Hilbert C^*- and W^*-Modules and Their Morphisms

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CONTENTS

Hilbert C^*-module is a natural generalization of a Hilbert space arising under replacement of the field of scalars $\mathbf{C}$ by a C^*-algebra. For commutative C^*-algebras such a generalization was for the first time discribed in the work of I. Kaplansky [30], however the noncommutative case looked at that time like a complicated one for study. The general theory of Hilbert C^*-modules has appeared 25 years ago in the basic papers of W. Paschke [52] and M. Rieffel [56]. This theory has proved to be a very convenient tool in the theory of operator algebras, allowing to study C^*-algebras by studying Hilbert modules over them. In particular, a series of results about such classes of C^*-algebras as AW^*-algebras and monotoneously complete C^*-algebras was obtained [21]. In terms of Hilbert modules the important notion of Morita-equivalence was formulated for C^*-algebras [57, 14]. This notion has also applications in theory of group representations. It turned to be possible to study group actions with the help of Hilbert modules arising from such actions [54, 58]. Some results about conditional expectations of finite index [7, 69] and about completely positive maps of C^*-algebras [4] were also obtained.

The theory of Hilbert C^*-modules may be considered also as a noncommutative generalization of the theory of vector bundles [19, 36]. This was the reason for Hilbert modules to become a tool in topological applications — namely in index theory of elliptic operators, in K-theory and in KK-theory of G. G. Kasparov [48, 46, 32, 34, 33, 35, 66] and in noncommutative geometry in whole [15, 16].

Among other applications it is necessary to emphasize the theory of quantum groups and unbounded operators [70, 71, 5, 6] and some physical applications [39, 2].

Alongside with these applications the theory of Hilbert C^*-modules itself has been developed too. A number of results about the structure of Hilbert modules and about operators on them was obtained [41, 22, 44, 42, 43, 65]. Besides that an axiomatic approach in theory of Hilbert modules based on the theory of operator spaces and tensor products was developed [10, 11].

The detailed bibliography of the theory of Hilbert C^*-modules can be found in [24].

A significant part of results presented here was only announced in the literature or the proofs were discussed only in brief. We have tried to fill such lacunae. We can not discuss here all aspects of the theory of Hilbert modules, but we have tried explicitly to explain the basic notions and theorems of this theory, a number of important examples, and also some results, related to the authors' interest.

The major part of the presented material formed the content of lecture course presented by the authors at the Department of Mechanics and Mathematics of Moscow State University in 1996.

We are grateful to A. S. Mishchenko for introducing us to the theory of Hilbert C^*-modules. Together with Yu. P. Solovyov he has acquainted us with the circle of problems related to its applications in topology.

While working in this field and in the process of writing the present text a significant influence on us was made by our friend and co-author M. Frank.

We have discussed a number of problems of the theory of Hilbert C^*-modules with L. Brown, A. A. Irmatov, G. G. Kasparov, R. Nest, G. K. Pedersen, W. Paschke, some applications were considered also with J. Kaminker, J. Cuntz, V. Nistor, J. Rosenberg, A. Ya. Helemskii, B. L. Tsygan.

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1 Basic definitions

1.1 C^*-algebras

The basic information about C^*-algebras can be found in the books [17, 55, 62, 29]. We will present some results on C^*-algebras, which will be necessary for us further on.

Remind that an involutive Banach algebra is called a *C^*-algebra*, if for each its element a the following relation

$$||a^*a|| = ||a||^2$$

is fulfilled. Any such algebra can be realized as a norm-closed subalgebra of the algebra of bounded operators $\mathcal{B}(H)$ on Hilbert space H. We do not assume the presence of the unity element in C^*-algebras. By A^+ we denote the C^*-algebra obtained from C^*-algebra A by unitization (taking direct sum with complex numbers).

We need also the notion of a *positive element* of a C^*-algebra. First of all we remind that the *spectrum* of an element a of a unital C^*-algebra is the set $\mathrm{Sp}(a)$ of complex numbers z such that $a - z \cdot 1$ is not invertible. If a C^*-algebra A has no unity, then the spectrum of the element $a \in A$ is its spectrum in the C^*-algebra $A^+ \supset A$. Spectrum is a compact subset of $\mathbf{C}$. An element $a \in A$ is called *positive* (we write $a \geq 0$), if it is *Hermitian*, i. e. satisfies the condition $a^* = a$, and if one of the following equivalent [17, 1.6.1] conditions is carried out

(i) $\mathrm{Sp}(a) \subset [0, \infty)$;

(ii) $a = b^*b$ for some $b \in A$;

(iii) $a = h^2$ for some Hermitian $h \in A$.

The set of all positive elements $P^+(A)$ forms a closed convex cone in A and $P^+(A) \cap (-P^+(A)) = 0$. Among the elements h existing by the item (iii) there exists only one positive, called the *positive square root* of a (we write $h = a^{1/2}$).

We remind also that a linear functional $\varphi : A \longrightarrow \mathbf{C}$ is called *postive* if $\varphi(a) \geq 0$ for any positive element $a \in P^+(A)$. A positive linear functional is called a *state* if $||\varphi|| = 1$. We have $||a|| = \sup \varphi(a)$, where $a \geq 0$, and sup is taken over all states.

A C^*-homomorphism of an algebra A into the C^*-algebra $\mathcal{B}(H)$ of all bounded operators on a Hilbert space H is called a *representation*. A vector $\xi \in H$ is called *cyclic* for the representation $\pi : A \longrightarrow \mathcal{B}(H)$, if the set of all vectors of the form $\pi(a)\xi$, $a \in A$, is dense in H. The vector $\xi \in H$ is called *separating* for the representation $\pi : A \longrightarrow \mathcal{B}(H)$, if the equality $\pi(a)\xi = 0$ implies $a = 0$.

We can associate with each positive linear functional ω on a C^*-algebra A a unique (up to unitary equivalence) representation π_ω of the algebra A on some Hilbert space H_ω and a vector $\xi_\omega \in H_\omega$ such that $\omega(a) = (\pi_\omega(a)\xi_\omega, \xi_\omega)$ for all $a \in A$ and the vector ξ_ω is cyclic. The construction of such a representations is called the *GNS-construction.*

An *approximate unit* of a C^*-algebra A is an increasing net $e_\alpha \in A$, $\alpha \in \mathcal{A}$, such that $\|e_\alpha\| \leq 1$ and $\lim \|a - ae_\alpha\| = 0$ for any $a \in A$. Each C^*-algebra has an approximate unit e_α such that $e_\alpha \geq 0$ and $e_\alpha \geq e_\beta$ for $\alpha \geq \beta$ [17].

Definition 1.1.1 A C^*-algebra possessing countable approximate unit, is called σ-unital.

Definition 1.1.2 An element $h \in A$ is called *strictly positive* if for any positive nonzero φ (or, equally, for any state) one has $\varphi(h) > 0$.

Remark 1.1.3 These two conditions are equivalent. It is possible to consider $e_i \geq 0$. Then $h := \sum_i e_i/2^i$ is strictly positive. Conversely, $e_i := h^{1/i}$ is a countable approximate unit. Separable algebra always satisfies these conditions. The details can be found in [55].

We will often use the following statements.

Lemma 1.1.4 [55, Lemma 1.4.4] *Let x, y and a be elements of a C^*-algebra A such that $a \geq 0$ and*

$$x^*x \leq a^\alpha, \qquad yy^* \leq a^\beta, \qquad \alpha + \beta > 1.$$

Then the sequence $u_n = x[(1/n) + a]^{-1/2}y$ is norm convergent in A to such u, for which $\|u\| \leq \|a^{(\alpha+\beta-1)/2}\|$.

Proof: Put $d_{nm} := [(1/n) + a]^{-1/2} - [(1/m) + a]^{-1/2}$. Then

$$\|u_n - u_m\|^2 = \|xd_{nm}y\|^2 = \|y^*d_{nm}x^*xd_{nm}y\| \leq \|y^*d_{nm}a^\alpha d_{nm}y\| = \|a^{\alpha/2}d_{nm}y\|^2 =$$

$$= \|a^{\alpha/2}d_{nm}yy^*d_{nm}a^{\alpha/2}\| \leq \|a^{\alpha/2}d_{nm}a^\beta d_{nm}a^{\alpha/2}\| = \|d_{nm}a^{(\alpha+\beta)/2}\|^2.$$

Studying the convergence of a monotone sequence on a spectrum a, we obtain by the Dini theorem the uniform convergence of

$$[(1/n) + t]^{-1/2}t^{(\alpha+\beta)/2} \longrightarrow t^{(\alpha+\beta-1)/2}, \qquad t \in \mathrm{Sp}(a).$$

Therefore, $\|d_{nm}a^{(\alpha+\beta)/2}\| \to 0$, so that by the Cauchy criterion $\{u_n\}$ is norm convergent to an element $u \in A$. Then reasoning as above we obtain

$$\|u_n\| = \|x[(1/n) + a]^{-1/2}y\| \leq \|a^{\alpha/2}[(1/n) + a]^{-1/2}a^{\beta/2}\| \leq \|a^{(\alpha+\beta-1)/2}\|,$$

So that $\|u\| \leq \|a^{(\alpha+\beta-1)/2}\|$. $\square$

Proposition 1.1.5 [55, Prop. 1.4.5] *Let x and a be elements of a C^*-algebra A such that $a \geq 0$ and $x^*x \leq a$. For any $0 < \alpha < \frac{1}{2}$ there exists an element $u \in A$ such that $\|u\| \leq \|a^{\frac{1}{2}-\alpha}\|$ and $x = ua^\alpha$.*

Proof: Let us define $u_n := x[(1/n) + a]^{-\frac{1}{2}}a^{\frac{1}{2}-\alpha}$. By Lemma 1.1.4 $\{u_n\}$ is norm convergent to an element $u \in A$ such that

$$\|u\| \leq \|a^{\frac{1}{2}(1+1-2\alpha-1)}\| = \|a^{\frac{1}{2}-\alpha}\|.$$

Further,

$$\|x - u_na^\alpha\|^2 = \|x(1 - [(1/n) + a]^{-1/2}a^{1/2})\|^2 \leq \|a^{1/2}(1 - [(1/n) + a]^{-1/2}a^{1/2})\|^2 \longrightarrow 0$$

for $n \longrightarrow \infty$ by the Dini theorem being applied to appropriate functions on the spectrum. Thus, $x = ua^\alpha$. $\square$

1.2 Pre-Hilbert modules

Let $\mathcal{M}$ be a module over C^*-algebra A. The action of an element $a \in A$ on $\mathcal{M}$ we denote by $x \cdot a$, where $x \in \mathcal{M}$.

Definition 1.2.1 *Pre-Hilbert A-module* is a (right) A-module $\mathcal{M}$ equipped with a sesquilinear form $\langle \cdot, \cdot \rangle : \mathcal{M} \times \mathcal{M} \longrightarrow A$ satisfying the following properties

(i) $\langle x, x \rangle \geq 0$ for any $x \in \mathcal{M}$;

(ii) $\langle x, x \rangle = 0$ only in the case, when $x = 0$;

(iii) $\langle y, x \rangle = \langle x, y \rangle^*$ for all $x, y \in \mathcal{M}$;

(iv) $\langle x, y \cdot a \rangle = \langle x, y \rangle a$ for all $x, y \in \mathcal{M}$, $a \in A$.

The map $\langle \cdot, \cdot \rangle$ is called an *A-valued scalar (or inner) product.*

Let us consider a few examples.

Example 1.2.2 If $J \subset A$ is a right ideal, then J can be equipped with the structure of pre-Hilbert A-modle if we define the inner product of elements $x, y \in J$ by the equality $\langle x, y \rangle := x^* y$.

Example 1.2.3 If $\{J_i\}$ is some countable set of right ideals in C^*-algebra A, then the linear space $\mathcal{M}$ of sequences (x_i), $x_i \in J_i$, satisfying the condition $\sum_i \|x_i\|^2 < \infty$, becomes a right A-module if $(x_i) \cdot a := (x_i a)$ for $(x_i) \in \mathcal{M}$, $a \in A$, and becomes a pre-Hilbert A-module if we define the inner product of elements $(x_i), (y_i) \in \mathcal{M}$ by the equality $\langle (x_i), (y_i) \rangle := \sum_i x_i^* y_i$.

Let $\mathcal{K}$ be a right A-module equipped with a sesquilinear map $[\cdot, \cdot] : \mathcal{K} \times \mathcal{K} \longrightarrow A$, satisfying all properties of Definition 1.2.1 except (ii). Put

$$N := \{x \in \mathcal{K} : [x, x] = 0\}.$$

For each positive linear functional φ on the C^*-algebra A the map $(x, y) \mapsto \varphi([x, y])$ is a (degenerate) inner product on $\mathcal{K}$, and hence the set $N_\varphi = \{x \in \mathcal{K} : \varphi([x, x]) = 0\}$ is a linear subspace in $\mathcal{K}$. By taking the intersection of all such subspaces we obtain that $N = \cap_\varphi N_\varphi$ is also a linear subspace in $\mathcal{K}$. From properties (iii) and (iv) of Definition 1.2.1 it follows that $N \cdot A \subset N$, therefore N is a submodule in $\mathcal{K}$. The quotient module $\mathcal{M} = \mathcal{K}/N$ is equipped with the obvious structure of a pre-Hilbert A-module with the inner product $\langle x + N, y + N \rangle := [x, y]$.

Let $\mathcal{M}$ be a pre-Hilbert A-module, $x \in \mathcal{M}$. Put $\|x\|_{\mathcal{M}} := \|\langle x, x \rangle\|^{1/2}$. We will omit the index $\mathcal{M}$ if it will not lead to a confusion of norms.

