of four positive elements with norms not exceeding $\|a\|$, see (1.7)). Thus, for every $k \in \mathbb{N}$ there is a positive element $a_k \in A$ such that $\|a_k\| \le 1$ and $\varphi(a_k) > 2^k$. Let us put $a := \sum_{k=1}^{\infty} \frac{a_k}{2^k} \in A$. Then for any $n \in \mathbb{N}$, we have $a \geqslant \sum_{k=1}^{n} \frac{a_k}{2^k}$ and

$$\varphi(a) \geqslant \varphi\left(\sum_{k=1}^{n} \frac{a_k}{2^k}\right) = \sum_{k=1}^{n} \frac{\varphi(a_k)}{2^k} > n,$$

that is impossible. Thus, φ is bounded in the non-unital case as well.

Let $m := \lim_{\lambda\in\Lambda} \varphi(u_\lambda)$, and the limit exists since the directed net is increasing and bounded by $\|\varphi\|$ from above. Then, for any $x \in A$ with $\|x\| \leqslant 1$ we have $|\varphi(x)| = \lim_{\lambda\in\Lambda} |\varphi(u_\lambda x)|$ due to the continuity of φ. Therefore, by the Cauchy-Schwartz-Bunyakovsky inequality we have

$$|\varphi(x)|^2 = \lim_{\lambda\in\Lambda} |\varphi(u_\lambda x)|^2 \leqslant \sup_\lambda \varphi(u_\lambda^2)\varphi(x^*x) \leqslant \sup_\lambda \varphi(u_\lambda)\varphi(x^*x) \leqslant m\|\varphi\|.$$

For any $\varepsilon > 0$ we choose an element $x \in A$ such that $\|\varphi\|^2 < |\varphi(x)|^2 + \varepsilon$. Then $\|\varphi\|^2 < m\|\varphi\| + \varepsilon$. Hence, $\|\varphi\|^2 \leqslant m\|\varphi\|$ and $\|\varphi\| \leqslant m$. Since for any $\varepsilon > 0$ there is u_{λ_0} for which $\varphi(u_{\lambda_0}) > m - \varepsilon$, we come to the equality $\|\varphi\| = m$. □

Corollary 2.11. *If φ is a state on a unital C^*-algebra, then $\varphi(1) = 1$.*

Proof. By the previous lemma, $1 = \|\varphi\| = \varphi(1)$. □

Lecture 8

2.3 GNS-construction (Gelfand-Naimark-Segal)

Definition 2.12. A vector $\xi \in H$ is called *cyclic* for $\pi : A \to \mathbb{B}(H)$, if $\pi(A)\xi$ is dense in H.

Theorem 2.13. *Let φ be a positive linear functional on the C^*-algebra A. Then there exists a representation π_φ of the algebra A on the Hilbert space H and a cyclic vector $\xi_\varphi \in H$ such that $\|\xi_\varphi\|^2 = \|\varphi\|$ and $(\xi_\varphi, \pi_\varphi(a)\xi_\varphi) = \varphi(a)$ for all $a \in A$.*

Proof. Let $N := \{a \in A : \varphi(a^*a) = 0\}$. Then $N = \{a \in A : \varphi(b^*a) = 0 \text{ for all } b \in A\}$ by the Cauchy-Schwartz-Bunyakovsky inequality. Therefore N is closed as an intersection of kernels of continuous functionals $a \mapsto \varphi(b^*a))$. Besides, N is a left ideal, since $\varphi(b^*an) = \varphi((a^*b)^*n) = 0$ for any $a, b \in A$ for $n \in N$, so $an \in N$.

Let us define an Hermitian inner product on the Banach quotient space A/N by the formula $(\dot{a}, \dot{b}) = \varphi(a^*b)$, where $\dot{a}$ denotes the coset class $a+N$. This product is well defined because if $n_1, n_2 \in N$, then $\varphi((a+n_1)^*(b+n_2)) = \varphi(a^*b) + \varphi((a+n_1)^*n_2) + \overline{\varphi(b^*n_1)} = \varphi(a^*b)$. Also $(\dot{a}, \dot{a}) > 0$ holds for $\dot{a} \neq 0$. Let H be the Hilbert space obtained from A/N by the completion w.r.t. the norm given by this inner product. Let us denote by π_0 the representation of A on A/N (here we slightly expand the concept of representation to a pre-Hilbert space) by the formula $\pi_0(a)\dot{x} = (ax)^{\cdot}$, where $\dot{x} \in A/N$. If $n \in N$ then $(a(x+n))^{\cdot} = (ax)^{\cdot}$, so π_0 is well defined. It is involutive, because $(\pi_0(a)\dot{x}, \dot{y}) = \varphi((ax)^*y) = \varphi(x^*(a^*y)) = (\dot{x}, \pi_0(a^*)\dot{y}) = (\pi_0(a^*)^*\dot{x}, \dot{y})$ and $\pi_0(a^*)^* = \pi_0(a)$. In this case, $\|\pi_0\| \leqslant 1$. Indeed,

$$\begin{aligned} \|\pi_0(a)\|^2 &= \sup_{\|\dot{x}\|\leqslant 1} \|\pi_0(a)\dot{x}\|^2 = \sup_{\|\dot{x}\|\leqslant 1} \varphi(x^*a^*ax) \leqslant \\ &\leqslant \sup_{\|\dot{x}\|\leqslant 1} \|a^*a\|\varphi(x^*x) \leqslant \|a\|^2. \end{aligned}$$

Therefore π_0 can be extended by continuity to a representation π_φ of the algebra A on H.

If the algebra A is unital, then we set $\xi_\varphi := \dot{1}$. Then $(\xi_\varphi, \pi_\varphi(a)\xi_\varphi) = \varphi(a)$ and ξ_φ is cyclic since $\pi_\varphi(A)\xi_\varphi = A/N$ is dense in H. Finally, $\|\varphi\| = \varphi(1) = \|\xi_\varphi\|^2$.

For a general algebra A, consider its approximate unit u_λ. Let us show that $\dot{u}_\lambda$ is a Cauchy directed net. Let us choose an $\varepsilon > 0$. Then there is an index $\alpha \in \Lambda$ such that $\varphi(u_\alpha) > \|\varphi\| - \varepsilon$ (since $\|\varphi\| = \lim_{\lambda\in\Lambda} \varphi(u_\lambda)$ by Lemma 2.10). Now let us find an index $\beta \in \Lambda$ such that $\beta \geqslant \alpha$ and $\|u_\lambda u_\alpha - u_\alpha\| < \varepsilon$ for any $\lambda \geqslant \beta$. Then

$$\mathrm{Re}(\varphi(u_\lambda u_\alpha)) = \varphi(u_\alpha) + \mathrm{Re}(\varphi(u_\lambda u_\alpha - u_\alpha)) > \|\varphi\| - 2\varepsilon.$$

That is why

$$\begin{aligned} \|\dot{u}_\lambda - \dot{u}_\alpha\|^2 &= \varphi((u_\lambda - u_\alpha)^2) = \varphi(u_\lambda^2) + \varphi(u_\alpha^2) - 2\mathrm{Re}(\varphi(u_\lambda u_\alpha)) \leqslant \\ &\leqslant \varphi(u_\lambda^2) + \varphi(u_\alpha^2) - 2(\|\varphi\| - 2\varepsilon) \leqslant 4\varepsilon. \end{aligned}$$

This means that for $\lambda, \mu \geqslant \beta$, we have

$$\|\dot{u}_\lambda - \dot{u}_\mu\| \leqslant \|\dot{u}_\lambda - \dot{u}_\alpha\| + \|\dot{u}_\alpha - \dot{u}_\mu\| \leqslant 4\varepsilon^{1/2}.$$

Thus, $\dot{u}_\lambda$ is a Cauchy net. Let $\xi_\varphi := \lim_{\lambda\in\Lambda} \dot{u}_\lambda \in H$. Then $(\xi_\varphi, \pi_\varphi(a)\xi_\varphi) = \lim_{\lambda\in\Lambda} \varphi(u_\lambda a u_\lambda) = \varphi(a)$. Since $\pi_\varphi(A)\xi_\varphi = A/N$, then ξ_φ is cyclic. From $\dot{a} = \pi_\varphi(a)\xi_\varphi$ it follows that

$$\lim_{\lambda\in\Lambda} \pi_\varphi(u_\lambda)\dot{a} = \lim_{\lambda\in\Lambda} \pi_\varphi(u_\lambda)\pi_\varphi(a)\xi_\varphi = \pi_\varphi(a)\xi_\varphi = \dot{a}$$

for any $\dot{a} \in A/N$, so the directed net $\pi_\varphi(u_\lambda)$ strongly converges to 1 on H, because it is bounded. Therefore $\|\varphi\| = \lim_{\lambda\in\Lambda} \varphi(u_\lambda) = \lim_{\lambda\in\Lambda} (\xi_\varphi, \pi_\varphi(u_\lambda)\xi_\varphi) = \|\xi_\varphi\|^2$. □

2.4 Realization of C^*-algebras as operator algebras on Hilbert space

Corollary 2.14. *Any state φ on a non-unital C^*-algebra A admits a unique extension to a state on A^+.*

Proof. Let π_φ be the representation of A given by the GNS construction. Let us set $\pi_\varphi(1) = 1$. Then π_φ can be extended to a representation of A^+ and $\tilde{\varphi}(a) := (\xi_\varphi, \pi(a)\xi_\varphi)$ is a state. It is unique, since $\tilde{\varphi}(1) = 1$ must hold (Corollary 2.11). □

Problem 45. Let u_λ, $\lambda \in \Lambda$, be some approximate unit in a unital algebra. Prove that $1 = \lim_{\lambda\in\Lambda} u_\lambda$.

