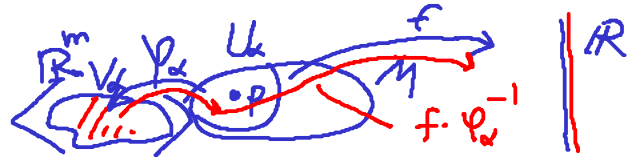
smooth m.



19.09.2022 **Definition 2.9.** A function $f : M \to \mathbf{R}$ is called *smooth,* if, for any point $P \in M$ and some chart $(U_\alpha, \varphi_\alpha)$ with $P \in U_\alpha$, the function $f \circ \varphi_\alpha^{-1} : V_\alpha \to \mathbf{R}$, defined on an open set in $\mathbf{R}^m$, is smooth.

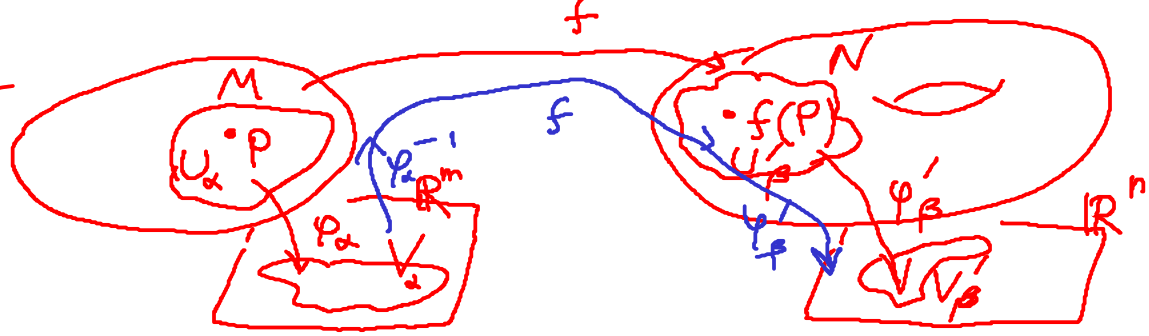
Home **Problem 2.10.** Prove that this definition does not depend on the choice of a chart (from the same maximal atlas).

Definition 2.11. A continuous mapping $f : M \to N$ of smooth manifolds is called *smooth,* if for any point $P \in M$ and some charts $(U_\alpha, \varphi_\alpha)$, $P \in U_\alpha$, and $(U'_\beta, \varphi'_\beta)$, $f(P) \in U'_\beta$, (these are charts on M and N, respectively) the mapping $\boxed{\varphi'_\beta \circ f \circ \varphi_\alpha^{-1}} : V_\alpha \to V'_\beta \subset \mathbf{R}^n$ defined on an open set in $\mathbf{R}^m$, is smooth, where $\dim M = m$ and $\dim N = n$.

This mapping is called *local* or *coordinate representative map~~s~~* for f

Home **Problem 2.12.** Verify that if a mapping is continuous w.r.t. some pair of charts, then it is smooth w.r.t. any other (compatible) pair.

$f(U_\alpha) \subseteq U'_\beta$



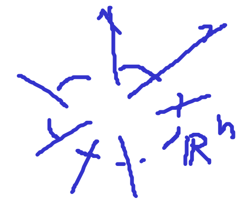
Definition 2.13. A bijective smooth mapping $f : M \to N$ of smooth manifolds is called a *diffeomorphism* , if f^{-1} is smooth.

homeomorphism

Problem 2.14. Verify that the following formulas Home

$$y^k = \frac{x^k}{\sqrt{\varepsilon^2 - (x^1)^2 - (x^2)^2 - \cdots - (x^n)^2}}, \qquad k = 1, \ldots, n,$$

$$x^k = \frac{\varepsilon\, y^k}{\sqrt{1 + (y^1)^2 + (y^2)^2 + \cdots + (y^n)^2}}, \qquad k = 1, \ldots, n,$$



define a diffeomorphism $B_\varepsilon(0) \subset \mathbf{R}^n$ and $\mathbf{R}^n$.