Proposition 1.2.4 ([52]) *The function $\|\cdot\|_{\mathcal{M}}$ is a norm on $\mathcal{M}$ and satisfies the following properties*

(i) $\|x \cdot a\|_{\mathcal{M}} \leq \|x\|_{\mathcal{M}} \cdot \|a\|$ *for all* $x \in \mathcal{M}$, $a \in A$;

(ii) $\langle x, y \rangle \langle y, x \rangle \leq \|y\|^2_{\mathcal{M}} \langle x, x \rangle$ *for all* $x, y \in \mathcal{M}$;

(iii) $\|\langle x, y \rangle\| \leq \|x\|_{\mathcal{M}} \|y\|_{\mathcal{M}}$ *for all* $x, y \in \mathcal{M}$.

Proof: For any positive linear functional φ on A the function $x \mapsto \varphi(\langle x, x \rangle)^{1/2}$ defines a seminorm on $\mathcal{M}$. For each $x \in \mathcal{M}$

$$\|x\|_{\mathcal{M}} = \|\langle x, x \rangle\|^{1/2} = \sup\{\varphi(\langle x, x \rangle)^{1/2}\},$$

where the supremum is taken over all states φ on A. Therefore $\|\cdot\|_{\mathcal{M}}$ is a seminorm, and by the property (ii) of Definition 1.2.1 $\|\cdot\|_{\mathcal{M}}$ is a norm on $\mathcal{M}$. Statement (i) follows from the equality

$$\|x \cdot a\|^2_{\mathcal{M}} = \|\langle x \cdot a, x \cdot a \rangle\| = \|a^* \langle x, x \rangle a\| \leq \|a\|^2 \|\langle x, x \rangle\| = \|x\|^2_{\mathcal{M}} \|a\|^2 .$$

To prove the statement (ii) we will take $x, y \in \mathcal{M}$ and a positive linear functional φ on A. Applying the Cauchy-Bunyakovskii inequality for the (degenerate) inner product $\varphi(\langle \cdot, \cdot \rangle)$ on $\mathcal{M}$ we obtain

$$\begin{aligned} \varphi(\langle x, y \rangle \langle y, x \rangle) &= \varphi(\langle x, y \cdot \langle y, x \rangle \rangle) \leq \varphi(\langle x, x \rangle)^{1/2} \cdot \varphi(\langle y \cdot \langle y, x \rangle, y \cdot \langle y, x \rangle \rangle)^{1/2} \\ &= \varphi(\langle x, x \rangle)^{1/2} \cdot \varphi(\langle x, y \rangle \langle y, y \rangle \langle y, x \rangle)^{1/2} \leq \varphi(\langle x, x \rangle)^{1/2} \cdot \|\langle y, y \rangle\|^{1/2} \cdot \varphi(\langle x, y \rangle \langle y, x \rangle)^{1/2}. \end{aligned}$$

Thus, for any positive linear functional φ we have $\varphi(\langle y,x\rangle\langle x,y\rangle) \le \|y\|^2_{\mathcal{M}} \cdot \varphi(\langle x,x\rangle)$. Therefore, the statement (ii) is proved. It evidently implies the statement (iii). $\square$

The inequality (ii) (and also its concequence — the inequality (iii)) of Proposition 1.2.4 we will call the Cauchy-Bunyakovskii inequality for Hilbert modules.

Remark 1.2.5 For any C^*-pre-Hilbert module, or more precisely, for any sesquilinear form $\langle .,. \rangle$, the following *polarization equality* is obviously satisfied

$$4\langle y,x\rangle = \sum_{k=0}^{3} i^k \langle x + i^k y, x + i^k y\rangle \qquad \text{for all } x,\ y \in \mathcal{M}.$$

1.3 Hilbert C^*-modules

Definition 1.3.1 An A-module $\mathcal{M}$ being at the same time a Banach space with a norm $\|\cdot\|$ satisfying the inequality $\|x \cdot a\| \le \|x\|\,\|a\|$, $x \in \mathcal{M}$, $a \in A$, is called a *Banach* A-module.

Definition 1.3.2 A pre-Hilbert A-module $\mathcal{M}$, which is complete with respect to the norm $\|\cdot\|_{\mathcal{M}}$, is called a *Hilbert C^*-module.*

If $\mathcal{M}$ is a pre-Hilbert A-module then the action of the C^*-algebra A and the inner product on $\mathcal{M}$ extend to the completion $\widetilde{\mathcal{M}}$, which thus becomes a Hilbert module. Let us consider some examples.

Example 1.3.3 If $J \subset A$ is a right ideal, then the pre-Hilbert module J is complete with respect to the norm $\|\cdot\|_J = \|\cdot\|$. In particular, the C^*-algebra A itself is a *free* Hilbert A-module with one generator.

Example 1.3.4 If $\{\mathcal{M}_i\}$ is a finite set of Hilbert A-modules, then it is possible to define their direct sum $\oplus\mathcal{M}_i$. The inner product on $\oplus\mathcal{M}_i$ is defined by the formula $\langle x,y\rangle := \sum_i \langle x_i, y_i\rangle$, where $x = (x_i)$, $y = (y_i) \in \oplus\mathcal{M}_i$. The direct sum of n copies of a Hilbert Module $\mathcal{M}$ we will denote by $\mathcal{M}^n$ or $L_n(\mathcal{M})$.

Example 1.3.5 If $\{\mathcal{M}_i\}$, $i \in \mathbf{N}$, is a countable set of Hilbert A-modules then it is possible to define their direct sum $\oplus\mathcal{M}_i$. We shall define the inner product on the A-module $\oplus\mathcal{M}_i$ of all sequences $x = (x_i) : x_i \in \mathcal{M}_i$ such that the series $\sum_i \langle x_i, x_i\rangle$ is norm convergent in the C^*-algebra A, by the formula

$$\langle x,y\rangle := \sum\nolimits_i \langle x_i, y_i\rangle \quad \text{for } x,y \in \oplus\mathcal{M}_i.$$

Let us show that the mentioned series converges. From the convergence of series $\sum_i \langle x_i, x_i\rangle$ and $\sum_i \langle y_i, y_i\rangle$ it follows that for any $\varepsilon > 0$ there exists a number N such that for all $n > 0$ the following estimate holds

$$\left\| \sum_{i=N}^{N+n} \langle x_i, x_i\rangle \right\| < \varepsilon, \qquad \left\| \sum_{i=N}^{N+n} \langle y_i, y_i\rangle \right\| < \varepsilon.$$

Then

$$\left\| \sum_{i=N}^{N+n} \langle x_i, y_i\rangle \right\| \le \left\| \sum_{i=N}^{N+n} \langle x_i, x_i\rangle \right\| \cdot \left\| \sum_{i=N}^{N+n} \langle y_i, y_i\rangle \right\| < \varepsilon^2,$$

This proves that the inner product is well-defined.

Let us verify completeness of the module $\oplus\mathcal{M}_i$. Let $x^{(n)} = (x_i^{(n)}) \in \oplus\mathcal{M}_i$ be a Cauchy sequence. Then for any $\varepsilon > 0$ there exists a number N such that for all $n, m \ge N$

$$\left\| \sum_i \langle x_i^{(n)} - x_i^{(m)}, x_i^{(n)} - x_i^{(m)}\rangle \right\| < \varepsilon. \tag{1}$$

Since all summands in (1) are positive, the inequality

$$\left\| \langle x_i^{(n)} - x_i^{(m)}, x_i^{(n)} - x_i^{(m)}\rangle \right\| < \varepsilon$$

holds for each number i separately. But then the sequences $x_i^{(n)} \in \mathcal{M}_i$ are the Cauchy sequences, and they have limits $x_i = \lim x_i^{(n)} \in \mathcal{M}_i$. Let us verify that a series $\sum_i \langle x_i, x_i \rangle$ is norm convergent in A. Let us fix $\varepsilon > 0$. There exists a number $n > N$ such that the estimate (1) is valid. Let us choose a number K satisfying the condition

$$\left\| \sum_{i=K}^{\infty} \langle x_i^{(n)}, x_i^{(n)} \rangle \right\| < \varepsilon.$$

Then for any $k > 0$ we have

$$\begin{aligned} &\left\| \sum_{i=K}^{K+k} \left(\langle x_i^{(m)}, x_i^{(m)} \rangle + \langle x_i^{(n)} - x_i^{(m)}, x_i^{(m)} \rangle + \langle x_i^{(m)}, x_i^{(n)} - x_i^{(m)} \rangle + \langle x_i^{(n)}, x_i^{(n)} \rangle \right) \right\| \\ &= \left\| \sum_{i=K}^{K+k} \langle x_i^{(n)} - x_i^{(m)}, x_i^{(n)} - x_i^{(m)} \rangle \right\| \le \left\| \sum_{i=1}^{\infty} \langle x_i^{(n)} - x_i^{(m)}, x_i^{(n)} - x_i^{(m)} \rangle \right\| < \varepsilon, \end{aligned}$$

Therefore,

$$\begin{aligned} \left\| \sum_{i=K}^{K+k} \langle x_i^{(m)}, x_i^{(m)} \rangle \right\| &< 2\varepsilon + \left\| \sum_{i=K}^{K+k} \langle x_i^{(n)} - x_i^{(m)}, x_i^{(m)} \rangle \right\| + \left\| \sum_{i=K}^{K+k} \langle x_i^{(m)}, x_i^{(n)} - x_i^{(m)} \rangle \right\| \\ &\le 2\varepsilon + 2 \left\| \sum_{i=K}^{K+k} \langle x_i^{(n)} - x_i^{(m)}, x_i^{(n)} - x_i^{(m)} \rangle \right\|^{1/2} \left\| \langle x_i^{(m)}, x_i^{(m)} \rangle \right\|^{1/2} \\ &\le 2\varepsilon + 2\varepsilon^{1/2} \left\| \langle x_i^{(m)}, x_i^{(m)} \rangle \right\|^{1/2}. \end{aligned}$$

Now by solving the square inequality, we obtain that

$$\left\| \sum_{i=K}^{K+k} \langle x_i^{(m)}, x_i^{(m)} \rangle \right\| < (1 + \sqrt{3})^2 \varepsilon < 8\varepsilon. \tag{2}$$

Passing in the inequality (2) to the limit $m \to \infty$, we obtain that

$$\left\| \sum_{i=K}^{K+k} \langle x_i, x_i \rangle \right\| < 8\varepsilon.$$

This proves that the series $\sum_i \langle x_i, x_i \rangle$ is norm convergent.

The direct sum of a countable number of copies of a Hilbert module $\mathcal{M}$ we shall denote by $l_2(\mathcal{M})$ or $H_{\mathcal{M}}$. The Hilbert C^*-module $l_2(A)$ (other denotation is H_A) we call the *standard* Hilbert module over A. If the C^*-algebra is unital then the Hilbert module H_A possesses the standard basis $\{e_i\}$, $i \in \mathbf{N}$, where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0, \ldots)$ with the unit at the i-th place.

Example 1.3.6 Let $B \subset A$ be a C^*-subalgebra of a C^*-algebra A having the common unit. Let us assume that there exists a linear map $E : A \longrightarrow B$ not increasing norms and being a projection, that is $E^2 = E$. Such a map is called a *conditional expectation* from A to B. Conditional expectation is a positive map, i.e. $E(a^*a) \ge 0$ for all $a \in A$, and it satisfies the equality

$$E(b_1 a b_2) = b_1 E(a) b_2 \quad \text{for } a \in A, b_1, b_2 \in B$$

(see [62]). A conditional expectation is called *exact,* if for any positive element $a \in P^+(A)$ the equality $E(a) = 0$ implies $a = 0$. In the case when the conditional expectation is exact, it is possible to introduce the structure of a pre-Hilbert B-module on the C^*-algebra A by putting

$$\langle x, y \rangle = E(x^* y), \quad x, y \in A.$$

We will give a condition for this module to be a Hilbert module (i.e. to be complete) in the section 4.4.

Let $\mathcal{N} \subset \mathcal{M}$ be a closed submodule of a Hilbert module $\mathcal{M}$. We define the orthogonal complement $\mathcal{N}^\perp$ by the equality

$$\mathcal{N}^\perp = \{y \in \mathcal{M} : \langle x, y\rangle = 0 \quad \text{for all } x \in \mathcal{N}\}.$$

Then $\mathcal{N}^\perp$ is a closed submodule of the Hilbert module $\mathcal{M}$ too. However, the equality $\mathcal{M} = \mathcal{N} \oplus \mathcal{N}^\perp$ is fulfilled not always, as shows the following

Example 1.3.7 Let $A = C[0,1]$ be the C^*-algebra of all continuous functions on the segment. Let us consider in the Hilbert A-module $\mathcal{M} = A$ the submodule $\mathcal{N} = C_0(0,1)$ of functions, vanishing at the end points of the segment $[0,1]$. Then, obviously, $\mathcal{N}^\perp = 0$.

If $\mathcal{M}$ is a Hilbert A-module then we denote by $\mathcal{M} \cdot A$ the closure in $\mathcal{M}$ of the linear span of the elements of the form $x \cdot a$, $x \in \mathcal{M}$, $a \in A$.

Lemma 1.3.8 *$\mathcal{M} \cdot A = \mathcal{M}$.*

Proof: Let $e_\alpha \in A$ be an approximate unit. Then for any $x \in \mathcal{M}$

$$\|x - x \cdot e_\alpha\|^2 = \|\langle x - x\cdot e_\alpha, x - x \cdot e_\alpha\rangle\| \le (1 + \|e_\alpha^*\|)\,\|\langle x, x\rangle - \langle x, x\rangle e_\alpha\| \to 0,$$

that proves that the elements of the form $x \cdot e_\alpha$ are dense in $\mathcal{M}$. $\square$

We will often use the following useful statement.