Lemma 2.15. *Let $\varphi : A \to \mathbb{C}$ be a continuous linear functional such that $\|\varphi\| = 1 = \lim_{\lambda\in\Lambda} \varphi(u_\lambda)$ for some approximate unit u_λ. Then φ is a state.*

Proof. Let us first reduce the proof to the unital case. Let $\tilde{\varphi}$ be some extension (by the Hahn-Banach theorem) functional φ to a continuous functional on A^+. Let $\tilde{\varphi}(1) =: \alpha$. Because the $\|\tilde{\varphi}\| = 1$, then $|\alpha| \leqslant 1$. From inequality $\|2u_\lambda - 1\| \leqslant 1$ it follows that $|2 - \alpha| = \lim_{\lambda\in\Lambda} |\varphi(2u_\lambda - 1)| \leqslant 1$. Thus, $\alpha = 1$. This means that we can assume that A is unital and $\varphi(1) = 1$ (if A was unital from the very beginning, then we use the problem 45).

Let us now show that $\varphi(a) \in \mathbb{R}$ if $a = a^*$ (and therefore contained in $[-\|a\|, \|a\|]$). Let a be a self-adjoint element of norm 1. Then $\|a \pm in1\|^2 = \|a^2 + n^2 1\| = n^2 + 1$, so $|\varphi(a) \pm in| \leqslant \sqrt{n^2+1}$ for any $n \in \mathbb{N}$. This means that $\varphi(a)$ is contained in the intersection of all disks with centers at $\pm in$ and radii $\sqrt{n^2+1}$. This intersection is equal to the real interval $[-1, 1]$.

If $0 \leq a \leq 1$, then $\|2a - 1\| \leq 1$. Applying the previous reasoning to the self-adjoint element $2a - 1$, we obtain that $-1 \leqslant 2\varphi(a) - 1 \leqslant 1$, so $\varphi(a) \geqslant 0$ and φ is positive. □

Lecture 9

Lemma 2.16. *Let $a \in A$ be a self-adjoint element. Then there is a state φ on A such that $|\varphi(a)| = \|a\|$.*

Proof. If A is non-unital, then we will work in A^+. Consider the commutative C^*-algebra $C^*(a)$. Then there is a multiplicative linear functional φ_0 on $C^*(a)$ such that $|\varphi_0(a)| = \|a\|$ (we must take as φ_0 the mapping, which is the taking of the value of functions at that point of $\mathrm{Sp}(a)$, where the function $\hat{a}$ reaches its maximum). Then $\varphi_0(1) = 1 = \|\varphi_0\|$. Consider the extension of φ_0 by the Hahn-Banach theorem to a functional φ on A^+. Then, since $\|\varphi\| = 1 = \varphi(1)$, then φ is a state by Lemma 2.15. □

Corollary 2.17. *For any $a \in A$ there exists a representation π and a unit vector ξ in the space of representation such that $\|\pi(a)\xi\| = \|a\|$.*

Proof. By the previous lemma, we find a state φ such that $\varphi(a^*a) = \|a\|^2$. Let $\pi = \pi_\varphi$ and $\xi = \xi_\varphi$ were obtained for φ using the GNS construction. Then $\|\pi(a)\xi\|^2 = (\xi, \pi(a^*a)\xi) = \varphi(a^*a) = \|a\|^2$. □

Theorem 2.18 (Gelfand-Naimark)**.** *Any C^*-algebra is isometrically $*$-isomorphic to a C^*-subalgebra of $\mathbb{B}(H)$ for some Hilbert space H. If A is separable, then H can be chosen to be separable.*

Proof. Let us set $\pi = \oplus_\varphi \pi_\varphi$, where the direct sum is taken over all states on A. More precisely, we consider the Hilbert direct sum $H := \oplus_\varphi H_\varphi$ (completion with respect to the ℓ_2 norm of the space of compactly supported mappings $\varphi \mapsto h_\varphi \in H_\varphi$, that is, the sets $h = \{h_\varphi\}$, $h_\varphi \in H_\varphi$, and only a finite number h_φ is nonzero, and the norm is defined as $\|h\|^2 = \sum_\varphi \|h_\varphi\|^2$) with diagonal action $\pi(a)(\{h_\varphi\}) = \{\pi_\varphi(a)(h_\varphi)\}$. Then, as can be seen from the proof of the previous consequences, $\|\pi(a)\| = \sup_\varphi \|\pi_\varphi(a)\| = \|a\|$. If A is separable, then it is sufficient to take the sum over a countable set $\{\varphi_n\}$, where $\|\pi_{\varphi_n}(a_n)\| = \|a_n\|$, for elements a_n forming a dense subset in A. Then $\pi = \oplus_{n\in\mathbb{N}} \pi_{\varphi_n}$, and the corresponding Hilbert space is separable, since each H_{φ_n} is separable (as a completion of a factor-space of a separable space). □

Definition 2.19. The representation constructed in the theorem (in its first part) is called the *universal representation* of A. The von Neumann algebra $\pi(A)''$, where π is the universal representation, contains $\pi(A) \cong A$ as a dense subset and is called the *enveloping von Neumann algebra* for A.

2.5 Jordan decomposition

Lemma 2.20. *Let φ be a linear functional on A. Then $\varphi = \psi_1 + i\psi_2$, where ψ_1 and ψ_2 are self-adjoint.*

Proof. Let us take, in the same way as we did for elements of algebra, $\psi_1(a) = (\varphi(a) + \overline{\varphi(a^*)})/2$ and $\psi_2(a) = (\varphi(a) - \overline{\varphi(a^*)})/2i$. □

Let A_{sa} denote the set of all self-adjoint elements of A. Then it is evident that A_{sa} is a real Banach space.

Problem 46. There is a natural bijection between self-adjoint linear functionals on A and (real) linear functionals on A_{sa}.

To prove the Jordan decomposition theorem, we need the following statement, which is of independent interest.

Theorem 2.21 (on extension of positive functionals)**.** *Let $B \subset A$ be a C^*-subalgebra, and $\varphi : B \to \mathbb{C}$ be a positive functional. Then there exists a positive functional $\varphi' : A \to \mathbb{C}$ such that that $\varphi'|_B = \varphi$ and $\|\varphi'\| = \|\varphi\|$.*

Proof. The following cases are possible:

a) both algebras have a common unit,

b) A has one, but B does not,

c) both algebras do not have a unit,

d) B has one, but A does not.

e) both algebras with 1, but $1_A \neq 1_B$.

By Corollary 2.14, (c) and (b) can be reduced by adjoining 1 to (a) (for (b) it should be noted that $B^+ \cong B \oplus \mathbb{C}\, 1_A$). In turn, (d) obviously reduces to (e).

In case (a) we extend φ (using the Hahn-Banach theorem) to some $\varphi' : A \to \mathbb{C}$ of the same norm. Then by Lemma 2.10, $\|\varphi'\| = \|\varphi\| = \varphi(1) = \varphi'(1)$ and φ' is positive by Lemma 2.15.

In case (e), consider the C^*-subalgebra $B_1 := B \oplus \mathbb{C}\, 1_A = B \oplus \mathbb{C}\,(1_A - 1_B)$ and extend φ to $\varphi_1 : B_1 \to \mathbb{C}$, setting $\varphi_1(1_A - 1_B) = 0$. Then $\varphi_1(a) = \varphi(1_B \cdot a)$, where $a \in B_1$. Indeed, if $a \in B$, then $\varphi_1(a) = \varphi(1_B \cdot a) = \varphi(a)$, and if $a = 1_A - 1_B$, then $\varphi_1(a) = \varphi(1_B(1_A - 1_B)) = \varphi(0) = 0$. In this case, the unit of B_1 is 1_A. Moreover, $\|\varphi_1\| \leqslant \|\varphi\| \cdot \|1_B\| = \|\varphi\|$, and $\varphi_1(1_A) = \varphi(1_B) = \|\varphi\|$. This means that $\|\varphi_1\| = \|\varphi\| = \varphi_1(1_A) = \varphi_1(1_{B_1})$ and, by Lemma 2.15, φ_1 is positive. Thus, case (e) is also reduced to the proven case (a). □

The Jordan theorem about decomposition of a measure in the sum of positive and negative ones [8, Ch. VI, §5, Theorem 1] in the functional language (in the sense of the Riesz-Markov-Kakutani theorem [5, Ch. I, §6, Theorem 4]) can be written as: for any bounded real linear functional $\tau : C(\Omega, \mathbb{R}) \to \mathbb{R}$ there are positive linear functionals τ_+ and τ_- such that $\tau = \tau_+ - \tau_-$ and $\|\tau\| = \|\tau_+\| + \|\tau_-\|$, where Ω is a compact Haudorff space and $C(\Omega, \mathbb{R})$ is the real algebra of all real continuous functions on Ω.

Theorem 2.22 (Jordan decomposition)**.** *Let ψ be a self-adjoint linear functional on A. Then $\psi = \varphi_+ - \varphi_-$, where φ_+ and φ_- are positive linear functionals on A and $\|\psi\| = \|\varphi_+\| + \|\varphi_-\|$.*

Proof. Denote by K the set of all self-adjoint linear functionals of norm $\leqslant 1$, i.e., $K \subset (A^*)_{sa}$. Then K is a $*$-weak closed subset of the unit ball and hence it is $*$-weak compact. Define an $\mathbb{R}$-linear map

$$\theta : A_{sa} \to C(K, \mathbb{R}), \qquad \theta(a)(\tau) = \tau(a),$$

so, if $a \in A$, $a \geqslant 0$, then $\theta(a) \geqslant 0$ in K. By Lemma 2.16 the mapping θ is an isometry onto its image.