Problem 2.15. Find an example of smooth homeomorphism, which is not a diffeomorphism. Class

Lemma 2.16. *For any smooth manifold M, there exists an atlas such that all V_α (images of coordinate maps) are ~~homeomorphic to~~ open balls (hence by Problem 2.14, to the entire space $\mathbf{R}^m$.)*

φ_α $B_\varepsilon(0) = V_\alpha$ Pr. 2.14 $\mathbb{R}^m$

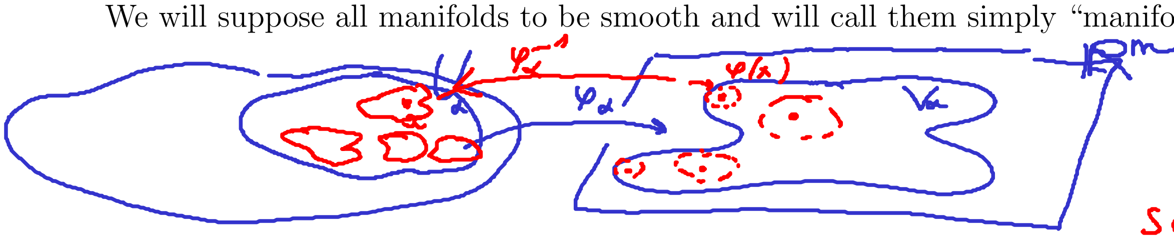
Proof. Let $(U_\alpha, \varphi_\alpha)$ be an atlas of M. For any $x \in M$, we can choose a chart $U_{\alpha(x)} \ni x$. Choose a small $\varepsilon(x)$ such that $B_{\varepsilon(x)}(\varphi_{\alpha(x)}(x)) \subseteq V_{\alpha(x)} \subseteq \mathbf{R}^m$. Then

$$(\widetilde{U}_x, \widetilde{\varphi}_x), \quad x \in M, \qquad \widetilde{U}_x := \varphi^{-1}_{\alpha(x)}(B_{\varepsilon(x)}(\varphi_{\alpha(x)}(x))), \qquad \widetilde{\varphi}_x := \varphi_{\alpha(x)}|_{\widetilde{U}_x},$$

is the desired atlas. □

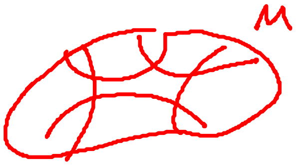
Remark 2.17. For any finite atlas of a compact manifold, there exists a subordinated partition of unity, because this manifold is normal as a topological space.

We will suppose all manifolds to be smooth and will call them simply "manifolds".

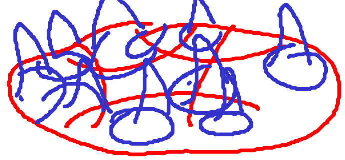


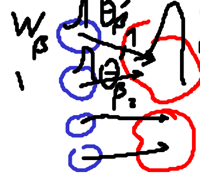
If M is compact $\longrightarrow$ take a finite sub-cover

M



a finite refinement of cover





$\theta'_{\beta_1}+\theta'_{\beta_2}$ U_α

s.t. $\alpha=\alpha(\beta)$ $W_\beta \subseteq U_\alpha$

Theorem 2.18. *For any finite atlas of a compact manifold M, there exists a subordinated smooth partition of unity.*

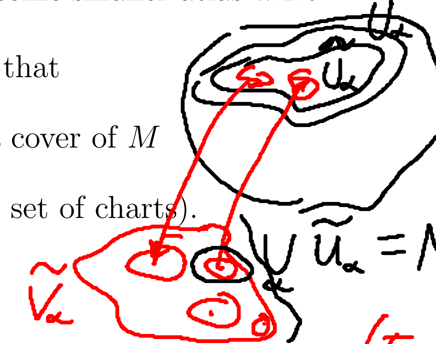
Proof. Remark that it is sufficient to find a smooth partition of unity for a finite refinement o the initial cover by charts (then we simply take some finite sums of functions as the desired partition).

Second, observe that Lemma 2.16 gives rise to a refinement of the initial atlas (we leave finitely many charts by compactness). Moreover, we can do this for some smaller atlas w.r.t the initial one (as in Theorem 1.55).