Lemma 1.3.9 *For any $x \in \mathcal{M}$*

$$x = \lim_{\varepsilon\to 0} x\langle x, x\rangle(\langle x, x\rangle + \varepsilon)^{-1}.$$

Proof: Let $\langle x, x\rangle = a$, then

$$\|x\langle x, x\rangle(\langle x, x\rangle + \varepsilon)^{-1} - x\|^2 = \|\langle x(\langle x, x\rangle(\langle x, x\rangle + \varepsilon)^{-1} - 1), x(\langle x, x\rangle(\langle x, x\rangle + \varepsilon)^{-1} - 1)\rangle\| =$$
$$= \|a(a^2(a+\varepsilon)^{-2} - 2a(a+\varepsilon)^{-1} + 1)\| = \|a^3(a+\varepsilon)^{-2} - 2a^2(a+\varepsilon)^{-1} + a\| \longrightarrow 0,$$

because the following inequalities hold under the condition $t \ge 0$

$$|t^3(t+\varepsilon)^{-2} - t| = \left|t\left(\left(\frac{t}{t+\varepsilon}\right)^2 - 1\right)\right| = \left|t\left(\frac{-\varepsilon^2 - 2\varepsilon t}{(t+\varepsilon)^2}\right)\right| = \varepsilon\left|\frac{\varepsilon t + 2t^2}{(t+\varepsilon)^2}\right| \le \varepsilon(1/2 + 2) = \frac{3}{2}\,\varepsilon$$

and

$$|t^2(t+\varepsilon)^{-1} - t| = \left|\frac{t\varepsilon}{t+\varepsilon}\right| < \varepsilon. \qquad \square$$

The following statement is an analog of polar decomposition for Hilbert modules. We will see below that, as well as in the case of algebras, the exact polar decomposition exists only in the case of Hilbert modules over W^*-algebras.

Proposition 1.3.10 ([38]) *Let $\mathcal{M}$ be a Hilbert A-module, $x \in \mathcal{M}$, and $0 < \alpha < 1/2$. Then there exists an element $z \in \mathcal{M}$ such that $x = z \cdot \langle x, x\rangle^\alpha$.*

Proof: For $n \in \mathbf{N}$ let us define functions

$$g_n(\lambda) = \begin{cases} n^{-\alpha/2}, & \text{если } \lambda \le 1/n, \\ \lambda^{\alpha/2}, & \text{если } \lambda > 1/n. \end{cases}$$

Then by the spectral theorem

$$\begin{aligned} \|x \cdot (g_n(\langle x, x\rangle) - g_m(\langle x, x\rangle))\| &= \left\|\langle x, x\rangle(g_n(\langle x, x\rangle) - g_m(\langle x, x\rangle))^2\right\|^{1/2} \\ &= \sup\{|\lambda(g_n(\lambda) - g_m(\lambda))| : \lambda \in \mathrm{Sp}(\langle x, x\rangle)\}. \end{aligned}$$

Therefore the sequence $x \cdot g_n(\langle x, x\rangle)$ is a Cauchy sequence, so and it has a limit $z \in \mathcal{M}$. Then

$$\begin{aligned} \|z\langle x, x\rangle^\alpha - x\| &= \lim_{n\to\infty} \|x \cdot g_n(\langle x, x\rangle)\langle x, x\rangle^\alpha - x\| = \lim_{n\to\infty} \|x(g_n(\langle x, x\rangle)\langle x, x\rangle^\alpha - 1)\| \\ &= \lim_{n\to\infty} \sup\{|\lambda^{1/2}(g_n(\lambda)\lambda^\alpha - 1)| : \lambda \in \mathrm{Sp}\,\langle x, x\rangle\} = 0. \end{aligned}$$

This completes the proof. $\square$

A Hilbert C^*-module $\mathcal{M}$ is called *finitely generated* if there exists a finite set $\{x_i\} \subset \mathcal{M}$ such that $\mathcal{M}$ equals the linear span (over $\mathbf{C}$ and A) of this set. A Hilbert C^*-module $\mathcal{M}$ is called *countably generated* if there exists a countable set $\{x_i\} \subset \mathcal{M}$ such that $\mathcal{M}$ equals the norm-closure of the linear span (over $\mathbf{C}$ and A) of this set.

1.4 The standard Hilbert module H_A

Theorem 1.4.1 (Kasparov stabilization theorem, [34]**)** *Let A be a C^*-algebra, $\mathcal{M}$ be a countably generated Hilbert A-module. Then $\mathcal{M} \oplus H_A \cong H_A$.*

Proof: We start by proving the theorem for the case of unital C^*-algebra A. For this purpose it is convenient to use the procedure of almost-ortogonalization [20]. An element x of the Hilbert module $\mathcal{N}$ is called *non-singular* if the element $\langle x, x\rangle \in A$ is invertible. The set $\{x_i\} \in \mathcal{N}$ is called *orthonormal* if $\langle x_i, x_j\rangle = \delta_{ij}$. It is called *basis* of the module $\mathcal{N}$ if finite sums of the form $\sum_i x_i \cdot a_i$, $a_i \in A$, are dense in $\mathcal{N}$.

Lemma 1.4.2 ([20]**)** *Let $\mathcal{N}$ be a Hilbert A-module containing the orthonormal elements $e_1, \dots, e_n$, $x \in \mathcal{N}$, $\varepsilon > 0$. If an element $y \in \mathcal{N}$ satisfies $\langle y, y\rangle = 1$ and $y \perp \{x, e_1, \dots, e_n\}$ then there exists an element $e_{n+1} \in \mathcal{N}$ such that*

(i) *the elements $e_1, \dots, e_n, e_{n+1}$ are orthonormal,*

(ii) $e_{n+1} \in \mathrm{Span}_A(e_1, \dots, e_n, x, y)$,

(iii) $\mathrm{dist}(x, \mathrm{Span}_A(e_1, \dots, e_{n+1})) \leq \varepsilon$.

Proof: Let

$$x' = x - \sum_{i=1}^{n} e_i \langle e_i, x\rangle, \qquad x'' = x' + \varepsilon y.$$

Then

$$\langle x'', x''\rangle = \langle x', x'\rangle + \varepsilon^2 \geq \varepsilon^2 > 0,$$

therefore the element x'' is nonsingular. Let's put $e_{n+1} = x'' \cdot \langle x'', x''\rangle^{-1/2}$. Then

$$e_{n+1} \in \mathrm{Span}_A(x', y) \perp \{e_1, \dots, e_n\}.$$

Therefore the elements $e_1, \dots, e_n, e_{n+1}$ are orthonormal. Since $x' \in \mathrm{Span}_A(x, e_1, \dots, e_n)$ and $e_{n+1} \in \mathrm{Span}_A(x', y)$, we obtain the statement (ii). Finally, let us put

$$w = e_{n+1}\langle x'', x''\rangle^{1/2} + \sum_{i=1}^{n} e_i \langle e_i, x\rangle \in \mathrm{Span}_A(e_1, \dots, e_{n+1}),$$

and the equality $\|w - x\| = \|x'' - x'\| = \|\varepsilon y\| = \varepsilon$ proves the statement (iii) . □

We return now to the proof of Theorem 1.4.1. Let $\{y_n\}$ be the sequence of generators of module $\mathcal{M}$. By $\{e_n\}$ we denote the standard basis of the module H_A. Let $\{x_n\} \subset \{e_n\} \cup \{y_n\}$ be a sequence, in which one meets each element e_n and each element y_n *infinitely* many times. Then the set $\{x_n\}$ is generating for the module $\mathcal{M} \oplus H_A$. We will prove the theorem by induction. Let us assume that the orthonormal elements $\overline{e}_1, \dots, \overline{e}_n \in \mathcal{M} \oplus H_A$ and a number $m(n) \geq n$ are already constructed in such a way that

(i) $\{\overline{e}_1, \dots \overline{e}_n\} \subset \mathrm{Span}_A(x_1, \dots, x_n, e_1, \dots, e_{m(n)})$,

(ii) $\mathrm{dist}(x_k, \mathrm{Span}_A(\overline{e}_1, \dots, \overline{e}_k)) \leq \frac{1}{k}$, $1 \leq k \leq n$.

Since each element x_i is equal to e_j or y_k, it is possible to find a number $m' > m(n)$ such that $e_{m'} \perp \{x_1, \dots, x_{n+1}\}$. Since $e_{m'} \perp \{e_1, \dots, e_{m(n)}\}$, so it follows from the condition (i) that

$$e_{m'} \perp \{x_{n+1}, \overline{e}_1, \dots, \overline{e}_n\}.$$

By Lemma 1.4.2 there exists an element

$$\overline{e}_{n+1} \in \mathrm{Span}_A(\overline{e}_1, \dots, \overline{e}_n, x_{n+1}, e_{m'}) \tag{3}$$

such that the elements $\overline{e}_1, \dots, \overline{e}_n, \overline{e}_{n+1}$ are orthonormal and

$$\mathrm{dist}(x_{n+1}, \mathrm{Span}_A(\overline{e}_1, \dots, \overline{e}_{n+1})) \leq \frac{1}{n+1}.$$

It follows from (3) and from the condition (i) that

$$\{\overline{e}_1,\dots,\overline{e}_{n+1}\}\subset \mathrm{Span}_A(x_1,\dots,x_{n+1},e_1\dots,e_{m'}).$$

By putting $m(n+1)=m'$ we complete the step of induction. Thus, an orthonormal sequence $\overline{e}_n$ satisfying the properties (i) and (ii) has been constructed. But the property (ii) means that this sequence generates the whole module $\mathcal{M}\oplus H_A$, therefore $\mathcal{M}\oplus H_A\cong H_A$.

So, the theorem 1.4.1 is proved for unital C^*-algebras. Let A be a C^*-algebra without unit and A^+ be its unitization. By defining the action of A^+ on the Hilbert A-module $\mathcal{M}$ by the formula $x\cdot(a,\lambda):=x\cdot a+x\lambda$, $x\in\mathcal{M}$, $(a,\lambda)\in A^+$, $\lambda\in\mathbf{C}$, we turn $\mathcal{M}$ into a Hilbert A^+-module. Let us consider the A^+-module H_{A^+} and denote by $H_{A^+}A$ the closure in H_{A^+} of the linear span of elements of the form $x\cdot a$, $x\in H_{A^+}$, $a\in A$. It is easy to see that $H_{A^+}A=H_A$. The isomorphism $\mathcal{M}\oplus H_{A^+}\cong H_{A^+}$ implies the isomorphism

$$\mathcal{M}\oplus H_A=\mathcal{M}A\oplus H_{A^+}A=(\mathcal{M}\oplus H_{A^+})A\cong H_{A^+}A=H_A,$$

This completes the proof of the theorem. □

Definition 1.4.3 Let $\mathcal{M}$ be a Hilbert A-module such that there exists a Hilbert A-module $\mathcal{N}$, for which holds $\mathcal{M}\oplus\mathcal{N}\cong L_n(A)$ with finite n. Then $\mathcal{M}$ is called *finitely generated projective* A-module.

The following two theorems of Dupre and Fillmore show, that finite-dimensional projective submodules in Hilbert modules have the sinpliest location.

Theorem 1.4.4 (Dupre – Fillmore, [20]**)** *Let A be a unital C^*-algebra, $\mathcal{M}$ be a finite-dimensional projective A-submodule in the standard Hilbert A-module H_A. Then*

(i) *the nonsingular elements of the module $\mathcal{M}^\perp$ are dense in $\mathcal{M}^\perp$;*

(ii) $H_A=\mathcal{M}\oplus\mathcal{M}^\perp$*;*

(iii) $\mathcal{M}^\perp\cong H_A$.

Proof: We begin the proof of the theorem with the case when $\mathcal{M}\cong L_n(A)$. Let $g_1,\dots,g_n$ be an orthonormal basis in $\mathcal{M}$. We fix $\varepsilon>0$. For each m let's put

$$e'_m=e_m-\sum_{i=1}^n g_i\langle g_i,e_m\rangle,$$

in such a way that $e'_m\in\mathcal{M}^\perp$. Then

$$\langle e'_m,e'_m\rangle=1-\sum_{i=1}^n\langle e_m,g_i\rangle\langle g_i,e_m\rangle.$$

Since the eqality $\langle x,e_m\rangle\to 0$ is fulfilled for each $x\in H_A$, we conclude that $\langle e'_m,e'_m\rangle\to 1$, therefore there exists a number m_0 such that for $m\ge m_0$ the elements e'_m are nonsingular. Then it is possible to define

$$e''_m=e'_m\langle e'_m,e'_m\rangle^{-1/2},$$

such that $\langle e''_m,e''_m\rangle=1$. Let $x\in\mathcal{M}^\perp$. Then

$$\langle e''_m,x\rangle=\langle e'_m,e'_m\rangle^{-1/2}\langle e'_m,x\rangle=\langle e'_m,e'_m\rangle^{-1/2}\langle e_m,x\rangle\to 0.$$

Let us select a number $m\ge m_0$ such that $\|\langle e''_m,x\rangle\|<\varepsilon$ and let us put

$$x'=x+\varepsilon\, e''_m.$$

It is easy to see that

$$\|x'-x\|=\varepsilon. \tag{4}$$

Let us show that the element x' is nonsingular. Put

$$u = x - e''_m \langle e''_m, x\rangle, \qquad v = e''_m(\langle e''_m, x\rangle + \varepsilon\, 1).$$

Then $u \perp v$ (since $u \perp e''_m$) and $x' = u + v$. Therefore,

$$\langle x', x'\rangle = \langle u, u\rangle + \langle v, v\rangle = \langle u, u\rangle + (\langle e''_m, x\rangle + \varepsilon\, 1)^* (\langle e''_m, x\rangle + \varepsilon\, 1), \tag{5}$$

and the right side of the equality (5) is invertible, since $\|\langle e''_m, x\rangle\| < \varepsilon$. Therefore, $\langle x', x'\rangle$ is invertible too. Together with the estimate (4) this proves the statement (i).