There is a natural isometry $\tau \mapsto \tau'$ of real spaces $(A^*)_{sa}$ and $(A_{sa})^*_{\mathbb{R}}$ (real functionals) (see Problem46). By the Hahn-Banach theorem there is a functional $\rho \in (C(K,\mathbb{R}))^*_{\mathbb{R}}$ such that $\rho \circ \theta = \psi'$ and $\|\rho\| = \|\psi'\|$ (an extension of a functional from the closed subspace $\theta(A_{sa})$). Then by the Jordan theorem for measures (as it is explained above before the formulation) there are positive functionals ρ_+ and ρ_- such that $\rho = \rho_+ - \rho_-$ and $\|\rho\| = \|\rho_+\| + \|\rho_-\|$. Consider $\varphi'_+ := \rho_+ \circ \theta$ and $\varphi'_- := \rho_- \circ \theta$. These are functionals from $(A_{sa})^*_{\mathbb{R}}$. Let φ_+ and φ_- correspond to them under the identification with $(A^*)_{sa}$. Evidently they satisfy all the conditions, except maybe the norm property. Let us verify it:

$$\|\psi\| = \|\psi'\| = \|\rho\| = \|\rho_+\| + \|\rho_-\| \geqslant \|\varphi'_+\| + \|\varphi'_-\| = \|\varphi_+\| + \|\varphi_-\| \geqslant \|\psi\|.$$

□

2.6 Linear topological spaces

Definition 2.23. A subset M of a linear space is called *balanced*, if for any $v \in M$ the vector λv belongs to M for any $|\lambda| \leqslant 1$. In particular, M is a star set relative to the zero of space.

Definition 2.24. A subset M of a linear space is called *absorbing*, if for any vector v of the space there is a number $\alpha > 0$ such that $v \in \beta M$ for $|\beta| \geqslant \alpha$.

Definition 2.25. A linear space equipped with a topology is called *linear topological space* (LTS), if the operations of linear space are continuous.

In the basic course of functional analysis, the following simple statements are proved: (see [8, Chapter III, §5]):

Proposition 2.26. *1). A base of LTS consists of shifts of neighborhoods of zero.*

2). Any vector of LTS and a closed set not containing it have disjoint neighborhoods.

Definition 2.27. An LTS L satisfies the *homothety condition*, if for any neighborhood of zero W its homothety λW is also a neighborhood of zero for any $\lambda \neq 0$ from the main field.

Remark 2.28. Obviously, the topology of a normed space satisfies the homothety condition.

Proposition 2.29. *For any neighborhood of zero U of an LTS L with the homothety condition, there is a balanced neighborhood contained in it.*

Proof. Consider the continuous mapping $\mathbb{C} \times L \to L$ (multiplication) mapping $(0, 0_L) \mapsto 0_L$. Then, by virtue of continuity, there are $\delta > 0$ and a neighborhood of zero W such that $\lambda W \subseteq U$ for $|\lambda| \leqslant \delta$ (a non-strict inequality can be achieved by reducing δ from the standard definition). Let $W' := \cup_{0<|\lambda|\leqslant 1} \lambda W$. By virtue of 2.27, this W' is what we are looking for. □

Remark 2.30. In fact, it can be proven that the base of neighborhoods of zero of an LTS L can be chosen from absorbing balanced sets, and also that the homothety condition is in fact not a condition, but we will not need this (see [9, Chapter II, §4]).

We will need the following important result.

Theorem 2.31. *Let L be a finite-dimensional space, $\dim L = n$. Then any Hausdorff topology τ making L a linear topological space L_τ with the homothety condition coincides with the topology of the Euclidean norm $\|v\|^2 = \sum_{i=1}^n |v^i|^2$, where $e_1, \dots, e_n$ is some base of L, and $v = v^1 e_1 + \cdots v^n e_n$.*

Proof. The space L with Euclidean (or unitary) topology will be denoted by L_u, and neighborhoods of zero of two topologies (τ and Euclidean) will be denoted by T and U, respectively.

Consider an arbitrary T. Then there is a neighborhood T_0 such that $T_0 + \cdots + T_0 \subset T$ (n terms) due to the continuity of the addition operation. For every k there is $\varepsilon_k > 0$ such that $v^k e_k \in T_0$ for $|v_k| < \varepsilon_k$ ($k = 1, \dots, n$). Let $\varepsilon := \min_k \varepsilon_k$, and $U := \{v \in L \mid \|v\| < \varepsilon\}$. Then $v^k e_k \in T_0$ for any $v \in U$ and any $k = 1, \dots, n$. Thus, $U \subset T$. From what has been proved, in particular, it follows that the identity mapping $\iota : L_u \to L_\tau$ is continuous.

Conversely, let U be an arbitrary neighborhood, we can assume that $U = B(0, \varepsilon)$ is an open ball of radius ε with boundary (sphere) S, which is a compact set. Then $S = \iota(S)$ is compact in L_τ. This means that it is closed, since the topology is Hausdorff. Then there is a stellar neighborhood of zero T (for example, balanced) that does not intersect S by virtue of propositions 2.26 and 2.29. Moreover, $T \subseteq U$, since otherwise there exists a vector $v \in T$ such that $\|v\| \geqslant \varepsilon$, and if we put $\alpha := \varepsilon/\|v\|$, $w := \alpha v$, then $\alpha \leqslant 1$, so $w \in T$ by the star property. But $\|w\| = \varepsilon$, so $w \in T \cap S = \varnothing$. A contradiction. □

Lecture 10

2.7 Finite-dimensional C^*-algebras

Consider the $*$-weak topology on A defined by the seminorm system $a \mapsto |\varphi(a)|$ for all linear functionals φ. From Lemma 2.20 and Theorem 2.22 it follows that the same topology can be obtained by using only seminorms, defined by states.

Note also that the corresponding LTS has the homothety property 2.27.

Lemma 2.32. *A finite-dimensional C^*-algebra is always unital.*

Proof. If A is finite-dimensional, then the topology of the norm coincides with the $*$-weak topology according to Theorem 2.31. Let u_n be an approximate unit of the algebra A. Then for any state φ the sequence $\varphi(u_n)$ is non-decreasing and bounded, hence has a limit. Passing to linear combinations, we obtain the cinvergence for any functional on the finite-dimensional vector space A, in particular, for functionals $\varphi_1, \dots \varphi_k$ of the dual base for some base $a_1, \dots, a_k$. Then there exists an element $a \in A$ with $\varphi_i(a) = \lim_n \varphi_i(u_n)$. Considering linear combinations of φ_i, we conclude that $\varphi(a) = \lim_n \varphi(u_n)$ for any φ. Therefore u_n converges to a in $*$-weak topology, and therefore in norm. Then $ax = xa = x$ for any $x \in A$, so $a = 1$. $\square$

Lemma 2.33. *Let $I \subset A$ be an ideal in a finite-dimensional C^*-algebra A. Then $I = Ap$ for some central projector (=idempotent from the center) p.*

Proof. Since I is finite-dimensional, it is unital by Lemma 2.32. Let $p \in I$ be the unit of I. Then for every $x \in A$, one has $xp \in I$, so $p(xp) = xp$. Hence $px^*p = x^*p$ for any $x \in A$, whence $xp = pxp = px$ and p belongs to the center of A. Obviously, $p^2 = p$. $\square$

Lemma 2.34. *A simple finite-dimensional C^*-algebra A is isometrically $*$-isomorphic to the matrix algebra M_n for some n.*

Proof. First of all, note that $aAb \neq 0$ for any non-zero $a, b \in A$. Indeed, AaA is a non-zero ideal (since A is unital and $0 \neq a = 1 \cdot a \cdot 1 \in A$), so by simplicity, $AaA = A$. Therefore $1 = \sum_i x_i a y_i$ and $b = \sum_i x_i a y_i b$. Hence, if $ayb = 0$ for any $y \in A$, then $b = \sum_i x_i(ay_ib) = 0$. This contradicts the assumption.

Let B be some maximal commutative subalgebra of A. Then it can be identified with $C(X) = \mathbb{C}^n = \mathbb{C} \cdot e_1 \oplus \dots \oplus \mathbb{C} \cdot e_n$ for some n, where X consists of n points, and $e_i \in B$ denotes the element corresponding to the characteristic functions at point i. Here e_i are projections with the relations $e_ie_j = 0$ for $i \neq j$ and $\sum_{i=1}^n e_i = 1$. Since $e_iAe_i \cdot e_j = e_j \cdot e_iAe_i = 0$ and B is maximal, then $e_iAe_i \subset B$. Therefore $e_iAe_i = \mathbb{C} \cdot e_i$ (since, obviously, $0 \neq e_iAe_i \ni e_i$, or you can use the statement from the beginning of the proof).

For any i, j there is $x = x_{ij} \in A$ such that $x = e_ixe_j \neq 0$, $\|x\| = 1$. Indeed, by virtue of the statement from the beginning of the proof, $e_iAe_j \neq 0$, so we have $x = e_iye_j$ with $\|x\| = 1$. In this case $e_ixe_j = e_ie_iye_je_j = e_iye_j = x$. Then $x^*x = e_jx^*e_ie_ixe_j \in e_jAe_j$, and therefore, according to what has been proven, this element has the form αe_j, $\alpha \in \mathbb{C}$. Since

x^*x is a positive element with norm equal to one, then $\alpha = 1$, so $x^*x = e_j$. Likewise, $xx^* = e_i$. Let us denote such $x = x_{ij}$ for $j = 1$ by u_i, so that $u_i = e_i x e_1 = e_i u_i e_1$. Then $u_i^* u_i = e_1$, $u_i u_i^* = e_i$, $i = 1, \dots, n$. Let us set $u_{ij} := u_i u_j^*$. In this case, $u_i e_1 u_i^* = u_i u_i^* u_i u_i^* = e_i e_i = e_i$, So $u_{ij} u_{ji} = u_i u_j^* u_j u_i^* = u_i e_1 u_i^* = e_i$. Also $e_j u_{ji} = u_j u_j^* u_j u_i^* = u_j e_1 u_i^* = u_j u_i^* u_i u_i^* = u_{ji} e_i$, and $e_i u_{ij} = u_i u_i^* u_i u_j^* = u_i e_1 u_j^* = u_i u_j^* u_j u_j^*$.