$\tilde U_\alpha \Subset U_\alpha$

Thus, we need to prove the statement for an atlas (W_β, τ_β) such that

$$\tau_\beta(W_\beta) = B_1(0) \subset \mathbf{R}^m, \qquad W_\beta^\varepsilon := \tau_\beta^{-1}(B_{1-\varepsilon}(0)) \text{ is still a cover of } M$$



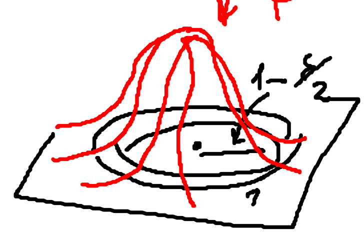
$\bigcup_\alpha \tilde U_\alpha = M$

(these ε's are distinct, but we can take the minimum over this finite set of charts).

Define the following smooth function on $\mathbf{R}^m$:

$$h(x) := \begin{cases} e^{-\dfrac{1}{(1-\varepsilon/2)^2 - \|x\|^2}}, & \text{for } \|x\|^2 < (1-\varepsilon/2)^2, \\ 0, & \text{for } \|x\|^2 \geq (1-\varepsilon/2)^2. \end{cases}$$

τ_β



Then

$$\operatorname{supp} h = \overline{B_{1-\varepsilon/2}(0)}, \qquad 0 \leq h(x) \leq 1, \qquad h(x) > 0 \text{ on } B_{1-\varepsilon}(0).$$

Define

$$\chi_\beta := \begin{cases} h(\tau_\beta(x)), & \text{for } x \in W_\beta, \\ 0, & \text{for } x \notin W_\beta. \end{cases}$$

Then $\chi_\beta \in C^\infty(M)$, $0 \leq \chi \leq 1$, $\operatorname{supp} \chi_\beta \subset W_\beta$ and $\chi_\beta > 0$ on W_β^ε. Hence, $\psi := \sum_\beta \chi_\beta > 0$ and $\psi_\beta := \chi_\beta/\psi$ is a desired C^∞-partition of unity. □

← Jacobi matrix of f

(in this particular case) rk $J(f)$ is maximal.

$M = f^{-1}(y_0) \to f$ $\mathbb{R}$ y_0

Problem 2.19. Prove the existence of a subordinated partition of unity for any locally finite atlas of a (non-compact) manifold. [Lee, Thm. 1.72] Home

$J(f)$

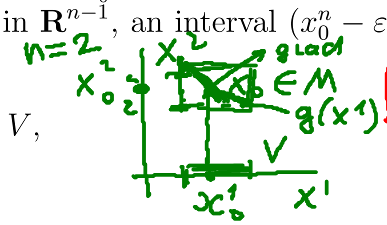
Theorem 2.20. *Let $f : \mathbf{R}^n \to \mathbf{R}$ be a smooth function such that* grad $f = \left(\frac{\partial f}{\partial x^1}, \dots, \frac{\partial f}{\partial x^n}\right) \neq \vec{0}$ *at any point of $M = f^{-1}(y_0)$. Then M is a smooth manifold. Some $n-1$ of $x^1, \dots, x^n$ can be taken as local coordinates (i.e., the corresponding projection is a chart). (Which ones — depends on point.) In particular,* $\dim M = n - 1$.

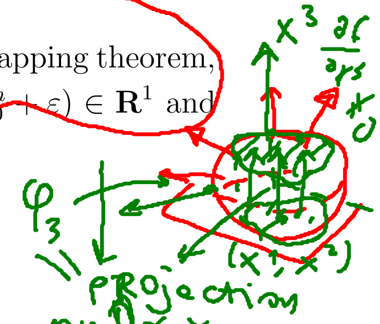
(Ex: $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x^1, x^2, x^3) = (x^1)^2 + (x^2)^2 + (x^3)^2 - 1$, $f^{-1}(0) = S^2$)

Proof. Apply the implicit mapping theorem. Namely, suppose that

$$\vec{x}_0 = (x_0^1, \dots, x_0^{n-1}, x_0^n) \in M, \qquad \text{grad } f_{\vec{x}_0} = \left(\frac{\partial f}{\partial x^1}, \dots, \frac{\partial f}{\partial x^n}\right)\bigg|_{\vec{x}_0} \neq \vec{0}.$$