Let $\{x_n\}$ be a sequence, in which each element e_m is repeated *infinitely* many times . Let us put $x = x_1 - \sum_{i=1}^n g_i \langle g_i, x_1\rangle$. Then (taking $\varepsilon = 1$) it is possible to find an element $g_{n+1} \in \mathcal{M}^\perp$ such that $\langle g_{n+1}, g_{n+1}\rangle = 1$, $\mathrm{dist}(x, g_{n+1}A) \le 1$, and, therefore $\mathrm{dist}(x_1, \mathrm{Span}_A(g_1, \dots, g_{n+1})) \le 1$. At the next step we replace the module $\mathcal{M}$ by $\mathrm{Span}_A(g_1, \dots, g_{n+1})$, x_1 by x_2, and $\varepsilon = 1$ by $\varepsilon = 1/2$. Going on with the indicated procedure, we will obtain an orthonormal basis $\{g_k\}$, $k \in \mathbf{N}$, extending the basis $g_1, \dots, g_n$ of submodule $\mathcal{M}$, and $\{g_k : k > n\}$ is a basis of the module $\mathcal{M}^\perp$. This proves the statements (ii) and (iii).

We pass now to the case of an arbitrary finitely generated projective module $\mathcal{M}$. Let $\mathcal{M} \oplus \mathcal{N} \cong L_n(A)$. By Theorem 1.4.1 $\mathcal{N} \oplus H_A \cong H_A$ holds, therefore

$$L_n(A) \cong \mathcal{N} \oplus \mathcal{M} \subset \mathcal{N} \oplus H_A \cong H_A.$$

Hence, if $\mathcal{K}$ is the orthogonal complement for submodule $\mathcal{N} \oplus \mathcal{M}$ in the module $\mathcal{N} \oplus H_A$, then $\mathcal{K} \cong H_A$ and $\mathcal{N} \oplus \mathcal{M} \oplus \mathcal{K} = \mathcal{N} \oplus H_A$. But obviously $\mathcal{K} = \mathcal{M}^\perp$ is the orthogonal complement of the submodule $\mathcal{M}$ in the module H_A. $\square$

Theorem 1.4.5 ([20]) *Let A be a unital C^*-algebra and let $\mathcal{M}$ be a finitely generated projective Hilbert submodule in an arbitrary Hilbert A-module $\mathcal{N}$. Then $\mathcal{N} = \mathcal{M} \oplus \mathcal{M}^\perp$.*

Proof: Similarly to the previous theorem the proof can be reduced to the case when $\mathcal{M} = L_n(A)$ is a free module. If $\{g_1, \dots, g_n\}$ is the standard basis of $\mathcal{M}$, and $x \in \mathcal{N}$, then put $x' = x - \sum_{i=1}^n g_i \langle g_i, x\rangle$. Then $x' \in \mathcal{M}$, and $x - x' \in \mathcal{M}^\perp$, therefore, $\mathcal{N} = \mathcal{M} \oplus \mathcal{M}^\perp$. $\square$

2 Operators on Hilbert modules

2.1 Bounded operators and operators admitting an adjoint

Let $\mathcal{M}, \mathcal{N}$ be Hilbert C^*-modules over a C^*-algebra A. The bounded $\mathbf{C}$-linear A-homomorphisms from the module $\mathcal{M}$ to the module $\mathcal{N}$ are called *operators* from $\mathcal{M}$ to $\mathcal{N}$. We denote by $\mathrm{Hom}_A(\mathcal{M}, \mathcal{N})$ the set of all operators from the module $\mathcal{M}$ to the module $\mathcal{N}$. If $\mathcal{N} = \mathcal{M}$, then $\mathrm{End}_A(\mathcal{M}) = \mathrm{Hom}_A(\mathcal{M}, \mathcal{M})$ is obviously a Banach algebra. However, we shall see soon that there is no natural involution on this algebra. Let $T \in \mathrm{Hom}_A(\mathcal{M}, \mathcal{N})$. We say that T *admits an adjoint* if there exists an operator $T^* \in \mathrm{Hom}_A(\mathcal{N}, \mathcal{M})$ such that for all $x \in \mathcal{M}$, $y \in \mathcal{N}$

$$\langle Tx, y\rangle = \langle x, T^* y\rangle.$$

Lemma 2.1.1 *Let $\mathcal{M}$ be a Hilbert A-module, $T : \mathcal{M} \to \mathcal{M}$ and $T^* : \mathcal{M} \to \mathcal{M}$ be maps such that for any $x, y \in \mathcal{M}$*

$$\langle x, Ty\rangle = \langle T^* x, y\rangle.$$

Then T (and T^* as well) is a bounded homomorphism from $\mathrm{End}_A(\mathcal{M})$. Therefore, $T \in \mathrm{End}^*_A(\mathcal{M})$.

Proof: For any x, y and z from $\mathcal{M}$, $w \in \mathbf{C}$ and $a \in A$ one has

$$\langle z, T(x+y)\rangle = \langle T^* z, x+y\rangle = \langle T^* z, x\rangle + \langle T^* z, y\rangle = \langle z, Tx\rangle + \langle z, Ty\rangle = \langle z, Tx + Ty\rangle,$$

$$\langle z, Twx\rangle = \langle T^* z, x\rangle w = \langle z, Tx\rangle w = \langle z, wTx\rangle,$$

$$\langle z, T(xa)\rangle = \langle T^* z, xa\rangle = \langle T^* z, x\rangle a = \langle z, Tx\rangle a = \langle z, (Tx)a\rangle.$$

Since z is an arbitrary element, it follows that

$$T(x+y) = Tx + Ty, \qquad T(wx) = wTx, \qquad T(xa) = (Tx)a,$$

and linearity properties hold.

To prove the continuity T we should verify that its graph is closed. Let $x_\alpha \to x$, $T(x_\alpha) \to y$ in $\mathcal{M}$, and $z \in \mathcal{M}$ be an arbitrary element. Then

$$0 = \langle T^*(y - Tx), x_\alpha \rangle - \langle T^*(y - Tx), x_\alpha \rangle$$

$$= \langle y - Tx, T(x_\alpha)\rangle - \langle T^*(y - Tx), x_\alpha \rangle \longrightarrow \langle y - Tx, y \rangle - \langle T^*(y - Tx), x \rangle = \langle y - Tx, y - Tx \rangle. \qquad \square$$

We show now that there exist operators without adjoint.

Example 2.1.2 Let A be a unital C^*-algebra. As above, the standard basis of the Hilbert module H_A consists of the elements $e_i = (0, \ldots, 0, 1, 0, \ldots)$, where 1 is at the i-th place. It is possible to associate with each operator $E \in \mathrm{End}_A(H_A)$ its matrix with respect to this basis:

$$\|t_{ij}\|, \qquad t_{ij} = \langle e_i, Te_j \rangle.$$

Then the adjoint operator has the matrix $\|t^*_{ji}\|$.

Let $A = C([0,1])$, and let the functions $\varphi_i \in A$, $i = 1, 2, \ldots$, be defined by the equality

$$\varphi_i = \begin{cases} 0 & \text{on } [0, \frac{1}{i+1}] \text{ and } [\frac{1}{i}, 1], \\ 1 & \text{at the point } x_i = \frac{1}{2}\left(\frac{1}{i} + \frac{1}{i+1}\right), \\ \text{islinear} & \text{on } [\frac{1}{i+1}, x_i] \text{ and } [x_i, \frac{1}{i}], \end{cases}.$$

Let an operator $T \in \mathrm{End}_A(H_A)$ has the matrix

$$\begin{pmatrix} \varphi_1 & \varphi_2 & \varphi_3 & \ldots \\ 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

(actually it is an operator from the module H_A to A, i.e. an A-functional). It is easy to verify that T is bounded. But the operator T^* is not well-defined, since it should have the matrix

$$\begin{pmatrix} \varphi_1^* & 0 & 0 & \ldots \\ \varphi_2^* & 0 & 0 & \ldots \\ \varphi_3^* & 0 & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix},$$

but the image of the basis element e_i should have the first column as its coordinates, i.e. an element of H_A, however the series $\sum \varphi_i \varphi_i^*$ does not converge with respect to the norm in C^*-algebra A.

The set of all operators from the module $\mathcal{M}$ to the module $\mathcal{N}$ admitting an adjoint we denote by $\mathrm{Hom}^*_A(\mathcal{M}, \mathcal{N})$. The algebra $\mathrm{End}^*_A(\mathcal{M}) = \mathrm{Hom}^*_A(\mathcal{M}, \mathcal{M})$ is a Banach involutive algebra. Moreover, it is a C^*-algebra; it follows from the estimate

$$\|T^*T\| \geq \sup_{x \in B_1(\mathcal{M})} \{\langle T^*Tx, x \rangle\} = \sup_{x \in B_1(\mathcal{M})} \{\langle Tx, Tx \rangle\} = \|T\|^2,$$

where the unit ball in the module $\mathcal{M}$ is denoted by $B_1(\mathcal{M})$.

The following statement will be used by us frequently without special reference.

Proposition 2.1.3 *For an operator $T : \mathcal{M} \to \mathcal{M}$ the following conditions are equivalent:*

(i) *T is a positive element of C^*-algebra* $\mathrm{End}^*(\mathcal{M})$*;*

(ii) *for any $x \in \mathcal{M}$ the inequality $\langle Tx, x \rangle \geq 0$ is fulfilled, i.e. this element is positive in the algebra A.*

Proof: The first condition is equivalent to the equality $T = S^*S$ for some $S \in \mathrm{End}^*(\mathcal{M})$. Therefore,

$$\langle Tx, x\rangle = \langle Sx, Sx\rangle \geq 0 \qquad \text{for any } x \in \mathcal{M}.$$

Let now $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{M}$. Then

$$\langle Tx, x\rangle = \langle Tx, x\rangle^* = \langle x, Tx\rangle \qquad \text{for all } x \in \mathcal{M}.$$

The map $(x, y) \mapsto \langle Tx, y\rangle$ defines a sesquilinear form on $\mathcal{M}$, therefore, by the polarization equality 1.2.5, $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all x and y from $\mathcal{M}$. This means by Lemma 2.1.1 that $T \in \mathrm{End}^*(\mathcal{M})$ and $T = T^*$. So, T is a selfadjoint element of the algebra $\mathrm{End}^*(\mathcal{M})$ and, therefore (see [17, 1.6.5]), it can be represented in the form of a difference $T = T_+ - T_-$ of two elements of $\mathrm{End}^*(\mathcal{M})$, $T_+ \geq 0$, $T_- \geq 0$ and $T_+T_- = T_-T_+ = 0$. Then $\langle T_-y, y\rangle \leq \langle T_+y, y\rangle$ for any $y \in \mathcal{M}$. In particular,

$$\langle T_-^3 x, x\rangle = \langle T_-^2 x, T_-x\rangle \leq \langle T_+T_-x, T_-x\rangle = 0.$$

On the other hand, $T_- \geq 0$ and $T_-^3 \geq 0$, therefore $\langle T_-^3 x, x\rangle \geq 0$ (as the statement in this direcrtion is already proved). So, we have the unique possibility: $\langle T_-^3 x, x\rangle = 0$ for any $x \in \mathcal{M}$. By the polarization equality, this implies $\langle T_-^3 x, y\rangle = 0$ for all x and y from $\mathcal{M}$, and $T_-^3 = 0$, $T_- = 0$. Hence, $T = T_+ \geq 0$. $\square$

Theorem 2.1.4 ([52]) *Let $\mathcal{M}$ and $\mathcal{N}$ be Hilbert A-modules, $T : \mathcal{M} \longrightarrow \mathcal{N}$ be a linear map. Then the following conditions are equivalent:*

(i) *the operator T is bounded and A-linear, i.e. $T(x \cdot a) = Tx \cdot a$ for all $x \in \mathcal{M}$, $a \in A$;*

(ii) *there exists a constant $K \geq 0$ such that for all $x \in \mathcal{M}$ the operator inequality $\langle Tx, Tx\rangle \leq K\langle x, x\rangle$ holds.*

Proof: To obtain the second statement from the first one, let us assume that $T(x \cdot a) = Tx \cdot a$ and $\|T\| \leq 1$. If C^*-algebra A does not contain a unit, then we the consider modules $\mathcal{M}$ and $\mathcal{N}$ as modules over C^*-algebra A^+, obtained from A by unitizaton. For $x \in \mathcal{M}$ and $n \in \mathbf{N}$ let us put

$$a_n = \left(\langle x, x\rangle + \frac{1}{n}\right)^{-1/2}, \qquad x_n = x \cdot a_n.$$

Then $\langle x_n, x_n\rangle = a_n^* \langle x, x\rangle a_n = \langle x, x\rangle \big(\langle x, x\rangle + \frac{1}{n}\big)^{-1} \leq 1$, therefore, $\|x_n\| \leq 1$, hence $\|Tx_n\| \leq 1$. Then for all $n \in \mathbf{N}$ the operator inequality $\langle Tx_n, Tx_n\rangle \leq 1$ is valid. But

$$\langle Tx, Tx\rangle = a_n^{-1}\langle Tx_n, Tx_n\rangle a_n^{-1} \leq a_n^{-2} = \langle x, x\rangle + \frac{1}{n}. \tag{1}$$

Passing in the inequality (1) to the limit $n \to \infty$, we obtain $\langle Tx, Tx\rangle \leq \langle x, x\rangle$.

To derive the first statement from the second one we assume that for all $x \in \mathcal{M}$ the inequality $\langle Tx, Tx\rangle \leq \langle x, x\rangle$ is fulfilled. It obviously follows from it that the operator T is bounded, $\|T\| \leq 1$. Let $x \in \mathcal{M}$, $y \in \mathcal{N}$. Let us define a map $r : A^+ \longrightarrow A^+$ by the equality

$$r(a) = \langle y, T(x \cdot a)\rangle.$$

Then

$$\begin{aligned} r(a)^* r(a) &= \langle T(x \cdot a), y\rangle\langle y, T(x \cdot a)\rangle \leq \|y\|^2 \langle T(x \cdot a), T(x \cdot a)\rangle \leq \|y\|^2 \langle x \cdot a, x \cdot a\rangle = \|y\|^2 a^*\langle x, x\rangle a \\ &\leq \|y\|^2 \|x\|^2 a^* a. \end{aligned}$$

To complete the proof we use the following statement.