If $x \in e_i A e_j$, that is, $x = e_i a e_j$, then $x u_{ji} = e_i a e_j u_{ji} = e_i a u_{ji} e_i \in e_i A e_i$, so $x u_{ji} = \lambda e_i$ for some $\lambda \in \mathbb{C}$. Then $x = x e_j = x u_{ji} u_{ij} = \lambda e_i u_{ij} = \lambda u_{ij}$, so for any $x \in A$ there is a number $\lambda_{ij}(x) \in \mathbb{C}$ such that $e_i x e_j = \lambda_{ij}(x) u_{ij}$. Thus, $x = \sum_{i,j} e_i x e_j = \sum_{ij} \lambda_{ij}(x) u_{ij}$. The correspondence $x \mapsto (\lambda_{ij}(x))$ defines an isomorphism $\kappa : A \to M_n$ (Problem 47). □

Problem 47. Check the bijectivity and necessary algebraic properties of κ.

Theorem 2.35. *If A is finite-dimensional, then $A = \oplus_k A p_k$, where p_k are central projections, and each $A p_k$ is a matrix algebra $M_{n(k)}$.*

Proof. For a simple algebra, the result follows from Lemma 2.34. If A is not simple, then $I = Ap$ by Lemma 2.33, where p is a central projection. Then $A = I \oplus J$, where $J := A(1-p)$. Then J is also an ideal, since $(1-p)$ is also a central projection, so $A(1-p)A = AA(1-p) \subseteq A(1-p)$. In this case, the center of A, being a finite-dimensional commutative algebra, is isomorphic to $\mathbb{C}^m$ (functions on finite set), and characteristic functions correspond to the projections. Next, we argue by induction, reducing the dimension, until we arrive to the sum of simple algebras. □

2.8 Non-degenerate representations

Definition 2.36. Let π be a representation of a C^*-algebra A on a Hilbert space H. We denote by $\pi(A)H$ the (possibly non-closed) linear space of finite linear combinations of the form $\sum_i \pi(a_i)\xi_i$, where $a_1, \dots, a_n \in A$, $\xi_1, \dots, \xi_n \in H$. A representation π is called *non-degenerate*, if $\pi(A)H$ is dense in H.

Problem 48. If A is unital, then π is non-degenerate if and only if $\pi(1) = 1$.

Lemma 2.37. *Let $I \subset A$ be an ideal and π a non-degenerate representation of I on a Hilbert space H. Then there is a unique extension π to a representation $\tilde\pi$ of the entire algebra A on H.*

Proof. Let us first define $\tilde\pi$ on vectors from the dense subspace $\pi(I)H \subset H$ by the formula

$$\tilde\pi(a)\left(\sum_i \pi(j_i)\xi_i\right) := \sum_i \pi(a j_i)\xi_i. \tag{2.3}$$

This is well-defined because if $\sum_i \pi(j_i)\xi_i = \sum_i \pi(j_i')\xi_i'$, then

$$\tilde\pi(a)\left(\sum_i \pi(j_i)\xi_i\right) = \lim_{\lambda\in\Lambda} \tilde\pi(a)\left(\sum_i \pi(u_\lambda j_i)\xi_i\right) = \lim_{\lambda\in\Lambda} \pi(a u_\lambda)\left(\sum_i \pi(j_i)\xi_i\right)$$

and, similarly, $\tilde{\pi}(a)(\sum_i \pi(j_i')\xi_i') = \lim_{\lambda\in\Lambda} \pi(au_\lambda)(\sum_i \pi(j_i')\xi_i')$, where $u_\lambda \in I$ is an approximate unit of I. Note that the existence of the last limit in the chain follows from the existence of the penultimate limit. Hence, for each of the two cases it should be proved separately. Since

$$\begin{aligned}\left\|\tilde{\pi}(a)\left(\sum_i \pi(j_i)\xi_i\right)\right\| &= \lim_{\lambda\in\Lambda}\left\|\pi(au_\lambda)\left(\sum_i \pi(j_i)\xi_i\right)\right\| \leqslant \sup_{\lambda\in\Lambda}\|\pi(au_\lambda)\|\cdot\left\|\sum_i \pi(j_i)\xi_i\right\| \leqslant \\ &\leqslant \|a\|\cdot\sup_{\lambda\in\Lambda}\|u_\lambda\|\cdot\left\|\sum_i \pi(j_i)\xi_i\right\| = \|a\|\cdot\left\|\sum_i \pi(j_i)\xi_i\right\|,\end{aligned}$$

$\tilde{\pi}$ is bounded, i.e., $\tilde{\pi}(a)$ extends to a bounded operator in H.

At the same time, it is easy to check $\tilde{\pi}(ab) = \tilde{\pi}(a)\tilde{\pi}(b)$ and $\tilde{\pi}(a^*) = \tilde{\pi}(a)^*$ for any $a, b \in A$, so $\tilde{\pi}$ is a representation of A. Uniqueness follows from the fact that any extension of π has to satisfy (2.3). □

Lemma 2.38. *Under the conditions of Lemma 2.37 the representation π is irreducible if and only if $\tilde{\pi}$ is irreducible.*

Proof. Let π be reduced by a proper invariant subspace $L \subset H$. Then, due to non-degeneracy, $H = \overline{\pi(I)(L + L^\perp)} \subseteq \overline{\pi(I)L} + \overline{\pi(I)L^\perp}$. Since $L^\perp$ is also invariant, then $\pi(I)L^\perp \subset L^\perp$, so $\overline{\pi(I)L} = L$. Then $\tilde{\pi}(A)L = \tilde{\pi}(A)\overline{\pi(I)L} = \overline{\pi(I)L} = L$ and L reduces $\tilde{\pi}$. The opposite statement is trivial. □

Lemma 2.39. *Let π be a representation of A on a Hilbert space H, and $I \subset A$ is an ideal. Then the orthogonal projection p onto $\overline{\pi(I)H}$ lies in the center of $\pi(A)''$. If π is irreducible and $\pi(I) \neq 0$, then $\pi|_I$ is also irreducible.*

Proof. Since $\pi(A)\pi(I)H = \pi(I)H$, then $\overline{\pi(I)H}$ is an invariant space for $\pi(A)$, hence $p \in \pi(A)'$ (see the end of proof of Lemma 2.3). If $x \in \pi(I)'$, then $x\pi(j)\xi = \pi(j)x\xi \in \pi(I)H$ for any $j \in I$, $\xi \in H$, so pH is an invariant subspace of $\pi(I)'$ and, therefore, $p \in \pi(I)''$. So,

$$p \in \pi(I)'' \cap \pi(A)' \subset \pi(A)'' \cap \pi(A)',$$

that is the center of $\pi(A)''$.

If π is irreducible, then p is a scalar operator (that is, 0 or 1) (cf. Lemma 2.3), and since $\pi(I) \neq 0$, then $p = 1$. Thus, $\pi|_I$ is non-degenerate. So by Lemma 2.38 it is irreducible. □

Chapter 3

Special classes of C^*-algebras

Lecture 11

3.1 C^*-algebra of compact operators

In this section we will consider C^*-subalgebras of C^*-algebra $\mathbb{K}(H)$ of compact operators on the Hilbert space H. We will say that C^*-subalgebra of the algebra $\mathbb{B}(H)$ *irreducible*, if its identical representation is irreducible.

Definition 3.1. The projection p is called *minimal*, if there is no projection $q \neq 0$, $q \neq p$ such that $qp = q$. In other words, p does not *dominate* any non-trivial projection.

Lemma 3.2. *Any nonzero C^*-algebra A consisting of compact operators contains a minimal projection e and $eAe = \mathbb{C} \cdot e$. If A is irreducible, then e is a rank 1 projection (as a projection in Hilbert space).*

Proof. Since A is nonzero, it contains a nonzero positive operator (see (1.7)), which (as is known from the basic course of functional analysis, see [5, Theorem 1, p. 360]), has a discrete spectrum (except of 0) with eigenvalues of finite multiplicities. Let us consider the spectral projection for a non-zero point of the spectrum. Since the characteristic function of this isolated point is continuous on the spectrum, then this projection belongs to A. Then among the nonzero projections dominated by it there is some projection $e \in A$ of minimal rank among the dominated (since they have finite ranks). Then e is minimal (the uniqueness of the minimal and even the equality of ranks of different minimal projections is not supposed). If eAe consists not only of $\mathbb{C} \cdot e$, then in the same way we can construct a projection dominated by e and arrive to a contradiction.

Now suppose that A is irreducible, but the rank of e is greater than 1. Let us choose a pair of nonzero orthogonal vectors ξ, η in the image e. Since for any a there is a number $\lambda \in \mathbb{C}$ such that $eae = \lambda e$, we have $(\xi, a\eta) = (e\xi, ae\eta) = (\xi, eae\eta) = \lambda(\xi, \eta)$, that is $a\eta \perp \xi$ for any $a \in A$. Considering all ξ from the image e being orthogonal to η, we see that the subspace $\overline{A\eta}$ is a proper invariant subspace. A contradiction. $\square$

Lemma 3.3. *The only irreducible C^*-subalgebra of $\mathbb{K}(H)$ is itself.*

Proof. Let A be an irreducible C^*-subalgebra of $\mathbb{K}(H)$, and $e \in A$ a minimal projection of rank 1. Then there is a unit vector $\xi \in H$ such that $e\eta = \xi(\xi, \eta)$ for any η (we take ξ from the image of e). Due to irreducibility, for any $\eta, \zeta \in H$ there are elements $a, b \in A$ such that $a\xi = \eta$, $b\xi = \zeta$ (see Lemma 2.5). Moreover, $A \ni aeb^*$ and $aeb^*(\kappa) = a\xi(\xi, b^*\kappa) = \eta(\zeta, \kappa)$, $\kappa \in H$. Thus, A contains all operators of rank 1. Such operators generate $\mathbb{K}(H)$ (any compact operator is approximated by finite-dimensional), so $A = \mathbb{K}(H)$. □

Corollary 3.4. *The algebra* $\mathbb{K}(H)$ *is simple.*

Proof. Since $\mathbb{K}(H)$ is irreducible, then any non-zero ideal is also irreducible (by Lemma 2.39), so it coincides with $\mathbb{K}(H)$ (by Lemma 3.3). □

Corollary 3.5. *Let A be an irreducible C^*-subalgebra of $\mathbb{B}(H)$ containing a nonzero compact operator. Then $\mathbb{K}(H) \subseteq A$.*

Proof. Since $A \cap \mathbb{K}(H)$ is a nonzero ideal of A, it is irreducible by Lemma 2.39. By Lemma 3.3 this subalgebra of $\mathbb{K}(H)$ should coincide with the entire $\mathbb{K}(H)$. □

3.2 AF-algebras

Definition 3.6. Let us call a C^*-algebra an *AF-algebra* (approximately finite-dimensional), if it is the closure of the union of an increasing sequence of its finite-dimensional C^*-subalgebras.