Without loss of generality one can assume that $\frac{\partial f}{\partial x^n}\big|_{\vec{x}_0} \neq 0$. By the implicit mapping theorem, there is a neighborhood V of $(x_0^1, \dots, x_0^{n-1})$ in $\mathbf{R}^{n-1}$, an interval $(x_0^n - \varepsilon, x_0^n + \varepsilon) \in \mathbf{R}^1$ and C^∞-function $g : V \to \mathbf{R}^1$ such that





1. $f(x^1, \dots, x^{n-1}, g(x^1, \dots, x^{n-1})) \equiv 0$ on V,
2. $g(x_0^1, \dots, x_0^{n-1}) = x_0^n$,
3. $g(x^1, \dots, x^{n-1}) \in (x_0^n - \varepsilon, x_0^n + \varepsilon)$ for $(x^1, \dots, x^{n-1}) \in V$,
4. any point $(x^1, \dots, x^n) \in M \cap (V \times (x_0^n - \varepsilon, x_0^n + \varepsilon))$ is defined by $x^n = g(x^1, \dots, x^{n-1})$.

$g(x^1, x^2) = \sqrt{1 - (x^1)^2 - (x^2)^2}$

Define a chart:

$$U := M \cap (V \times (x_0^n - \varepsilon, x_0^n + \varepsilon)), \quad \varphi : U \to \mathbf{R}^{n-1}, \ \varphi(x^1, \dots, x^n) := (x^1, \dots, x^{n-1}) \in V.$$

$\varphi : U \approx V.$

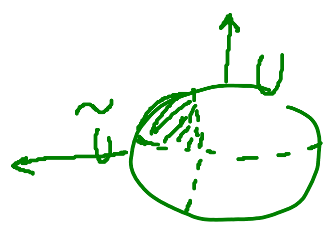
Then, by 1) and 4), the inverse mapping for φ is

$$\varphi^{-1}(x^1, \dots, x^{n-1}) = (x^1, \dots, x^{n-1}, g(x^1, \dots, x^{n-1})).$$

Verify that the atlas is smooth. Without loss of generality, suppose that $\vec{x}_0$ is contained in (U, φ) and also in $(\widetilde{U}, \widetilde{\varphi})$, where $\widetilde{\varphi} : (x^1, \dots, x^n) \mapsto (x^2, \dots, x^n)$. Then, on $\varphi(U \cap \widetilde{U})$ we have

$$\widetilde{\varphi}\varphi^{-1}(x^1, \dots, x^{n-1}) = \widetilde{\varphi}(x^1, \dots, x^{n-1}, g(x^1, \dots, x^{n-1})) = (x^2, \dots, x^{n-1}, g(x^1, \dots, x^{n-1})),$$

i.e., a smooth transition function. □



← only for surfaces! (i.e. embedded manifolds)

1 way — Only

Definition 2.21. **(Tensor definition of a tangent vector)** *A (tangent) vector* ξ at a point $P \in M$ to a manifold M is a correspondence which, to each chart $(U_\alpha, \varphi_\alpha)$ (i.e., a local coordinate system $(x^1_\alpha, \dots, x^m_\alpha)$ containing P) put in correspondence an n-tuple of numbers $(\xi^1_\alpha, \dots, \xi^m_\alpha)$. This correspondence is restricted to satisfy the *tensor transformation law*: if to another chart (U_β, φ_β) local coordinate system $(x^1_\beta, \dots, x^m_\beta)$) ξ put in correspondence an n-tuple $(\xi^1_\beta, \dots, \xi^m_\beta)$, then

$$\boxed{\xi^i_\beta = \frac{\partial x^i_\beta}{\partial x^j_\alpha}\xi^j_\alpha,} \Rightarrow \sum_{j=1}^{\dim} \qquad (1)$$

$\xi : (U_\alpha, \varphi_\alpha) \rightsquigarrow (\xi^1_\alpha, \dots, \xi^m_\alpha)$

$\xi : (U_\beta, \varphi_\beta) \rightsquigarrow (\xi^1_\beta, \dots, \xi^n_\beta)$

where the summation over repeated up and down indexes j is supposed (the Einstein summation convention).