Lemma 2.1.5 ([28, 52]) *Let A be a unital C^*-algebra let $r : A \longrightarrow A$ be a linear map such that for some constant $K \geq 0$ the inequality $r(a)^* r(a) \leq K a^* a$ is fulfilled for all $a \in A$. Then $r(a) = r(1)a$ for all $a \in A$.* $\square$

Therefore, $r(a) = r(1)a$, i.e.

$$\langle y, T(x \cdot a)\rangle = \langle y, Tx\rangle a = \langle y, Tx \cdot a\rangle$$

for all $y \in \mathcal{N}$, $x \in \mathcal{M}$. This implies the statement of the theorem. $\square$

Corollary 2.1.6 *Let $\mathcal{M}, \mathcal{N}$ be Hilbert A-modules, $T \in \mathrm{End}_A(\mathcal{M}, \mathcal{N})$. Then*

$$\|T\| = \inf\left\{K^{1/2} : \langle Tx, Tx\rangle \le K\langle x, x\rangle \quad \forall x \in \mathcal{M}\right\}. \qquad \square$$

Example 2.1.7 Let $\mathcal{M} = \mathcal{N} \oplus \mathcal{L}$ be a decomposition into an orthogonal direct sum of Hilbert modules. We define an operator $P : \mathcal{M} \longrightarrow \mathcal{M}$ as the operator of projection onto a submodule $\mathcal{N}$ along the module $\mathcal{L}$. Then P is bounded, $\|P\| = 1$, and $P^* = P$, therefore $P \in \mathrm{End}^*_A(\mathcal{M})$.

2.2 Compact operators in Hilbert module

Let $\mathcal{M}, \mathcal{N}$ be Hilbert A-modules, $x \in \mathcal{N}$, $y \in \mathcal{M}$. Let us define an action of an operator $\theta_{x,y} : \mathcal{M} \longrightarrow \mathcal{N}$ on an element $z \in \mathcal{M}$ by the formula

$$\theta_{x,y}(z) := x\langle y, z\rangle. \tag{2}$$

The operators of the form (2) are called *elementary* operators. They obviously satisfy the equalities

(i) $(\theta_{x,y})^* = \theta_{y,x}$;

(ii) $\theta_{x,y}\theta_{u,v} = \theta_{x\langle y,u\rangle,v} = \theta_{x,v\langle u,y\rangle}$ for $u \in \mathcal{M}$, $v \in \mathcal{N}$;

(iii) $T\theta_{x,y} = \theta_{Tx,y}$ for $T \in \mathrm{Hom}_A(\mathcal{N}, \mathcal{L})$;

(iv) $\theta_{x,y}S = \theta_{x,S^*y}$ for $S \in \mathrm{Hom}^*_A(\mathcal{L}, \mathcal{M})$.

We denote the closed linear span of the set of all elementary operators by $\mathcal{K}(\mathcal{M}, \mathcal{N})$. The elements of $\mathcal{K}(\mathcal{M}, \mathcal{N})$ we shall call *compact* operators. In the case $\mathcal{N} = \mathcal{M}$ the equalities (i)–(iv) mean that the algebra $\mathcal{K}(\mathcal{M}) = \mathcal{K}(\mathcal{M}, \mathcal{M})$ is a closed two-sided ideal in the C^*-algebra $\mathrm{End}^*_A \mathcal{M})$. Compact operators acting on Hilbert modules are not compact operators in the usual sence one considers them as operators from one Banach space to another. However, they are a natural generalization of compact operators on a Hilbert space.

Proposition 2.2.1 *Let H_A be the standard Hilbert module over a unital C^*-algebra A, $L_n(A) \subset H_A$ be free submodule generated by the first n basis elements. An operator $K \in \mathrm{End}^*_A(H_A)$ is compact if and only if norms of restrictions of the operator K onto orthogonal complements to submodules $L_n(A)$ tend to zero.*

Proof: Let us denote by P_n the projection in H_A onto the submodule $L_n(A)^\perp$. Then for any $z \perp L_n(A)$ we have

$$\begin{aligned}\|\theta_{x,y}(z)\|^2 &= \|\langle\theta_{x,y}(z), \theta_{x,y}(z)\rangle\| = \left\|\langle y, z\rangle^*\langle x, x\rangle\langle y, z\rangle\right\| \\ &\le \|x\|^2 \|\langle y, z\rangle\|^2 = \|x\|^2 \|\langle P_n y, z\rangle\|^2 \\ &\le \|x\|^2 \cdot \|P_n y\|^2 \cdot \|z\|^2 .\end{aligned}$$

Since $\|P_n y\|$ tends to zero, the same is true for the norm of the restriction of the operator $\theta_{x,y}$ to the submodule $L_n(A)^\perp$ and, therefore, for norm of any compact operator. Let us assume now that for some operator K $\left\|K|_{L_n(A)^\perp}\right\| \to 0$ holds. Then, since $\sum_{m=1}^n Ke_m\langle e_m, z\rangle = 0$ for any $z \perp L_n(A)$, we have for $\|z\| \le 1$ and $z \perp L_n(A)$

$$\sup_z \left\| Kz - \sum_{m=1}^n Ke_m\langle e_m, z\rangle \right\| = \sup_z \|Kz\| \longrightarrow 0 \tag{3}$$

as $n \to \infty$. If $z \in L_n(A)$ then $Kz = \sum_{m=1}^n Ke_m\langle e_m, z\rangle$. It means that (3) holds also if the supremum is taken over the unit ball of the whole module H_A, therefore the operator K is a norm limit of the operators $K_n = \sum_{m=1}^n \theta_{Ke_m,e_m}$. $\square$

Let us remark that in the case of modules over C^*-algebras without unit the statement of 2.2.1 is not valid.

We denote the C^*-algebra of compact operators of separable Hilbert space H by $\mathcal{K}$. Since the algebra $\mathcal{K}$ is nuclear [37], there is the unique C^*-seminorm on the algebraic tensor product of $\mathcal{K}$ by any C^*-algebra A, and we denote by $\mathcal{K}\otimes A$ the completion with respect to this seminorm. We denote by $\mathcal{M}_n(A) = \mathcal{M}_n \otimes A$ the C^*-algebra of all $n\times n$-matrices over A.

Proposition 2.2.2 *There exist the following natural isometric isomorphisms*

(i) $\mathcal{K}(A) \cong A$;

(ii) $\mathcal{K}(L_n(A)) \cong \mathcal{M}_n(A)$;

(iii) $\mathcal{K}(H_A) \cong \mathcal{K} \otimes A$.

Proof: If a C^*-algebra is unital then the statement (i) is obvious. In the general case we consider a map $\varphi : \mathrm{Span}_{\mathbf{C}}(\theta_{a,b} : a, b \in A) \longrightarrow A$, defined by the formula

$$\varphi\left(\sum_{i=1}^{n} \lambda_i \theta_{a_i,b_i}\right) = \sum_{i=1}^{\infty} \lambda_i a_i b_i^*.$$

Let us verify that this map is well-defined: if $\sum_i \lambda_i \theta_{a_i,b_i} = \sum_j \mu_j \theta_{c_j,d_j}$, then $\sum_i \lambda_i a_i b_i^* x = \sum_j \mu_j c_j d_j^* x$ for any $x \in A$, therefore, $\sum_i \lambda_i a_i b_i^* = \sum_j \mu_j c_j d_j^*$. The map φ is multiplicative and involutive:

$$\varphi(\theta_{a,b}\ \theta_{c,d}) = \varphi(\theta_{ab^*,dc^*}) = \varphi(\theta_{a,b})\varphi(\theta_{c,d}); \qquad \varphi(\theta_{a,b}^*) = \varphi(\theta_{b,a}) = \varphi(\theta_{a,b})^*.$$

Surjectivity of φ follows from the possibility of representation $a = u(a^*a)^{1/4}$ for any $a \in A$ (see 1.1.5). If (u_α), $\alpha \in \mathcal{A}$, is an approximate unit of the algebra A then

$$\lim_\alpha \left\| \sum_{i=1}^{n} \lambda_i \theta_{a_i,b_i}(u_\alpha) \right\| = \left\| \sum_{i=1}^{n} \lambda_i a_i b_i^* \right\|,$$

therefore $\|\varphi(k)\| \leq \|k\|$ for $k = \sum_{i=1}^{n} \lambda_i \theta_{a_i,b_i}$. It means that the map φ can be extended by continuity up to a map from the whole algebra $\mathcal{K}(A)$. The inequality

$$\left\| \sum_{i=1}^{\infty} \lambda_i \theta_{a_i,b_i} \right\| = \sup_{\|x\|\leq 1} \left\| \sum_{i=1}^{\infty} \lambda_i a_i b_i^* x \right\| \leq \left\| \sum_{i=1}^{\infty} \lambda_i a_i b_i^* \right\| = \left\| \varphi\left(\sum_{i=1}^{\infty} \lambda_i \theta_{a_i,b_i} \right) \right\|$$

shows that the map φ is an isometry, so the statement (i) is proved. The statement (ii) can be proved similarly with the use of the map

$$\varphi_n : \theta_{a_1\oplus\cdots\oplus a_n, b_1\oplus\cdots\oplus b_n} \longmapsto \begin{pmatrix} a_1 b_1^* & \dots & a_1 b_n^* \\ \vdots & & \vdots \\ a_n b_1^* & \dots & a_n b_n^* \end{pmatrix}.$$

Finally, as the isometric map from the linear space $\cup_n \mathcal{K}(L_n(A))$ to the linear space $\cup_n \mathcal{M}_n(A)$ is defined, and as these spaces are dense in C^*-algebras $\mathcal{K}(H_A)$ and $\mathcal{K} \otimes A$ respectively, so we obtain the statement (iii). $\square$

Lemma 2.2.3 *Let $x \in \mathcal{M}$ be an arbitrary element. Then there exists $z \in \mathcal{M}$ and $k = \theta_{u,v} \in \mathcal{K}(\mathcal{M})$ such that $x = kz$.*

Proof: Let us put

$$u := v := z := \lim_{\varepsilon\to 0} x\,(\varepsilon + \langle x, x\rangle^{1/3})^{-1}.$$

As $s^2(\varepsilon+s)^{-1}$ is uniformly convergent to s on bounded sets, so in order to prove that u is well-defined we should remark that for $t=\langle x,x\rangle$ one has

$$\begin{aligned}\langle x\,(\varepsilon+\langle x,x\rangle^{1/3})^{-1}-x\,(\mu+\langle x,x\rangle^{1/3})^{-1},x\,(\varepsilon+\langle x,x\rangle^{1/3})^{-1}-x\,(\mu+\langle x,x\rangle^{1/3})^{-1}\rangle\\ =[(\varepsilon+t^{1/3})^{-1}-(\mu+t^{1/3})^{-1}]t[(\varepsilon+t^{1/3})^{-1}-(\mu+t^{1/3})^{-1}]\\ =[(\varepsilon+t^{1/3})^{-1}-(\mu+t^{1/3})^{-1}]^2(t^{1/3})^4.\end{aligned}$$

The same argument shows that $x=kz$. $\square$

Remark that we have also proved that $\mathcal{M}\langle\mathcal{M},\mathcal{M}\rangle=\mathcal{M}$.

Theorem 2.2.4 *A Hilbert A-module $\mathcal{M}$ is countably generated if and only if the C^*-algebra $\mathcal{K}(\mathcal{M})$ is σ-ubital.*

Proof: Let the algebra $\mathcal{K}(\mathcal{M})$ be σ-unital and $æ_n$ be a countable approximate unit for it. Then

$$x=\lim_{n\to\infty}æ_n x \quad \text{for any } x\in\mathcal{M}. \tag{4}$$

Really, by Lemma 2.2.3, $x=kz$ holds for some $k\in\mathcal{K}(\mathcal{M})$, $z\in\mathcal{M}$. As $æ_n k\longrightarrow k$ with respect to the norm, so $æ_n x=æ_n kz\longrightarrow kz=x$.

By definition, any compact operator is close to a linear combination of elementary ones. Hence, for each $æ_n$ there exist elements x_i^n and y_i^n from $\mathcal{M}$ such that

$$\left\|\sum_{i=1}^{s(n)}\theta_{x_i^n,y_i^n}-æ_n\right\|<\frac{1}{n},\qquad n=1,2,\ldots.$$

Let us show that the countable set x_i^n, $i=1,\ldots,s(n)$, $n=1,2,\ldots$, generates the module $\mathcal{M}$. Let us consider an arbitrary element $x\in\mathcal{M}$ and arbitrary small $\varepsilon>0$. By (4) we can find n so big that

$$\|x-æ_n x\|<\frac{\varepsilon}{2}\qquad u\qquad \frac{1}{n}<\frac{\varepsilon}{2}.$$

Then

$$\left\|x-\sum_{i=1}^{s(n)}x_i^n\cdot\langle y_i^n,x\rangle\right\|\le\|x-æ_n(x)\|+\left\|æ_n(x)-\sum_{i=1}^{s(n)}\theta_{x_i^n,y_i^n}(x)\right\|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

Let now a module $\mathcal{M}$ be countably generated. It can be considered as a module over the algebra A^+ obtained by unitization of A (if it was not unital) with respect to the action $x\cdot(a,\mu):=x\cdot a+\mu x$, $x\in\mathcal{M}$, $a\in A$ и $\mu\in\mathbf{C}$. If it was countably generated over A then it should be countably generated over A^+ too. Since in the definition of elementary compact operators only the A-inner product is involved, then $\mathcal{K}_A(\mathcal{M})=\mathcal{K}_{A^+}(\mathcal{M})$. Thus we can restrict ourselves to the case of unital algebra A.