Problem 49. Prove that the matrix algebra M_n is simple for any n (this does not follow from Lemma 2.34, from which one can deduce that M_n is simple for some n). *Hint:* for any ideal $I \neq \{0\}$ consider a matrix from it with $a_{ij} \neq 0$. By multiplying on the left and right by matrices with 1's in one place and zeros in the rest, you obtain a matrix of I with a single nonzero element a_{ij}. Multiplying by permutation matrices, get similar matrices with all possible i, j. Their linear combinations give the entire M_n algebra.

Problem 50. Deduce from Problem 49 and Lemma 2.34 that the image of the matrix algebra M_n under a $*$-homomorphism is either a zero algebra or an algebra isomorphic to M_n.

Problem 51. Prove the following almost obvious fact: if p and q are projections of the same rank in M_n, then there exists a unitary matrix u such that $q = u^*pu$.

Lemma 3.7. *Let $\varphi : M_n \to M_k$ be a non-zero $*$-homomorphism, so that $p := \varphi(1_n)$ is a self-adjoint projection, where 1_n is the unit of M_n. Then $\mathrm{rk}(p) = \mathrm{Trace}(p)$ is divided by $n = \mathrm{rk}(1_n) = \mathrm{Trace}(1_n)$.*

Proof. Consider some one-dimensional orthogonal (self-adjoint) projection $e \in M_n$. Then $\varphi(e)$ is a self-adjoint projection in M_k. Its rank does not depend on the choice of e, since any other e' is equal to u^*eu (by problem 51), where u is unitary, so

$$\mathrm{Trace}(\varphi(e')) = \mathrm{Trace}(\varphi(u^*eu)) = \mathrm{Trace}(\varphi(u^*)\varphi(e)\varphi(u)) =$$

$$= \mathrm{Trace}(\varphi(u)\varphi(u^*)\varphi(e)) = \mathrm{Trace}(\varphi(uu^*e)) = \mathrm{Trace}(\varphi(e)).$$

If this (one for all) rank is zero, then φ would be zero. This means it is equal to $c \geqslant 1$. Let us now consider an orthonormal basis $e_1, \dots, e_n$ (for example, canonical) in $\mathbb{C}^n$ and denote the corresponding one-dimensional orthoprojections by $[e_i]$, so $[e_j]\,[e_i] = 0$ for $i \neq j$. Then, since $\varphi([e_i])\varphi([e_j]) = \varphi([e_i e_j]) = 0$ for $i \neq j$, we get

$$\mathrm{Trace}(p) = \mathrm{rk}(\varphi(1_n)) = \mathrm{rk}(\varphi([e_1] \oplus \cdots \oplus [e_n])) = \mathrm{rk}(\varphi([e_1])) + \cdots + \mathrm{rk}(\varphi([e_n])) = cn.$$

□

Definition 3.8. The ratio $c := \frac{\mathrm{rk}(p)}{n}$ will be called *multiplicity* of φ.

Along with the standard left action of M_n on $\mathbb{C}^n$, we consider the left action of M_n on itself by multiplication, so the canonical expansion

$$M_n \cong \underbrace{\mathbb{C}^n \oplus \cdots \oplus \mathbb{C}^n}_{n \text{ times}} = M_n[e_1] \oplus \cdots \oplus M_n[e_n]$$

is a decomposition into simple modules (=irreducible representations), where $[e_i] \in M_n$ is an orthogonal projection onto the basis vector e_i of the standard basis. Another way to write it is $[e_i] = e_i \otimes e_i^*$ (considering matrices as endomorphisms), where e^* is the Hermitian conjugate functional for e, so $[e_i]v = (e_i \otimes e_i^*)v = e_i(e_i, v)$. For different vectors we get the matrix unit $e_{ij} = e_i \otimes (e_j)^*$, so $[e_i] = e_{ii}$.

Lemma 3.9. *Any irreducible left module M in M_n has the form $M_n(g \otimes f^*) = \mathbb{C}^n \otimes f^*$, where g, f are some unit (can be taken to be unit) vectors.*

Proof. For the left action, the module $M_n(g \otimes f^*) = \mathbb{C}^n \otimes f^*$ is isomorphic to $\mathbb{C}^n$ with the standard action, and therefore is irreducible. Therefore, if $g \otimes f^* \in M$, then $M = M_n(g \otimes f^*)$. It remains to show that M contains an element of the form $g \otimes f^*$. But this form describes any operator of rank 1. Indeed, if a is an operator of rank 1, then we must take as f the unit vector perpendicular to its kernel, and $g = a(f)$. Finally, if M is nonzero and $0 \neq b \in M$, then choose $f \neq 0$ from its image. Then $(f \otimes f^*)b$ is an operator of rank 1 from M. □

Lecture 12

Theorem 3.10. *Let φ be a (unital) $*$-automorphism of the C^*-algebra M_n. Then it is inner: $\varphi(a) = vav^*$ for any a, where $v \in M_n$ is unitary.*

Proof. Note that φ is an isomorphism between M_n, considered as a left module over M_n with the standard action $a \cdot x$, and M_n, considered as a module with the action $a * x = \varphi(a) \cdot x$, since $\varphi(a \cdot x) = \varphi(a) \cdot \varphi(x) = a * \varphi(x)$. Since $\varphi(M_n) = M_n$, then the invariant and irreducible modules for both actions are the same (the latter are described by Lemma 3.9), and $\varphi(\mathbb{C} \otimes e_i) = \mathbb{C} \otimes h_i$, moreover, since the automorphism takes a direct sum to a direct sum, then h_i form a basis in $\mathbb{C}^n$ and, thus, an isomorphism $u : \mathbb{C}^n \to \mathbb{C}^n$ is defined by $u : e_i \to h_i$ (even if we assume $\|h_i\| = 1$, then u is uniquely defined only up to multiplication by a diagonal matrix of complex numbers modulo one). Thus, $\varphi(e_{ij}) = r_{ij} \otimes h_j$, where $r_{ij} \in \mathbb{C}^n$ are some elements. Similar reasoning with right modules shows that $\varphi(e_{ij}) = g_i \otimes (s_{ij})^*$, where $v : \mathbb{C}^n \to \mathbb{C}^n$, $v : e_i \to g_i$, is an isomorphism, and $s_{ij} \in \mathbb{C}^n$ are some elements. From these two relations we obtain that $\varphi(e_{ij}) = \lambda_{ij} g_i \otimes (h_j)^*$, where λ_{ij} are some numbers. Moreover (see the proof of Lemma 3.7) $\varphi([e_i]) = \lambda_{ii} g_i \otimes (h_i)^*$ is a self-adjoint projection, so $h_i = \mu g_i$ and $g_i = (\lambda_{ij}\overline{\mu})(g_i \otimes (g_i)^*)(g_i) = (\lambda_{ij}\overline{\mu})g_i$ and $\lambda_{ij}\overline{\mu} = 1$. Thus, we can assume that $h_i = g_i$, $u = v$ and (new) λ_{ij} satisfy $\lambda_{ii} = 1$ for any i. In this case, g_i form an orthonormal basis (see the proof of Lemma 3.7), so $u : e_i \mapsto g_i$ is unitary. Since the homomorphism φ preserves the equalities $e_{ij}e_{ji} = e_{ii} = [e_i]$ and $e_{ij}^* = e_{ji}$, we obtain that $\lambda_{ij}\lambda_{ji} = 1$, $\overline{\lambda_{ij}} = \lambda_{ji}$, so, in particular, λ_{ij} are complex numbers of norm 1.

Consider the matrix $\Lambda = \|\lambda_{ij}\|$. Passing, if necessary, from vectors g_i to $(\lambda_{1i})^{-1}g_i = \overline{\lambda_{1i}}g_i$ for $i = 2, \dots, n$, we can consider $\lambda_{1i} = \lambda_{i1} = 1$ for $i = 2, \dots, n$. At the same time, the image $\varphi(a)$ of a matrix $a = \|a_{ij}\|$ can be written down with respect to the orthonormal basis $\{g_i\}$ as $\|\lambda_{ij}a_{ij}\|$, and if the matrix a was unitary, then $\varphi(a)$ must also be unitary. Let some $\lambda_{ij} \neq 1$ (that is possible only for $i \neq j$, $i \neq 1$). Taking for these i, j the unitary matrix a with $a_{1i} = 1/\sqrt{2}$, $a_{1j} = 1/\sqrt{2}$, $a_{ii} = 1/\sqrt{2}$, $a_{ij} = -1/\sqrt{2}$ (and the rest in these rows and columns, of course, is formed with zeros), we obtain the orthogonality condition for these rows in the form $0 = (1/\sqrt{2} \cdot 1)(\overline{1/\sqrt{2} \cdot 1}) + (-1/\sqrt{2} \cdot \lambda_{ij})(\overline{1/\sqrt{2} \cdot 1}) = (1 - \lambda_{ij})/2$, so $\lambda_{ij} = 1$. Contradiction.