Class **Problem 2.22.** (a justification of the definition) Suppose that $\gamma : (-1; 1) \to M$ is a smooth mapping and $\gamma(0) = P$. Then the correspondence

$x^1 = x^1(t)$, …, $x^n = x^n(t)$ — $\varphi \circ \gamma$ — $\mathbb{R}^n$ — $(-1,1)$

$\varphi \circ \gamma$ — vector tangent — (U, φ) — $\in \mathbb{R}^n$

$$\xi_\gamma : (x^1, \dots, x^n) \rightsquigarrow \left(\frac{dx^1}{dt}, \dots, \frac{dx^n}{dt}\right)\Big|_{t=0}$$

$(x^1, \dots, x^n) = \dim M$ — φ — $\mathbb{R}^n$

is a vector at P, where, for a local coordinate system $(x^1, \dots, x^n)$, the mapping γ is defined as $(x^1(t), \dots, x^n(t))$.

Home **Problem 2.23.** Any tangent vector at P is uniquely defined by its components for any coordinate system. Moreover, any such n-tuple defines a vector.

$\xi + \tilde\xi$

$\eta =$

Hence, the set of tangent vectors at a point P (*tangent space* $T_P(M)$) is a finite dimensional **R**-linear space of dimension $\dim M$. The operations do not depend on the choice of local coordinate system by (1).

$\xi : U_\alpha \mapsto (\xi^1_\alpha, \dots, \xi^n_\alpha) \in \mathbb{R}^n$

wish: $\xi : U_\beta \mapsto ?$

$$\xi^i_\beta = \frac{\partial x^i_\beta}{\partial x^j_\alpha}\xi^j_\alpha$$

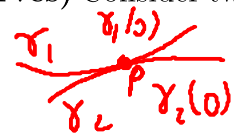
U_γ

$$\xi^k_\gamma = \frac{\partial x^k_\gamma}{\partial x^j_\alpha}\xi^j_\alpha$$

verify the tensor law

$$\eta^i_\beta = \frac{\partial x^i_\beta}{\partial x^j_\alpha}\eta^j_\alpha = \frac{\partial x^i_\beta}{\partial x^j_\alpha}(\xi^j_\alpha + \tilde\xi^j_\alpha) = \frac{\partial x^i_\beta}{\partial x^j_\alpha}\xi^j_\alpha + \frac{\partial x^i_\beta}{\partial x^j_\alpha}\tilde\xi^j_\alpha = \xi^i_\beta + \tilde\xi^i_\beta$$

Definition 2.24. (Definition of tangent vector via curves) Consider two smooth curves $\gamma_1 : (-1,1) \to M$ and $\gamma_2 : (-1,1) \to M$ such that



- $\gamma_i(0) = P$
- for some (hence, any) coordinate system $(x^1, \ldots, x^m)$ in a neighborhood of P the following holds:

$$\sum_{k=1}^{m} \left[x^k(\gamma_1(t)) - x^k(\gamma_2(t))\right]^2 = o(t^2), \qquad (t \to 0).$$

$\frac{\cdots}{t^2} \to 0 \quad (t \to 0)$

Such curves are called (tangentially) *equivalent*: $\gamma_1 \sim \gamma_2$.

All curves satisfying the first condition form non-intersecting equivalence classes called *tangent vectors* to M at point P.

Problem 2.25. Verify that the above equivalence is really an equivalence relation. Home

III

Definition 2.26. (Definition of a tangent vector via differentiation operators) A linear map $D : C^\infty(M) \to \mathbf{R}$, i.e., a linear functional on the space of smooth functions, is called a *differentiation operator* at some point $P \in M$, if

- its values are determined only by values of functions in an arbitrary small neighborhood of P. More precisely, if $f, g \in C^\infty(M)$ satisfy $f \equiv g$ over some neighborhood U of P, then $D(f) = D(g)$ (they say "operator is defined on germs of functions");

$(fg)' = f' \cdot g + f \cdot g'$

- the Newton-Leibniz property

$$D(fg) = f(P)D(g) + g(P)D(f) \text{ is fulfilled for any } f, g \in C^\infty(M).$$

Such operator is called a *tangent vector* to M at point P.