So, $\mathcal{M}$ is a countably generated module over the unital algebra A. By the Kasparov stabilization theorem $\mathcal{M}\oplus H_A\cong H_A$. Let $\iota:\mathcal{M}\to H_A$ be the corresponding inclusion and $\pi:H_A\to\mathcal{M}$ be the corresponding selfajoint projection. Let $\{e_i\}$ be the standard basis of H_A. Remind that for a C^*-algebra the property of being σ-unital is equivalent to that of having a strictly positive element. Let us consider

$$æ:=\sum_{n=1}^{\infty}\frac{\theta_{e_n,e_n}}{n},$$

or in a matrix form

$$æ:=\operatorname{diag}(1,\frac{1}{2},\frac{1}{3}\ldots).$$

Then $æ$ is a strictly positive element in $\mathcal{K}(H_A)$. Indeed, on the one hand, by Proposition 2.2.1 we have $æ\in\mathcal{K}(H_A)$. On the other hand, if $\rho:\mathcal{K}(H_A)\to\mathbf{C}$ is a state such that $\rho(æ)=0$, then $\rho(\theta_{e_n,e_n})=0$ for any n, since all $\theta_{e_n,e_n}\ge 0$. Then for any $x=(x_1,x_2,\ldots)\in H_A$

$$\rho(\theta_{e_n,x}\theta_{x,e_n})=\rho\left(\left(\sum_{j=1}^{\infty}\theta_{e_n,e_jx_j}\right)\left(\sum_{j=1}^{\infty}\theta_{e_n,e_jx_j}\right)^*\right)=\rho\left(\sum_{j=1}^{\infty}\theta_{e_n,e_jx_j}\theta^*_{e_n,e_jx_j}\right)$$

$$= \rho\left(\sum_{j=1}^{\infty}\theta_{e_n,e_jx_j}\theta_{e_jx_j,e_n}\right) = \rho\left(\sum_{j=1}^{\infty}\theta_{e_n\cdot\langle e_jx_j,e_jx_j\rangle,e_n}\right) \le \|x\|^2\sum_{j=1}^{\infty}\rho(\theta_{e_n,e_n}) = 0,$$

since the last inequality follows from

$$\langle\theta_{e_n\cdot\langle e_jx_j,e_jx_j\rangle,e_n}(z),z\rangle = \langle e_n\cdot\langle e_jx_j,e_jx_j\rangle\langle e_n,z\rangle,z\rangle = \langle z,e_n\rangle x_j^*x_j\langle e_n,z\rangle \le \|x\|^2\langle\theta_{e_n,e_n}z,z\rangle.$$

Thus for any x, y and z from $\mathcal{M}$,

$$\theta_{x,y}(z) = x\cdot\langle y,z\rangle = \theta_{x,e_n}\theta_{e_n,y}(z)$$

and

$$|\rho(\theta_{x,y})| = |\rho(\theta_{x,e_n}\theta_{e_n,y})| \le \rho^{1/2}(\theta_{x,e_n}\theta_{e_n,x})\,\rho^{1/2}(\theta_{e_n,y}\theta_{y,e_n}) = 0,$$

as the second multiplicand vanishes. So ρ vanishes on a dense subset, consequently everywhere on $\mathcal{K}(H_A)$. We have shown that æ is a strictly positive element of $\mathcal{K}(H_A)$. Then $æ_n := æ^{1/n}$ is a countable approximate unit of $\mathcal{K}(H_A)$, and $\pi æ_n\iota$ is a countable approximate unit of $\mathcal{K}(\mathcal{M})$. Indeed, if $k\in\mathcal{K}(\mathcal{M})$, $\pi\iota k = k$, then $\iota k\pi\in\mathcal{K}(H_A)$ and

$$\|k-\pi æ_n\iota\| = \|\pi(\iota k\pi - æ_n)\iota\| = \|\iota k\pi - æ_n\| \longrightarrow 0 \qquad (n\longrightarrow\infty). \qquad \square$$

2.3 Complemented submodules and projections in Hilbert C^*-modules

Let us remind that a closed submodule $\mathcal{N}$ of Hilbert C^*-module $\mathcal{M}$ is called orthogonally complemented, if $\mathcal{M} = \mathcal{N}\oplus\mathcal{N}^\perp$. As we have already seen, a closed submodule of Hilbert C^*-module can be orthogonally uncomplemented.

Definition 2.3.1 A closed submodule $\mathcal{N}$ of Hilbert C^*-module $\mathcal{M}$ is called (*topologically*) *complemented,* if there exists a closed submodule $\mathcal{L}$ in $\mathcal{M}$ such that $\mathcal{N}+\mathcal{L} = \mathcal{M}$, $\mathcal{N}\cap\mathcal{L} = \{0\}$.

The following example shows that there exist topologically complemented but orthogonally uncomplemented submodules.

Example 2.3.2 Let $J\subset A$ be an ideal such that the equality $Ja = 0$, $a\in A$, implies $a = 0$. Let us put $\mathcal{M} = A\oplus J$,

$$\mathcal{N} = \{(b,b) : b\in J\}.$$

Then

$$\mathcal{N}^\perp = \{(c,-c) : c\in J\}.$$

Therefore, $\mathcal{N}\oplus\mathcal{N}^\perp = J\oplus J \ne \mathcal{M}$. However, the submodule $\mathcal{L} = \{(a,0) : a\in A\}\subset\mathcal{M}$ is a topological complement to $\mathcal{N}$ in $\mathcal{M}$.

We denote non-ortogonal direct sum of Hilbert modules by $\mathcal{N}\widetilde{\oplus}\mathcal{L}$. A decomposition in a direct sum $\mathcal{M} = \mathcal{N}\widetilde{\oplus}\mathcal{L}$ allows to define a projection P onto $\mathcal{N}$ along $\mathcal{L}$. The operator P is A-linear and, by the closed graph theorem, is bounded, therefore $P\in\mathrm{End}_A(\mathcal{M})$. However, as it is clesr from Example 2.3.2, the projection P can not admit an adjoint. But if $\mathcal{M} = \mathcal{N}\oplus\mathcal{L}$, then the corresponding projection is selfadjoint, $P\in\mathrm{End}^*_A(\mathcal{M})$. Since it is more convenient to work with orthogonal decompositions, we would like to know, when such decomposition exists.

Theorem 2.3.3 ([46]) *Let $\mathcal{M}$, $\mathcal{N}$ be Hilbert A-modules, $T\in\mathrm{Hom}^*_A(\mathcal{M},\mathcal{N})$ is an operator with closed image. Then*

(i) $\mathrm{Ker}\,T$ *is an orthogonally complemented submodule in* $\mathcal{M}$,

(ii) $\mathrm{Im}\,T$ *is an orthogonally complemented submodule in* $\mathcal{N}$.

Proof: (i) Let $\mathrm{Im}\,T = \mathcal{N}_0$ and let $T_0 : \mathcal{M}\longrightarrow\mathcal{N}_0$ be an operator such that its action coincides with the action of T. By the open mapping theorem, the image of the unit ball of $T_0(B_1(\mathcal{M}))$ contains some ball of radius $\delta > 0$ in $\mathcal{N}_0$. Therefore, for each $y\in\mathcal{N}_0$ it is possible to find $x\in\mathcal{M}$ such that $T_0x = y$ and $\|x\|\le\delta^{-1}\|y\|$.

$$\|T_0^*y\|^2 = \|\langle y,T_0T_0^*y\rangle\| \le \|y\|\cdot\|T_0T_0^*y\|,$$

and hence,

$$\|y\|^2 = \|\langle T_0 x, y\rangle\| = \|\langle x, T_0^* y\rangle\| \le \|x\| \cdot \|T_0^* y\| \le \delta^{-1} \|y\| \cdot \|y\|^{1/2} \|T_0 T_0^* y\|^{1/2}.$$

We obtain, that for any $y \in \mathcal{N}_0$

$$\|y\| \le \delta^{-2} \|T_0 T_0^* y\|.$$

Let us show that the spectrum of the operator $T_0 T_0^*$ does not contain the origin. Suppose the opposite, i.e. that $0 \in \mathrm{Sp}(T_0 T_0^*)$. Let f be a continuous function on $\mathbf{R}$ such that

$$f(0) = 1 = \|f\|, \qquad f(t) = 0 \text{ при } |t| \ge \frac{1}{2}\delta^{-2}.$$

Using functional calculus in the C^*-algebra $\mathrm{End}_A^*(\mathcal{M})$, we define the operator $S \in \mathrm{End}_A^*(\mathcal{M})$ by the equality $S = f(T_0 T_0^*)$. Then $\|S\| = 1$ and $\|T_0 T_0^* S\| \le \frac{1}{2}\delta^{-2}$. We can choose an element $x \in \mathcal{M}$ such that $\|x\| = 1$, $\|Sx\| > \frac{1}{2}$. Then

$$\|T_0 T_0^* Sx\| \le \frac{1}{2}\delta^{-2} < \delta^{-2} \|Sx\|$$

is a contradiction to the supposition (with $y = Sx$). So $0 \notin \mathrm{Sp}(T_0 T_0^*)$, therefore the operator $T_0 T_0^*$ is invertible, and, in particular, surjective. For any $z \in \mathcal{M}$ it is possible to find an element $w \in \mathcal{N}_0$ such that $T_0 z = T_0 T_0^* w$. Then $z - T_0^* w \in \mathrm{Ker}\, T$ and

$$z = (z - T_0^* w) + T_0^* w \in \mathrm{Ker}\, T + \mathrm{Im}\, T_0^*.$$

Since the module $\mathrm{Im}\, T_0^*$ is obviously orthogonal to $\mathrm{Ker}\, T$, it is a complement of $\mathrm{Ker}\, T$, that completes the proof of (i).

(ii) Since $\mathcal{M} = \mathrm{Ker}\, T \oplus \mathrm{Im}\, T_0^*$, the submodule $\mathrm{Im}\, T_0^*$ is closed. Let us remark that $\mathrm{Im}\, T_0^* = \mathrm{Im}\, T^*$, therefore it is possible to apply the proof of (i) to the case of the operator T^* instead of T, and it gives the orthogonal decomposition $\mathcal{N} = \mathrm{Ker}\, T^* \oplus \mathrm{Im}\, T$. □

Corollary 2.3.4 *If $P \in \mathrm{End}_A^*(\mathcal{M})$ is an idempotent, then its image $\mathrm{Im}\, P$ is an orthogonally complemented submodule in $\mathcal{M}$.* □

Corollary 2.3.5 *Let $\mathcal{M}, \mathcal{N}$ be Hilbert A-modules, $F : \mathcal{M} \to \mathcal{N}$ be a topologically injective A-homomorphism (i.e. $\|Fx\| \ge \delta \|x\|$ for some $\delta > 0$ and for all $x \in M$) admitting an adjoint operator, then $F(\mathcal{M}) \oplus F(\mathcal{M})^\perp = \mathcal{N}$.* □

Corollary 2.3.6 *Let $\mathcal{M}$ be a Hilbert A-module, $J : \mathcal{M} \to \mathcal{M}$ be a selfadjoint topologically injective A-homomorphism. Then J is an isomorphism.* □

Lemma 2.3.7 ([46]) *Let $\mathcal{M}$ be a finitely generated Hilbert submodule in a Hilbert module $\mathcal{N}$ over a unital C^*-algebra. Then $\mathcal{M}$ is an orthogonal direct summand in $\mathcal{N}$.*

Proof: Let $x_1, \dots, x_n \in \mathcal{M}$ be a finite set of generators. Let us define an operator $F : L_n(A) \longrightarrow \mathcal{N}$ by the formula $F(e_i) = x_i$, where $e_i \in L_n(A)$ is the standard basis, $i = 1, \dots, n$. It is easy to see that the operator F admits the adjoint $F^* : \mathcal{N} \longrightarrow L_n(A)$ acting by the formula $F^*(x) = (\langle x_1, x\rangle, \dots, \langle x_n, x\rangle)$, where $x \in \mathcal{N}$. By Theorem 2.3.3 the module $\mathrm{Im}\, F = \mathcal{M}$ is an orthogonal direct summand. □

Lemma 2.3.8 *Let A be a unital C^*-algebra and let $H_A = \mathcal{M} \widetilde{\oplus} \mathcal{N}$, $\quad p : H_A \to \mathcal{M}$ be a projection, $\mathcal{N}$ be a projective module. Then $H_A = \mathcal{M} \oplus \mathcal{M}^\perp$ if and only if p admits an adjoint.*

Proof: If there exists p^*, then $(1-p)^* = 1 - p^*$ exists too. Therefore, by Theorem 2.3.3 $\mathrm{Ker}(1-p) = \mathcal{M}$ is the image of a selfadjoint projection.