So, for any i, j we get that $\varphi(e_i \otimes (e_j)^*) = v(e_i) \otimes (v(e_j))^* = v \cdot (e_i \otimes (e_j)^*) \cdot v^*$. Since any matrix is a linear combination of such matrix units, by linearity we obtain the required equality $\varphi(a) = vav^*$. □

Lemma 3.11. *Let the homomorphism φ be unital under the conditions of Lemma 3.7. Then φ is determined by multiplicity up to a unitary equivalence (conjugation) in M_k.*

Proof. Let $f_i^1, \dots, f_i^c$ be an orthonormal basis in the image of the projection $\varphi(e_i)$, $s = 1, \dots, n$. Denoting by $[f_i^j]$ the corresponding one-dimensional pairwise orthogonal projections, we have $\varphi(e_i) = [f_i^1] + \dots + [f_i^c]$. Then $\{f_i^j\}$, $i = 1, \dots, n$, $j = 1, \dots, c$, is an orthobasis $\mathbb{C}^k$, $1_k = \sum_{i,j}[f_i^j]$. If $u \in M_k$ is a unitary matrix taking $\{f_i^j\}$ to the canonical basis of $\mathbb{C}^k$, where $uf_i^j = e_{(i-1)n+j}$, then

$$[e_{(i-1)n+j}]x = e_{(i-1)n+j}(e_{(i-1)n+j}, x) = uf_i^j(uf_i^j, x) = uf_i^j(f_i^j, u^*x) = u[f_i^j]u^*x \qquad (3.1)$$

for any $x \in \mathbb{C}^k$. That is why

$$\varphi([e_i]) = [f_i^1] + \cdots + [f_i^c] = \sum_{j=1}^{c} u^*[e_{(i-1)n+j}]u = u^* \left(\sum_{j=1}^{c} [e_{(i-1)n+j}] \right) u. \tag{3.2}$$

Thus,

$$u\varphi(a)u^* = \begin{pmatrix} \varphi_1(a) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \varphi_c(a) \end{pmatrix} \text{ (block diagonal matrix)}, \tag{3.3}$$

where $\varphi_i : M_n \to M_n$ is a non-zero homomorphism, and therefore an isomorphism ($i = 1, \ldots, c$). Therefore, we can apply Theorem 3.10 to it and find a unitary $v_i \in M_n$, such that $\varphi_i(a) = v_i^* a v^i$. Denoting $v = v_1 \oplus \cdots \oplus v_c$ (a unitary element from M_k), we obtain that $vu\varphi(a)u^*v^* = S_c(a)$ for any $a \in M_n$, where $S_c : M_n \to M_k$, $k = cn$, is the standard homomorphism of multiplicity c:

$$S_c(a) = \begin{pmatrix} a & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a \end{pmatrix} \text{ (block diagonal matrix with } c \text{ blocks equal to } a),$$

as desired. □

Lemma 3.12. *Let φ be a unital $*$-homomorphism of a finite-dimensional C^*-algebra $A = M_{n_1} \oplus \ldots \oplus M_{n_k}$ into a finite-dimensional C^*-algebra $B = M_{m_1} \oplus \ldots \oplus M_{m_l}$. Then φ is given (up to unitary equivalence in B) by some $l \times k$-matrix $C = (c_{ij})$ with nonnegative elements, and $\sum_{j=1}^{k} c_{ij} n_j = m_i$.*

Proof. Let $\epsilon_i : B \to M_{m_i}$ be the canonical epimorphism, and $\sigma_j : M_{n_j} \to A$ the canonical embedding, $i = 1, \ldots, l$, $j = 1, \ldots, k$. Then $\epsilon_i \circ \varphi$ is a unital homomorphism of A into M_{m_i}. Let c_{ij} be the multiplicity of $\varphi_{ij} = \epsilon_i \circ \varphi \circ \sigma_j : M_{n_j} \to M_{m_i}$ in the sense of Definition 3.8.

Note that $\sigma_j(1_{n_j})$ are pairwise orthogonal self-adjoint projections in A with their sum equal to one, so $p_{ij} := \varphi_{ij}(1_{n_j})$ are pairwise orthogonal self-adjoint projections in M_{m_i}, also with their sum equal to one. Therefore, as before, their direct sum is unitarily equivalent with the help of u_i in M_{m_i} to the sum of the corresponding diagonal projections $u_i p_{ij} u_i^*$. Applying Lemma 3.11 to each of $a \mapsto u_i \varphi_{ij}(a) u_i^*$, we find that $\epsilon_i \circ \varphi$ is unitarily equivalent (in M_{m_i}) to the homomorphism $\mathrm{id}_{n_1}^{c_{i1}} \oplus \ldots \oplus \mathrm{id}_{n_k}^{c_{ik}} = S_{c_{i1}} \oplus \cdots \oplus S_{c_{ik}}$. Comparison of dimensions gives the equality $\sum_{j=1}^{k} c_{ij} n_j = m_i$. Since φ is determined by the direct sum $\epsilon_i \circ \varphi$, $i = 1, \ldots, l$, the statement is proven. □

Problem 52. Suppose that in the previous lemma we exclude the requirement of unitality. Prove an analogue of the lemma in this case. Namely, take instead of B the subalgebra $\varphi(A) = p\varphi(A)p$ in B, where $p = \varphi(1_A)$, apply the previous lemma to $\varphi : A \to \varphi(A)$ and obtain the statement of the lemma with inequalities $\sum_{j=1}^{k} c_{ij} n_j \leqslant m_i$ instead of equalities.

Definition 3.13. Any A $*$-homomorphism between finite-dimensional C^*-algebras can be represented in the following graphical way. Let's represent A in the form k-tuple $\{(1,1)=n_1,\dots,(1,k)=n_k\}$ corresponding $A\cong M_{n_1}\oplus\dots\oplus M_{n_k}$, and B — in the form l-tuple $\{(2,1)=m_1,\dots,(2,l)=m_l\}$ corresponding to $B\cong M_{m_1}\oplus\dots\oplus M_{m_l}$. Let us represent φ using arrows between the elements of sets, and from $(1,j)$ to $(2,i)$ we draw c_{ij} arrows, the number is equal to the partial multiplicity. A sequence of such pictures for a sequence of homomorphisms $A_1\subset A_2\subset\dots\subset A_p\subset\cdots$ is called the *Bratteli diagram of this sequence*. It is sometimes called the Bratteli diagram of an algebra, but the same algebra can be obtained from different sequences.

Problem 53. Draw Bratteli diagrams (for some defining sequences) of the following AF-algebras:

1) of the algebra of compact operators $\mathbb{K}(H)$;

2) of its unitization $\mathbb{K}(H)^+$;

3) the closure of the union of $A_p=M_{2^p}$, with embeddings $A_p\subset A_{p+1}$ of multiplicity 2 according to the formula $a\mapsto\begin{pmatrix} a & 0\\ 0 & a\end{pmatrix}$ (CAR algebra);

4) $C(K)$, where K is the Cantor set obtained from $[0,1]$ by successive removing the middle third of the corresponding intervals. If K_p is a set, obtained at the pth step of this process, then A_p is an algebra of continuous functions constant on intervals of K_p;

5) $C(X)$, where $X:=\{0\}\cup\{\frac{1}{n}:n\in\mathbb{N}\}$, and A_k consists of all functions constant on $[0,1/2^k]$.

Lemma 3.14. *If two Bratteli diagrams coincide, then the corresponding AF-algebras are isometrically $*$-isomorphic.*

Proof. Let A_n and B_n be two sequences of finite-dimensional C^*-algebras with inclusions $\alpha_n:A_n\to A_{n+1}$, $\beta_n:B_n\to B_{n+1}$. Since the Bratteli diagrams are the same, then for each n there is an isomorphism $\varphi_n:A_n\to B_n$. Consider $\varphi_{n+1}\circ\alpha_n$ and $\beta_n\circ\varphi_n:A_n\to B_{n+1}$. They can differ only by unitary $u_{n+1}\in B_{n+1}$, that is $\beta_n\circ\varphi_n=\operatorname{Ad}_{u_{n+1}}\varphi_{n+1}\circ\alpha_n$, where $\operatorname{Ad}_{u_{n+1}}(a)=u_{n+1}a(u_{n+1})^*$.

Let's put $\psi_1=\varphi_1$, $v_1=1$. Let us define inductively $v_{n+1}=\beta_n(v_n)u_{n+1}\in B_{n+1}$, $\psi_{n+1}=\operatorname{Ad}_{v_{n+1}}\varphi_{n+1}$. Then

$$\begin{aligned}\beta_n\psi_n &= \beta_n\operatorname{Ad}_{v_n}\varphi_n=\operatorname{Ad}_{\beta_n(v_n)}\beta_n\varphi_n=\operatorname{Ad}_{\beta(v_n)}\operatorname{Ad}_{u_{n+1}}\varphi_{n+1}\alpha_n\\ &= \operatorname{Ad}_{\beta_n(v_n)u_{n+1}}\varphi_{n+1}\alpha_n=\psi_{n+1}\alpha_n.\end{aligned}$$

In this case $\cup_{n=1}^\infty\psi_n:\cup_{n=1}^\infty A_n\to\cup_{n=1}^\infty B_n$ is an isometric $*$-isomorphism, so the closures are also isomorphic. $\square$

One should not think that AF-algebras are "small" and that C^*-subalgebras of AF-algebras are AF-algebras again. For example, $C[0,1]$ is not an AF-algebra (since its only finite-dimensional C^*-subalgebra consists of constant functions), but it is a C^*-subalgebra of the AF-algebra $C(K)$ functions on the Cantor set. Indeed, let f be a function on K that has a dense set of rational numbers in the interval $[0,1]$ as its values. For example, the restriction of the Cantor function $f : [0,1] \to [0,1]$ [8, Ch. VI, §4] on K has all rationals of the form $p/2^s$ as its values. Its spectrum is a closure of this set, that is, equal to the entire interval $[0,1]$. Thus, $C^*(1,f) \subset C(K)$ is isometrically $*$-isomorphic to $C(\mathrm{Sp}(f)) = C[0,1]$.

Lecture 13

3.3 Multipliers

We will call a C^*-subalgebra $\mathbb{B}(H)$ *non-degenerate*, if its natural representation in the Hilbert space H is non-degenerate.

Everywhere in this section $A, B \subset \mathbb{B}(H)$.

Definition 3.15. An operator $x \in \mathbb{B}(H)$ is called *left multiplier* A if $xA \subset A$. It is called *right multiplier*, if $Ax \subset A$ and *double (or double-sided) multiplier* or simply *multiplier*, if both conditions are met.