Evidently, they form a linear space.

Home **Problem 2.27.** Suppose that $(x^1, \ldots, x^n)$ is a local coordinate system in a neighborhood of $P \in M$, $P = (x_0^1, \ldots, x_0^n)$, and $\xi \in T_P M$ (in the tensor sense) has components ξ^i. Then the mapping

I $\quad (\xi^1, \ldots, \xi^n)$

$$C^\infty(M) \ni f \mapsto \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(x_0^1, \ldots, x_0^n)\xi^i \in \mathbb{R} \text{ a number}$$

(*directional derivative* w.r.t. ξ) does not depend on the choice of a local coordinate system and defines a differentiation operator.

III

evidently linear. $\dfrac{\partial(f_1 + f_2)}{\partial x^i} = \dfrac{\partial f_1}{\partial x^i} + \dfrac{\partial f_2}{\partial x^i}$

Theorem 2.28. *These three definitions are equivalent in the sense that the following natural correspondences*

$\gamma \xrightarrow{\Gamma} \left(\frac{dx^1}{dt}, \dots, \frac{dx^n}{dt}\right) \to D_3$

$$\textit{a curve} \to \textit{its tangent vector in a coordinate system} \to$$

II I III

$$\to \textit{the directional derivative w.r.t. this vector}$$

gives rise to a bijection of the tangent spaces in three senses (the second map is a linear isomorphism of linear spaces).

Proof. Let us prove the first bijection. Keeping in mind Problem 2.22 we see that to prove that Γ (defined in the problem) is well defined on equivalence classes, it is sufficient to verify in one coordinate system that $\gamma_1 \sim \gamma_2$ imples $\xi_{\gamma_1} = \xi_{\gamma_2}$. Indeed,

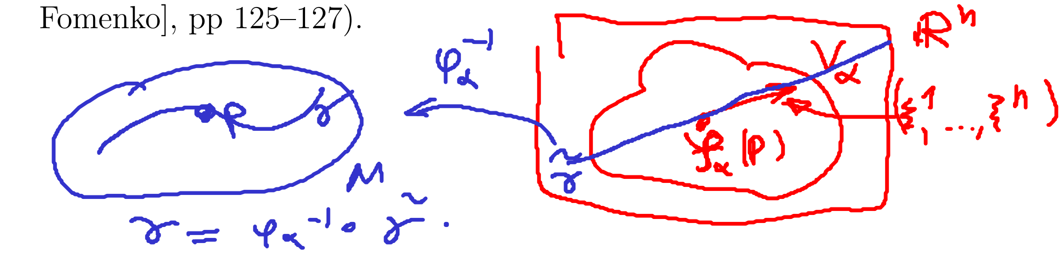
def of $\sim$

$$0 = \lim_{t\to 0} \sum_{k=1}^{m} \left[\frac{x^k(\gamma_1(t)) - x^k(\gamma_2(t))}{t} \right]^2 =$$

$$= \sum_{k=1}^{m} \left[\lim_{t\to 0} \frac{(x^k(\gamma_1(t)) - x^k(P)) - (x^k(\gamma_2(t)) - x^k(P))}{t} \right]^2,$$

so $\xi_{\gamma_1} = \xi_{\gamma_2}$. The same calculation shows that two curves are equivalent iff they have the same tangent vector in their intersection point P. Thus, Γ is well defined and injective. Fix a coordinate system x^i in a neighborhood of P. Define a map Δ (may be depending on the choice of coordinates) in the inverse direction by sending a vector ξ with coordinates ξ^i in this system, to a "straight line", i.e. to the following curve: $x^i(t) = x^i(P) + t \cdot \xi^i$. Then $\left.\frac{dx^i}{dt}\right|_{P_0} = \xi^i$ and $\Gamma \circ \Delta = \mathrm{Id}$. Hence, Γ is a surjection. $\square$

Problem 2.29. Prove the second equivalence in the above theorem ([Mishchenko, Fomenko], pp 125–127). Home



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