To prove the converse, let us verify at first, that $H_A = \mathcal{N}^\perp + \mathcal{M}^\perp$. By the Kasparov stabilization theorem, it is possible to suppose without loss of generality, that $\mathcal{N} = \mathrm{span}_A\langle e_1, \dots, e_n\rangle$, $\mathcal{N}^\perp = \mathrm{span}_A\langle e_{n+1}, e_{n+2}, \dots\rangle$. Let g_i be the images of e_i under the projection $\mathcal{N}$ onto $\mathcal{M}^\perp$:

$$e_1 = f_1 + g_1, \dots, e_n = f_n + g_n, \quad f_i \in \mathcal{M},\ g_i \in \mathcal{M}^\perp.$$

Since the projection realizes an isomorphism of A-modules $\mathcal{N} \cong \mathcal{M}^\perp$, the elements $g_1, \ldots, g_n$ are free generators and $\langle g_k, g_k \rangle > 0_A$. So, if

$$f_k = \sum_{i=1}^{\infty} f_k^i e_i, \quad \text{then} \quad e_k - f_k^k e_k = \sum_{i \neq k} f_k^i e_i + g_k.$$

On the other hand,

$$1 = \langle e_k, e_k \rangle = \langle f_k, f_k \rangle + \langle g_k, g_k \rangle, \quad 1 - (f_k^k)(f_k^k)^* \geq \langle g_k, g_k \rangle > 0,$$

i.e. the spectrum is separated from the origin. Then the element $1 - f_k^k$ is invertible in A,

$$e_k = \frac{1}{1 - f_k^k} \left(\sum_{i \neq k} f_k^i e_i + g_k \right) \in \mathcal{N}^\perp + \mathcal{M}^\perp \quad (k = 1, \ldots, n),$$

and, therefore $\mathcal{N}^\perp + \mathcal{M}^\perp = H_A$. Let $x \in \mathcal{N}^\perp \cap \mathcal{M}^\perp$. Since any element $y \in H_A = \mathcal{M} \widetilde{\oplus} \mathcal{N}$ has the form $y = m + n$, then $\langle x, y \rangle = \langle x, m \rangle + \langle x, n \rangle = 0$; in particular, $\langle x, x \rangle = 0$ and, therefore $x = 0$. Thus $H_A = \mathcal{N}^\perp \widetilde{\oplus} \mathcal{M}^\perp$. Let us consider the following map $q = \begin{cases} 1 & \text{on } \mathcal{N}^\perp \\ 0 & \text{on } \mathcal{M}^\perp \end{cases}$, which is a bounded projection, since $H_A = \mathcal{N}^\perp \widetilde{\oplus} \mathcal{M}^\perp$. Let $x + y \in \mathcal{M} \widetilde{\oplus} \mathcal{N}$, $x_1 + y_1 \in \mathcal{N}^\perp \widetilde{\oplus} \mathcal{M}^\perp$. Then

$$\begin{aligned} \langle p(x + y), x_1 + y_1 \rangle &= \langle x, x_1 + y_1 \rangle = \langle x, x_1 \rangle, \\ \langle x + y, q(x_1 + y_1) \rangle &= \langle x + y, x_1 \rangle = \langle x, x_1 \rangle. \end{aligned}$$

Therefore, there exists $p^* = q$. $\square$

2.4 Full Hilbert C^*-modules

Let $\mathcal{M}$ be a Hilbert A-module . We denote by $\langle \mathcal{M}, \mathcal{M} \rangle \in A$ the closure of the linear span of the elements of the form $\langle x, x \rangle$, $x \in \mathcal{M}$. It is obvious, that the set $\langle \mathcal{M}, \mathcal{M} \rangle$ is a closed two-sided involutive ideal in the C^*-algebra A.

Definition 2.4.1 A Hilbert A-module is called *full*, if $\langle \mathcal{M}, \mathcal{M} \rangle = A$.

It is possible to consider any Hilbert module as a full Hilbert module over the C^*-algebra $\langle \mathcal{M}, \mathcal{M} \rangle$. The following statement gives an example showing that this is a useful notion.

Theorem 2.4.2 ([38, 20]) *Let A be a σ-unital C^*-algebra, $\mathcal{M}$ be a full Hilbert A-module. Then*

(i) *there exists a Hilbert A-module $\mathcal{N}$ such that $l_2(\mathcal{M}) \cong H_A \oplus \mathcal{N}$. If a C^*-algebra A is unital, then there exists a number n and a Hilbert A-module $\mathcal{N}'$ such that $\mathcal{M} \oplus \ldots \oplus \mathcal{M} = \mathcal{M}^n \cong A \oplus \mathcal{N}'$;*

(ii) *if the module $\mathcal{M}$ is countably generated then $l_2(\mathcal{M}) \cong H_A$.*

Proof: Let us consider the following set

$$S = \left\{ c \in A : \|c\| \leq 1, c = \sum_{i=1}^{k} \langle x_i, x_i \rangle, k \in \mathbf{N}, x_i \in \mathcal{M} \right\}.$$

For the proof of the theorem two following lemmas will be necessary.

Lemma 2.4.3 ([13]) *For any $a \in A$, $a \geq 0$, and any $\varepsilon > 0$ there exists $c \in S$ such that $\|(1 - c)a\| < \varepsilon$.*

Proof: Since the module $\mathcal{M}$ is full, it is possible to find a finite set of elements $y_i \in \mathcal{M}$ such that

$$\left\| a - \sum_{i=1}^{k} \langle y_i, y_i \rangle \right\| < \varepsilon/2. \tag{5}$$

Let us put

$$x_i = y_i(\varepsilon^2 + \sum_{j=1}^{k} \langle y_j, y_j\rangle)^{-1/2}, \quad i = 1, \dots, k; \qquad c = \sum_{i=1}^{k} \langle x_i, x_i\rangle, \qquad b = \sum_{i=1}^{k} \langle y_i, y_i\rangle.$$

Then $\|c\| = \left\|(\varepsilon^2 + b)^{-1/2} b(\varepsilon^2 + b)^{-1/2}\right\| \le 1$, therefore $c \in S$. Let $f(t) := \varepsilon^4 t^2 (\varepsilon^2 + t)^{-2}$. Applying this function to the element b, we obtain the estimate

$$\|f(b)\| = \left\|\varepsilon^4 b^2 (\varepsilon^2 + b)^{-2}\right\| = \left\|\varepsilon^2(\varepsilon^2 + b)^{-1} b^2 \varepsilon^2 (\varepsilon^2 + b)^{-1}\right\| = \left\|(1 - c)b^2(1 - c)\right\| =\le \varepsilon^2/4.$$

Therefore, $\|(1 - c)b\| \le \varepsilon/2$, and together with the estimate (5) it proves the lemma. □

Lemma 2.4.4 ([13]) *In the module $\mathcal{M}$ there exists a sequence (x_i), $x_i \in \mathcal{M}$, such that the sequence of partial sums of the series $\sum_{i=1}^{k} \langle x_i, x_i\rangle$ is a countable approximate unit of the algebra A. If is unital then there exists a finite number k and elements $x_1, \dots, x_k \in \mathcal{M}$ such that $\sum_{i=1}^{k} \langle x_i, x_i\rangle = 1$.*

Proof: We shall consider at first the case of a unital C^*-algebra. Then by Lemma 2.4.3 it is possible to find an element $c \in S$ such that $\|1 - c\| < 1/2$. Therefore the element c is invertible and $c = \sum_{j=1}^{k} \langle y_j, y_j\rangle$ for some k and $y_j \in \mathcal{M}$. By putting $x_j = y_j \cdot c^{-1/2}$, we get $\sum_{j=1}^{k} \langle x_j, x_j\rangle = 1$.

In the case of a C^*-algebra without unit let $h \in A$ be a strictly positive element. By induction we shall construct a sequence (c_j) in S such that

$$\sum_{j=1}^{k} c_j \le 1; \qquad \left\| \left(1 - \sum_{j=1}^{k} c_j\right) h \right\| < \frac{1}{2^k}.$$

By Lemma 2.4.3 we can find an element $c_1 \in S$ such that $\|(1 - c_1)h\| < \frac{1}{2}$. Under assumption that the elements $c_1, \dots, c_k$ are already found, by Lemma 2.4.3 we can find an element $d \in S$ such that

$$\left\| (1 - d)\left(1 - \sum_{j=1}^{k} c_j\right)^{1/2} h \right\| < \frac{1}{2^{k+1}} \tag{6}$$

and let us put

$$c_{k+1} = \left(1 - \sum_{j=1}^{k} c_j\right)^{1/2} d \left(1 - \sum_{j=1}^{k} c_j\right)^{1/2}.$$

Since $\left\| \left(1 - \sum_{j=1}^{k} c_j\right)^{1/2} \right\| \le 1$ and $d \in S$, we have $c_{k+1} \in S$. Since $\|d\| \le 1$, $c_{k+1} \le 1 - \sum_{j=1}^{k} c_j$, then $\sum_{j=1}^{k+1} c_j \le 1$. Finally, it follows from the inequality (6) that

$$\begin{aligned}
\left\| \left(1 - \sum_{j=1}^{k+1} c_j\right) h \right\| &= \left\| \left(1 - \sum_{j=1}^{k} c_j\right)^{1/2} (1 - d)\left(1 - \sum_{j=1}^{k} c_j\right)^{1/2} h \right\| \\
&\le \left\| \left(1 - \sum_{j=1}^{k} c_j\right)^{1/2} \right\| \left\| (1 - d)\left(1 - \sum_{j=1}^{k} c_j\right)^{1/2} h \right\| < \frac{1}{2^{k+1}},
\end{aligned}$$

that completes the step of induction. So we obtain that

$$\left\| \left(1 - \sum_{j=1}^{k} c_j\right) h \right\| \longrightarrow 0$$

as $k \to \infty$. Since the strictly positive element h generates the whole C^*-algebra A [55], the lemma is proved. □

Let us continue the proof of the theorem. By Lemma 2.4.4 we can choose a sequence (x_i) in the module $\mathcal{M}$ such that the partial sums of $\sum_{j=1}^{k} \langle x_i, x_i \rangle$ form an approximate unit in A. Define a map $T : A \longrightarrow l_2(\mathcal{M})$ by the equality

$$T(a) = (x_1 a, \ldots, x_k a, \ldots), \qquad a \in A. \tag{7}$$

As the series $\langle Ta, Ta \rangle = \sum_{i=1}^{\infty} a^* \langle x_i, x_i \rangle a = a^* a$ converges uniformly in A, so $T(a) \in l_2(\mathcal{M})$. Moreover, the adjoint operator is well-defined, $T^* : l_2(\mathcal{M}) \longrightarrow A$, $T^*(y_i) = \sum_{i=1}^{\infty} \langle x_i, y_i \rangle \in A$ for $(y_i) \in l_2(\mathcal{M})$. Since T^*T acts identically on A, the operator T is an isometry, and

$$l_2(\mathcal{M}) = \operatorname{Im} T \oplus \operatorname{Ker} T^* \cong A \oplus \mathcal{N},$$

where $\mathcal{N} = \operatorname{Ker} T^*$. This finishes the proof of statement (i) of the theorem for the case of a C^*-algebra without unit. In the case of a unital C^*-algebra the previous reasonings can be applied literally if we replace the module $l_2(\mathcal{M})$ by $\mathcal{M}^k$ and replace infinite sequence in (7) by a finite one.

We pass to the proof of the statement (ii). For this purpose let us renumber the sequences in the module $l_2(\mathcal{M})$ with the help of a bijection $\mathbf{N} \longrightarrow \mathbf{N} \times \mathbf{N}$. Then elements of the module $l_2(\mathcal{M})$ are realised as sequences (m_{ij}), $m_{ij} \in \mathcal{M}$ and for each $i \in \mathbf{N}$ the set of sequences $(m_{ij}), j \in \mathbf{N}$, is isomorphic to the module $l_2(\mathcal{M})$. Such a renumbering defines an isomorphism $l_2(\mathcal{M}) \cong l_2(l_2(\mathcal{M}))$. By the isomorphism $l_2(\mathcal{M}) \cong A \oplus \mathcal{N}$ we conclude that

$$l_2(\mathcal{M}) \cong l_2(l_2(\mathcal{M})) \cong l_2(A \oplus \mathcal{N}) = l_2(A) \oplus l_2(\mathcal{N}).$$

Notice that the Hilbert module $l_2(\mathcal{N})$ is countably generated, therefore

$$l_2(\mathcal{M}) \cong l_2(A) \oplus l_2(\mathcal{N}) \cong l_2(A)$$

by the Kasparov stabilization theorem. $\square$

2.5 Dual modules. Self-duality

For a Hilbert A-module $\mathcal{M}$ let us denote by $\mathcal{M}'$ the set of all bounded A-linear maps from $\mathcal{M}$ to A. The structure of a vector space over the field $\mathbf{C}$ is introduced by the formula $(\lambda \cdot f)(x) := \overline{\lambda} f(x)$, where $\lambda \in \mathbf{C}$, $f \in \mathcal{M}'$, $x \in \mathcal{M}$. This definition seems artificial, however it is convenient, because it allows to define a linear inclusion of the module $\mathcal{M}$ into $\mathcal{M}'$ (there is also the alternate approach: to define $\mathcal{M}'$ as the set of all *anti*-linear maps from $\mathcal{M}$ into A). The formula

$$(f \cdot a)(x) = a^* f(x),$$

where $a \in A$, introduces a structure of a right A-module on $\mathcal{M}'$. This module is complete with respect to the norm $\|f\| = \sup\{\|f(x)\| : \|x\| \leq 1\}$. Such modules we shall call *dual* (*Banach*) modules. The elements of the module $\mathcal{M}'$ are called *functionals* on the Hilbert module $\mathcal{M}$. Let us remark that there is an obvious isometric inclusion $\mathcal{M} \subset \mathcal{M}'$, which is defined by the formula $x \mapsto \langle x, \cdot \rangle = \widehat{x}$. Sometimes, if it will not cause a confusion, we will write $\langle f, x \rangle$ instead of $f(x)$.

Definition 2.5.1 A Hilbert module $\mathcal{M}$ is called *self-dual* if $\mathcal{M} = \mathcal{M}'$.

The condition of autoduality is very strong. Below we will see that there exist only a few self-dual modules: each module over a C^*-algebra A is self-dual iff A is finite dimensional. Auto-dual Hilbert C^*-modules behave quite like Hilbert spaces. In the same way as and in the case of Hilbert spaces, the following statements can be obviously checked.