If A is unital, then every left (right) multiplier lies in A.

Since A is weakly dense in A'', we can proceed to the closure $xA \subset A$ and get $xA'' \subset A''$. If A'' is equal to one, then $x \in A''$.

Definition 3.16. The linear mapping $\sigma : A \to A$ is called *left centralizer*, if $\sigma(ab) = \sigma(a)b$ for any $a, b \in A$. Linear mapping $\sigma : A \to A$ called *right centralizer*, if $\sigma(ab) = a\sigma(b)$ for any $a, b \in A$. Pair (σ_1, σ_2) called *double centralizer*, if σ_1 is a right centralizer, σ_2 is a left centralizer and $\sigma_1(a)b = a\sigma_2(b)$ for any $a, b \in A$.

Lemma 3.17. *Any left centralizer is bounded.*

Proof. Let us assume the opposite. Then for any $n \in \mathbb{N}$ there is an element $x_n \in A$ such that $\|x_n\| < 1/n$ and $\|\sigma(x_n)\| > n$. This means that the series $a = \sum_{n=1}^{\infty} x_n x_n^*$ converges, so $a \in A$ and $x_n x_n^* \leqslant a$. According to Lemma 1.45, x_n can be written as $x_n = a^{1/4}u_n$, where $\|u_n\| \leqslant \|a\|^{1/4}$. Therefore, for any n we have $\|\sigma(x_n)\| = \|\sigma(a^{1/4})u_n\| \leqslant \|\sigma(a^{1/4})\| \cdot \|a^{1/4}\|$. A contradiction. $\square$

Theorem 3.18. *If A is non-degenerate, then there is a bijective correspondence between left (right, double) multipliers and left (right, double) centralizers.*

Proof. If x is a left multiplier, then the mapping $A \ni a \mapsto xa \in A$ is a left centralizer. If $xa = ya$ for any $a \in A$, then $(x - y)a = 0$ for any $a \in A''$, so $x = y$ in A''.

Let σ be a left centralizer, and u_λ be an approximate unit for A. Since the directed net $\{\sigma(u_\lambda)\}$ is bounded, then it has a point of accumulation in A'' (bounded sets are weakly compact in $\mathbb{B}(H)$ and accumulation points must lie in the closure of A). Let us denote one of the accumulation points by x. For any $a \in A$, the directed net $\{u_\lambda a\}$ converges in norm to a, so that $\sigma(u_\lambda a) = \sigma(u_\lambda)a$ converges to $\sigma(a)$. Then $xa = \sigma(a) \in A$, so x is a left multiplier. If $xA = 0$, then $\sigma = 0$. Note that if $y \in A''$ is another accumulation point, then $xa = \sigma(a) = ya$ for any $a \in A$, and $(x - y)a = 0$ for any $a \in A''$ (due to the strong density of A in A''), so $x = y$ in A''. Therefore, in this case there is only one point of accumulation.

A similar proof works also for right multipliers and right centralizers.

Let now x be a double multiplier. Then the mappings $\sigma_2 : a \mapsto xa$ and $\sigma_1 : a \mapsto ax$ are left and right multipliers, with $\sigma_1(a)b = (ax)b = a(xb) = a\sigma_2(b)$ for any $a, b \in A$, so x

defines a double centralizer. Conversely, if (σ_1, σ_2) is a double centralizer, then, by what has been proven, σ_1 determines a right multiplier x_1, and σ_2 a left multiplier x_2. Since $ax_1b = \sigma_1(a)b = a\sigma_2(b) = ax_2b$ for any $a, b \in A$, we have $x_1 = x_2$, and $x_1 = x_2$ is a double multiplier. $\square$

Problem 54. Let $\pi : A \to \mathbb{B}(H)$ be a degenerate representation. Let us denote by H_0 the invariant subspace $H_0 := \{\xi \in H \colon \pi(a)(\xi)0 \text{ for any } a \in A\}$. Prove that π induces a representation $\pi' : A \to \mathbb{B}(H/H_0)$, and if π was a faithful representation (an injective homomorphism), then so is π'.

Remark 3.19. Accordingly, until the end of this section we will consider non-degenerate $A \subseteq \mathbb{B}(H)$, so that (double) multipliers coincide with double centralizers.

The set of all left (right) multipliers of A is denoted by $LM(A)$ ($RM(A)$), and the set of all multipliers of A by $M(A)$.

Problem 55. Check that $RM(A) = (LM(A))^*$ and that $M(A) = LM(A) \cap RM(A)$, so that $M(A)$ is symmetric with respect to the involution.

It follows directly from the definition that all three sets are norm closed. Thus, $M(A)$ is a C^*-algebra (and the other two are, in the general case, only Banach algebras).

Problem 56. Let X be a locally compact space, and let $C_0(X)$, as before, be the C^*-algebra of continuous functions tending to 0 at infinity. Prove that the algebra $M(C_0(X)) \subset L^\infty(X)$ can be identified with the C^*-algebra $C_b(X)$ of all bounded continuous functions on X.

Example 3.20. If $A = \mathbb{K}(H)$, then $M(\mathbb{K}(H)) \subseteq \mathbb{B}(H)$, but any bounded operator is a multiplier (since $\mathbb{K}(H)$ is an ideal in $\mathbb{B}(H)$), so $M(\mathbb{K}(H)) = \mathbb{B}(H)$.

Definition 3.21. An ideal $A \subset B$ is said to be *essential* if any nonzero ideal B has a nontrivial intersection with A.

Let $A^\perp \subset B$ denote the set $A^\perp = \{x \in B \colon Ax = 0\}$.

Lemma 3.22. *An ideal $A \subset B$ is essential if and only if $A^\perp = 0$.*

Proof. Suppose that $A^\perp = 0$, but A is not essential. Then there is a nonzero ideal $J \subset B$ such that $A \cap J = \{0\}$. Let us take $j \in J$, $j \neq 0$. For any $a \in A$ we have $ja \in J \cap A$, so $ja = 0$ and $0 \neq j \in A^\perp$. A contradiction.

Conversely, let A be essential, but $A^\perp \neq 0$. Then there is an element $x \in A^\perp$ such that $x \neq 0$. Consider the ideal BxB (the closure of the set of all linear combinations of elements of the form $\sum_i b_i x b_i'$, $b_i, b_i' \in B$) and take an arbitrary $y \in BxB \cap A$. As it is known (for example, from Lemma 1.45), any element of the C^*-algebra admits a decomposition into the product of two of its elements, so we can write $y = z \cdot a$, where $z, a \in BxB \cap A$. We write $z = \sum_i b_i x b_i'$, so $y = za = \sum_i b_i x (b_i' a) = 0$, since $b_i' a \in A$, hence $x b_i' a = 0$, because $x \in A^\perp$. Therefore, $BxB \cap A = 0$ and we arrive to a contradiction. $\square$

Lemma 3.23. *Let $A \subset B$ be an essential ideal. Then there is an embedding $B \subset M(A)$ that is identical on A.*

Proof. Consider $b \in B$. Then b defines the left and right centralizer A (since A is an ideal) $\sigma_2 : a \mapsto ba$ and $\sigma_1 : a \mapsto ab$, and $\sigma_1(a)a' = (ab)a' = a(ba') = a\sigma_2(a')$ for any $a, a' \in A$, so b defines a double centralizer, and hence a multiplier. So the mapping $\pi : B \to M(A)$ is defined, identical on A. This mapping is obviously a $*$-homomorphism. It remains to check whether π is injective. If $b \in \operatorname{Ker} \pi$, then $\sigma_1 = 0$ and $\sigma_2 = 0$ (see proof of theorem 3.18), so $bA = 0$, $Ab = 0$ and $b \in A^\perp$, which means $b = 0$. □

Note that the correspondence $A \mapsto M(A)$ is not a functor. For example, for $A = \mathbb{K}(H)$ and $B = A^+$, consider the embedding $A \subset B$. Wherein $M(A) = \mathbb{B}(H)$, and $M(B) = B$. Obviously, the embedding does not continue to these multiplier algebras. However, in some cases the transition to multipliers has some functorial properties.

Lemma 3.24. *Let $\varphi : A \to B$ be a surjective $*$-homomorphism of two C^*-algebras. Then it continues to a $*$-homomorphism $\bar\varphi : M(A) \to M(B)$.*

Proof. Let $\sigma \in LM(A)$ be a left centralizer. For any $b \in B$ we set $\bar\varphi(\sigma)(b) := \varphi(\sigma(\varphi^{-1}(b)))$. It is necessary check that the map is well defined, that is, its independence of the choice of a representative in $\varphi^{-1}(b) \subset A$. Due to linearity, it suffices to check that σ maps $\operatorname{Ker} \varphi$ to itself. Let $a \in \operatorname{Ker} \varphi$. Let us represent it in the form $a = a_1 \cdot a_2$, $a_1, a_2 \in \operatorname{Ker} \varphi$. Then $\varphi(\sigma(a)) = \varphi(\sigma(a_1)a_2) = 0$, since $\operatorname{Ker} \varphi$ is an ideal.

Thus, a left centralizer σ defines the mapping $\bar\varphi(\sigma)$, which is the left centralizer of B. The same is done for right and double centralizers (Problem 57). □

Problem 57. Prove that $\bar\varphi(\sigma)$ is a left centralizer. Construct an extension of a right centralizer in a similar way. Check that for a double centralizer we get a double centralizer.

Problem 58. Check that in the situation of the previous lemma the extension $\bar\varphi$ is also surjective.

Problem 59. Prove that a representation $\pi : A \to \mathbb{B}(H)$ is non-degenerate if and only if for some approximate unit u_λ of the algebra A the following condition is satisfied: for any vector $\xi \in H$ there is a λ such that that $u_\lambda(\xi) \neq 0$.

Lemma 3.25. *Let $B \subset A$ be an algebra and its C^*-subalgebra with a common approximate unit u_λ. Then $M(B) \subset M(A)$.*

Proof. If A is non-degenerate, then B is also non-degenerate by Problem 59. So $M(B) \subset B'' \subset A''$. For any $x \in A$, $y \in M(B)$, $yx = \lim yu_\lambda x \in A$, and similarly, $xy \in A$, so y is a multiplier of A. □

Lecture 12

3.4 Hilbert C^*-modules

Definition 3.26. Let M be a Banach space (with norm $\|\cdot\|$), which at the same time is a right module over a C^*-algebra A (the action of A on M is assumed to be continuous). Let $\langle \cdot, \cdot \rangle : M \times M \to A$ be a sesquilinear form (called an *inner product*) with the properties:

1. $\langle m, na \rangle = \langle m, n \rangle a$ for all $m, n \in M$, $a \in A$;

2. $\langle m, n \rangle = \langle n, m \rangle^*$ for all $m, n \in M$;

3. $\langle m, m \rangle \in A$ is positive for all $m \in M$, and if it is equal to 0, then $m = 0$.

We call M a *Hilbert C^*-module* if $\|m\|^2 = \|\langle m, m \rangle\|$ for every $m \in M$.

Example 3.27. The algebra A is a Hilbert C^*-module over A if we define its inner product by the formula $\langle a, b \rangle := a^* b$, $a, b \in A$.

Example 3.28. The module A^n is a Hilbert C^*-module over A with an inner product, given by the formula $\langle (a_1, \dots, a_n), (b_1, \dots, b_n) \rangle := \sum_{i=1}^n a_i^* b_i$.

Lemma 3.29 (Cauchy(-Schwarz-Bunyakovsky) inequality)**.** *We have $\|\langle n, m \rangle\|^2 \leqslant \|n\|^2 \|m\|^2$ for any $n, m \in M$.*

Proof. For all $A \in A$ we have $\langle m - na, m - na \rangle \geqslant 0$, so $\langle m, m \rangle - a^* \langle n, m \rangle - \langle m, n \rangle a + a^* \langle n, n \rangle a \geqslant 0$. Let's take $a = \frac{1}{\|n\|^2} \langle n, m \rangle$. Then

$$\langle m, m \rangle - \frac{2}{\|n\|^2} \langle m, n \rangle \langle n, m \rangle + \frac{1}{\|n\|^4} \langle m, n \rangle \langle n, n \rangle \langle n, m \rangle \geqslant 0.$$

Since $\|n\|^2 = \|\langle n, n \rangle\| \geqslant \langle n, n \rangle$, we obtain that $\langle m, m \rangle - \frac{1}{\|n\|^2} \langle m, n \rangle \langle n, m \rangle \geqslant 0$, so $\langle m, n \rangle \langle n, m \rangle \leqslant \|n\|^2 \langle m, m \rangle$. From this we obtain the required inequality. $\square$

Example 3.30. Let M consist of all sequences (a_i), $a_i \in A$ for which the series $\sum_{i=1}^\infty a_i^* a_i$ converges in A (in norm). The inner product is given by formula $\langle (a_i), (b_i) \rangle := \sum_{i=1}^\infty a_i^* b_i$. By the previous lemma, this series converges. This Hilbert C^* module is very important for applications. It is usually denoted by $l_2(A)$ and is called the *standard* Hilbert C^*-module.

A mapping $T : M \to M$ is called a (bounded) *operator* on a Hilbert C^*-module M if it is linear and A-linear (that is, $T(ma) = T(m)a$ for any $m \in M$, $a \in A$). If $M = A$, then this definition coincides with the definition of left centralizers.

Definition 3.31. An operator T is called *adjointable* if there is an operator S such that $\langle m, T(n) \rangle = \langle S(m), n \rangle$ for any $m, n \in M$. In this case, S is called the *adjoint operator* for T and is denoted by $T^\star$. Let $\mathbb{B}_A^\star(M)$ denote the set of operators admitting an adjoint.

Unlike Hilbert spaces, in Hilbert C^*-modules there are bounded operators that do not admit an adjoint.

Problem 60. Construct an example of an operator that does not admit an adjoint.

Problem 61. Prove that $\|x\| = \sup_{y\in B_1(M)} |\langle x, y\rangle|$, where $B_1(M) \subset M$ is the unit ball.

Theorem 3.32. *The algebra $\mathbb{B}_A^\star(M)$ is a C^*-algebra.*

Outline of proof. The following points are the key ones.

1) The involution $\star : \mathbb{B}_A^\star(M) \to \mathbb{B}_A^\star(M)$ is an isometry. Indeed, by Problem 61,

$$\|T\| = \sup_{x\in B_1(M)} \|Tx\| = \sup_{x,y\in B_1(M)} \|\langle Tx, y\rangle\| =$$

$$= \sup_{x,y\in B_1(M)} \|\langle x, T^\star y\rangle\| = \sup_{y\in B_1(M)} \|T^\star y\| = \|T^\star\|.$$

2) The norm satisfies the C^*-property. This follows from the equality

$$\|T^\star T\| \geqslant \sup_{x\in B_1(M)} \|\langle T^\star Tx, x\rangle\| = \sup_{x\in B_1(M)} \|\langle Tx, Tx\rangle\| = \|T\|^2$$

(in the opposite direction is a general property of the operator norm, taking into account item 1).

3) The algebra $\mathbb{B}_A^\star(M)$ is closed as a subalgebra of the Banach algebra B of all bounded $\mathbb{C}$-linear operators $M \to M$ (with operator norm). Really, first of all, note that the Banach algebra $\mathbb{B}_A(M)$ of all operators is closed in B as the intersection over all $x \in M$, $a \in A$, of closed sets $\operatorname{Ker}(f_{x,a})$, where $f_{x,a} : B \to M$, $f_{x,a}(T) = T(xa) - T(x)a$, is a bounded linear mapping, $\|f_{x,a}\| \leqslant 2\|x\|\,\|a\|$. Let a directed net of elements $T_\alpha \in \mathbb{B}_A^\star(M)$ converge to $T \in \mathbb{B}_A(M)$, in particular, be a Cauchy directed net. According to item 1), the directed net $T_\alpha^\star$ is also Cauchy directed net, and therefore has a limit $S \in \mathbb{B}_A(M)$. It is easy to see that $S = T^\star$ and $T \in \mathbb{B}_A^\star(M)$. $\square$

Problem 62. Complete the proof of Theorem 3.32.

If $M = A$, then the definition of an operator admitting an adjoint coincides with the definition of a double centralizer.

Definition 3.33. The operator $\theta_{x,y}$, defined by the formula $\theta_{x,y}(z) = x\langle y, z\rangle$, called *elementary.*

Problem 63. Prove that $\theta_{x,y} \in \mathbb{B}_A^\star(M)$ by providing an explicit formula for the adjoint.

Definition 3.34. The closure of the set of $\mathbb{C}$-linear combinations of elementary operators is denoted by $\mathbb{K}_A(M)$. Its elements are called *A-compact* operators.

Problem 64. Prove that $\mathbb{K}_A(M)$ is an ideal in $\mathbb{B}_A^\star(M)$.

Problem 65. Prove that $\mathbb{K}_A(A) = A$. Note that if A is non-unital, then $\mathbb{K}_A(A) = A \neq \mathbb{B}_A^\star(A) = DC(A)$ (algebra of double centralizers).

3.5 Calkin algebra

If a Hilbert space H is not separable, then $\mathbb{B}(H)$ can be quite complex, in particular, have more than just one proper ideal. For example, the set of operators with separable image is an ideal. This does not happen in the case of separable H, which we will usually restrict ourselves to.

Definition 3.35. The factor-C^*-algebra $Q(H) = \mathbb{B}(H)/\mathbb{K}(H)$ is called *Calkin algebra.*

Definition 3.36. The factor-C^*-algebra $M(A)/A$ is called *algebra of external multipliers,* or *generalized Calkin algebra.*

Lemma 3.37. *Calkin algebra is simple.*

Proof. We must prove that $\mathbb{K}(H)$ is the only ideal of $\mathbb{B}(H)$. Let an ideal $I \subset \mathbb{B}(H)$ contain at least one non-compact operator. We can assume that this operator is positive (since any operator is a linear combination of four positive ones). Let us denote it by t. Note that since $\mathbb{K}(H)$ is simple (by Corollary 3.4), then either $\mathbb{K}(H) \subseteq I$ or $\mathbb{K}(H) \cap I = \{0\}$. Since $t \notin \mathbb{K}(H)$, then we can assume that there is a number $\alpha > 0$ such that both spectral projections $p_1 = p_{[0,\alpha)}$ and $p_2 = p_{[\alpha,\infty)}$ are of infinite rank. Indeed, if there is no p_2 with the specified property, then t is compact, contrary to assumption. If $p_1 = 0$, then t is invertible, which means $I = \mathbb{B}(H)$, as required. If $p_1 \neq 0$, but its rank is finite, then we have a finite number of eigenvalues, less than α, so that for some function f continuous on the spectrum, we have $0 \neq f(t) \in \mathbb{K}(H)$. Hence, $\mathbb{K}(H) \subseteq I$. Therefore $p_1 \in I$ and $t + p_1 \in I$ is invertible and $I = \mathbb{B}(H)$. So, both projections p_1 and p_2 are of infinite rank. Let $H_i = \operatorname{Im} p_i$, $i = 1, 2$. Then $H_1 \perp H_2$ and $H = H_1 \oplus H_2$. Since H_1 and H_2 are isomorphic due to the separability of H, then there is a partial isometry $w \in \mathbb{B}(H)$ such that $w|_{H_2} = 0$ and $w|_{H_1}$ maps H_1 isometrically onto H_2. Since spectral projections commute with t, then $tp_2 \geqslant \alpha \cdot 1_{H_2}$, where 1_{H_2} is the identity operator in H_2. Also, $w^*tw \geqslant \alpha \cdot 1_{H_1}$, so $tp_2 + w^*tw \geqslant \alpha \cdot 1$. Therefore $tp_2 + w^*tw \in I$ is invertible, so $I = \mathbb{B}(H)$. $\square$

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