Proposition 2.5.2 ([52]) *Let $\mathcal{M}$ be a self-dual Hilbert A-module, $\mathcal{N}$ be an arbitrary Hilbert A-module, and $T : \mathcal{M} \longrightarrow \mathcal{N}$ is a bounded operator, $T \in \operatorname{Hom}_A(\mathcal{M}, \mathcal{N})$. Then there exists an operator $T^* : \mathcal{N} \longrightarrow \mathcal{M}$ such that for all $x \in \mathcal{M}$, $y \in \mathcal{N}$ the equality $\langle x, T^* y \rangle = \langle Tx, y \rangle$ is valid.* $\square$

Corollary 2.5.3 *Let $\mathcal{M}$ be a self-dual Hilbert A-module. Then $\operatorname{End}_A(\mathcal{M}) = \operatorname{End}^*_A(\mathcal{M})$.* $\square$

Proposition 2.5.4 *Let $\mathcal{M} \subset \mathcal{N}$, $\mathcal{M}$ be a self-dual Hilbert A-module. Then $\mathcal{N} = \mathcal{M} \oplus \mathcal{M}^{\perp}$.*

Proof: Since $\mathcal{M}$ is self-dual, $i : \mathcal{M} \to \mathcal{N}$ is an isometric inclusion, admitting an adjoint. Therefore, $\mathcal{M} = i\mathcal{M}$ has an orthogonal complement by Lemma 2.3.5. $\square$

If A is a unital C^*-algebra then the Hilbert module $L_n(A)$ is obviously self-dual. For an arbitrary module it is not true, moreover, the Banach module $\mathcal{M}'$ can not admit a structure of a Hilbert module at all. Description of the dual module for the standard Hilbert module H_A is given by the following

Proposition 2.5.5 *Let us consider the set of sequences $f = (f_i)$, $f_i \in A$, $i \in \mathbf{N}$, such that the norms of partial sums $\left\| \sum_{i=1}^N f_i^* f_i \right\|$ are uniformly bounded. If A is a unital C^*-algebra, this set coincides with H'_A, the action of f on elements of the module H_A is defined by the formula*

$$f(x) = \sum_{i=1}^{\infty} f_i^* x_i, \tag{8}$$

where $x = (x_i) \in H_A$, and the norm of f is defined by the equality

$$\|f\|^2 = \sup_N \left\| \sum_{i=1}^N f_i^* f_i \right\|. \tag{9}$$

Proof: Let $f \in H'_A$, e_i be the standard basis in H_A. Let us put $f_i = (f(e_i))^*$. We show that the sequence (f_i) determines a functional f in a unique way. Let us assume that there exists a functional $g \neq f$, $g(e_i) = f(e_i)$. Let us choose $x \in H_A$ such that $\|f(x) - g(x)\| = C \neq 0$. Denote by $x^{(N)}$ the image of x under the projection onto the submodule $L_N(A) \subset H_A$, $x^{(N)} = \sum_{i=1}^N e_i x_i = (x_1, \ldots, x_N, 0, \ldots)$. Let ud find a number N such that

$$\left\| x - x^{(N)} \right\| = \left\| \sum_{i=N+1}^{\infty} x^* x \right\|^{1/2} < \frac{C}{2(\|f\| + \|g\|)}.$$

Since $f(x^{(N)}) = g(x^{(N)})$, we have $\left\| f(x - x^{(N)}) - g(x - x^{(N)}) \right\| = C$. But, on the other hand, one has

$$\left\| f(x - x^{(N)}) - g(x - x^{(N)}) \right\| \leq (\|f\| + \|g\|) \left\| x - x^{(N)} \right\| < (\|f\| + \|g\|) \frac{C}{2(\|f\| + \|g\|)} \leq C/2.$$

The obtained contradiction shows that $f = g$. The Cauchy – Bunyakovskii inequality

$$\left\| \sum_{i=1}^N f_i^* x_i \right\|^2 \leq \left\| \sum_{i=1}^N f_i^* f_i \right\| \left\| \sum_{i=1}^N x_i^* x_i \right\| \tag{10}$$

shows that

$$\|f\|^2 \leq \sup_N \left\| \sum_{i=1}^N f_i^* f_i \right\|. \tag{11}$$

Remark that if we take $x_i = f_i / \left\| \sum_{i=1}^N f_i^* f_i \right\|^{1/2}$, then the equality is reached in (10). Let us put $f^{(N)} = (f_1, \ldots, f_N, 0, \ldots)$, $f^{(N)} \in L_N(A)' \cong L_N(A)$. It is obvious that

$$\|f\| \geq \left\| f^{(N)} \right\|. \tag{12}$$

But $\left\| f^{(N)} \right\|^2 = \left\| \sum_{i=1}^N f_i^* f_i \right\|$, therefore (9) follows from (11) and (12). The convergence of the series (8) follows from the fact, that for any $\varepsilon > 0$ it is possible to find a number N such that for all $n > 0$ the following estimate holds

$$\left\| \sum_{i=N}^{N+n} f_i^* x_i \right\| \leq \left\| \sum_{i=N}^{N+n} f_i^* f_i \right\| \cdot \left\| \sum_{i=N}^{N+n} x_i^* x_i \right\| \leq \|f\|^2 \left\| \sum_{i=N}^{N+n} x_i^* x_i \right\| < \|f\|^2 \varepsilon. \qquad \square$$

Remark that for the functional $f = (\varphi_i)$ from Example 2.1.2 the partial sums $\sum_{i=1}^N \varphi_i^* \varphi_i$ are uniformly bounded, however the appropriate series is not convergent.

Let us describe an interesting example of a dual module.

Example 2.5.6 [23] Let $A = \mathcal{B}(H)$ be the algebra of all bounded operators on a separable Hilbert space H. Let us consider pairwise orthogonal projections $p_i \in A$, $i \in \mathbf{N}$ such that the series $\sum_i p_i$ converges w*-weakly to $1 \in A$, and each projection p_i is equivalent to 1. We can consider $H = \oplus_i H_i$ as an orthogonal sum of Hilbert spaces isomorphic to H, $u_i : H \to H_i$ being isometries, so that

$$p_i = u_i u_i^*, \qquad 1 = u_i^* u_i.$$

As it was shown above (see Proposition 2.5.5),

$$l_2(A)' = \left\{ \{a_i\} \,\middle|\, a_i \in A, i \in \mathbf{N}, \left\{ \sup_{N \in \mathbf{N}} \| \sum_{i=1}^N a_i a_i^* \| \right\} < \infty \right\}$$

is an A-Hilbert module with respect to the inner product

$$\langle \{a_i\}, \{b_i\} \rangle := w^*\text{–}\lim \sum_i a_i b_i^*.$$

The maps

$$S : A \to l_2(A)', \qquad S : a \mapsto a \cdot \{u_i\},$$

$$S^{-1} : l_2(A)' \to A, \qquad S^{-1} : \{a_i\} \mapsto w^*\text{–}\lim \sum_i a_i u_i^*$$

define an isometric isomorphism of A and $l_2(A)'$.

Let φ be a positive linear functional on A. If $\mathcal{M}$ is a Hilbert A-module and if $N_\varphi = \{x \in \mathcal{M} : \varphi(\langle x, x \rangle) = 0\}$ is its linear subspace, then $\mathcal{M}/N_\varphi$ is a pre-Hilbert space with the inner product $(\cdot, \cdot)_\varphi$ given by the formula

$$(x + N_\varphi, y + N_\varphi)_\varphi = \varphi(\langle x, y \rangle), \quad x, y \in \mathcal{M}.$$

The norm defined by this scalar product we denote by $\|\cdot\|_\varphi$, and the Hilbert space, obtained by completion of $\mathcal{M}/N_\varphi$ with respect to this norm, we denote by H_φ. Let $f \in \mathcal{M}'$. In accordance with Proposition 2.1.4 we have for all $x \in \mathcal{M}$

$$f(x)^* f(x) \le \|f\|^2 \langle x, x \rangle,$$

therefore, if $x \in N_\varphi$, then

$$\varphi(f(x)^* f(x)) = 0 = \varphi(f(x)).$$

Hence, the map

$$x + N_\varphi \longmapsto \varphi(f(x)) \tag{13}$$

defines a linear functional on $\mathcal{M}/N_\varphi$. Since

$$|\varphi(f(x))| \le \|\varphi\|^{1/2} \varphi(f(x)^* f(x))^{1/2} \le \|\varphi\|^{1/2} \|f\| \varphi(\langle x, x \rangle)^{1/2} = \|\varphi\|^{1/2} \|f\| \|x + N_\varphi\|_\varphi,$$

the functional (13) is bounded. Then there exists a unique vector $f_\varphi \in H_\varphi$ such that $\|f_\varphi\|_\varphi \le \|f\| \|\varphi\|^{1/2}$ and $(f_\varphi, x + N_\varphi)_\varphi = \varphi(f(x))$ for all $x \in \mathcal{M}$. For $x \in \mathcal{M}$ we shall denote by $\widehat{x}$ the functional $\langle x, \cdot \rangle \in \mathcal{M}'$. Let us remark that $\widehat{y}_\varphi = y + N_\varphi$ for all $y \in \mathcal{M}$.

Let ψ be another positive linear functional on A such that $\psi \le \varphi$. Then $N_\varphi \subset N_\psi$ and the natural map $x + N_\varphi \longmapsto x + N_\psi$ can be extended to the map

$$V_{\varphi,\psi} : H_\varphi \longrightarrow H_\psi, \qquad \|V_{\varphi,\psi}\| \le 1.$$

It is easy to see that $V_{\varphi,\psi}(\widehat{x}_\varphi) = \widehat{x}_\psi$. It appears that the same holds for all functionals on $\mathcal{M}'$.

Proposition 2.5.7 *Let $\mathcal{M}$ be a Hilbert A-module, φ and ψ be positive linear functionals on A and $\psi \le \varphi$. Then $V_{\varphi,\psi}(f_\varphi) = f_\psi$ for any functional $f \in \mathcal{M}'$.*

Proof: Let $f \in \mathcal{M}'$. Since the quotient space $\mathcal{M}/N_\varphi$ is dense in H_φ, it is possible to choose a sequence $\{y_n + N_\varphi\}$ of the elements from $\mathcal{M}/N_\varphi$ such that $\|y_n + N_\varphi - f_\varphi\|_\varphi \to 0$. Then

$$V_{\varphi,\psi}(f_\varphi) = \lim_n V_{\varphi,\psi}(y_n + N_\varphi) = \lim_n (y_n + N_\psi).$$

To prove the statement, it is sufficient to show that $\psi(\langle y_n, x\rangle) \to \psi(f(x))$ for all $x \in \mathcal{M}$. But

$$\begin{aligned}
|\psi(\langle y_n, x\rangle - f(x))|^2 &\le \|\psi\| \cdot \psi(\langle y_n, x\rangle\langle x, y_n\rangle - \langle y_n, x\rangle f(x)^* - f(x)\langle x, y_n\rangle + f(x)f(x)^*) \\
&\le \|\varphi\| \cdot \varphi(\langle y_n, x\rangle\langle x, y_n\rangle - \langle y_n, x\rangle f(x)^* - f(x)\langle x, y_n\rangle + f(x)f(x)^*)
\end{aligned}$$

for each $n \in \mathbf{N}$. Since

$$\varphi(\langle y_n, x\rangle f(x)^*) = \varphi(\langle y_n, x \cdot (f(x))^*\rangle) \to \varphi(f(x \cdot (f(x))^*)) = \varphi(f(x)f(x)^*),$$

it will be sufficient to show, that $\varphi(\langle y_n, x\rangle\langle x, y_n\rangle - f(x)\langle x, y_n\rangle) \to 0$. Notice that

$$\begin{aligned}
\varphi(\langle y_n, x\rangle\langle x, y_n\rangle - f(x)\langle x, y_n\rangle) &= \varphi(\langle y_n, x \cdot \langle x, y_n\rangle\rangle - f(x \cdot \langle x, y_n\rangle)) \\
&= (y_n + N_\varphi - f_\varphi, x \cdot \langle x, y_n\rangle + N_\varphi)_\varphi,
\end{aligned}$$

and the sequence $\{x \cdot \langle x, y_n\rangle + N_\varphi\}$ is norm bounded with respect to the norm $\|\cdot\|_\varphi$. Indeed,

$$\begin{aligned}
\|x \cdot \langle x, y_n\rangle + N_\varphi\|_\varphi^2 &= \varphi(\langle x \cdot \langle x, y_n\rangle, \langle x \cdot \langle x, y_n\rangle\rangle) = \varphi(\langle y_n, x\rangle\langle x, x\rangle\langle x, y_n\rangle) \\
&\le \|x\|^2 \cdot \varphi(\langle y_n, x\rangle\langle x, y_n\rangle) \le \|x\|^2 \cdot \varphi(\|x\|^2 \cdot \langle y_n, y_n\rangle) = \|x\|^4 \cdot \|y_n + N_\varphi\|^2,
\end{aligned}$$

and the sequence $\{y_n + N_\varphi\}$ is bounded. Since $\|y_n + N_\varphi - f_\varphi\|_\varphi \to 0$, the statement is proved. □

2.6 Banach-compact operators

Definition 2.6.1 Let $\mathcal{M}, \mathcal{N}$ be Hilbert A-modules, $\mathcal{M}'$ be the dual module. Consider the closure $\mathcal{BK}(\mathcal{M}, \mathcal{N})$ in the Banach space $\mathrm{Hom}_A(\mathcal{M}, \mathcal{N})$ of the linear span of operators of the form

$$\theta_{y,f}(x) = y \cdot f(x),$$

where $x \in \mathcal{M}$, $y \in \mathcal{N}$, $f \in \mathcal{M}'$. The elements of the set $\mathcal{BK}(\mathcal{M}, \mathcal{N})$ we call *Banach-compact operators.*

In the case $\mathcal{N} = \mathcal{M}$ the set $\mathcal{BK}(\mathcal{M}, \mathcal{N})$ is equipped with a natural structure of a Banach algebra. If $T \in \mathrm{End}_A(\mathcal{M})$ is an operator, generally speaking, not admitting an adjoint, then the equalities

$$\theta_{y,f} T x = y \cdot f(Tx) = \theta_{y,f\circ T}(x), \qquad T\theta_{y,f}(x) = Ty \cdot f(x) = \theta_{Ty,f}(x)$$

show that $\mathcal{BK}(\mathcal{M})$ is a two-sided ideal in the Banach algebra $\mathrm{End}_A(\mathcal{M})$.

In the case of the standard Hilbert module over a unital C^*-algebra we shall give one more description of compact and Banach-compact operators, a geometric one. Let $S \subset H_A$ be a bounded set. We shall name it *A-pre-compact,* if for each $\varepsilon > 0$ there exists a free finitely generated A-module $\mathcal{N} \cong L_n(A)$; $\mathcal{N} \subset H_A$ such that $\mathrm{dist}(S, \mathcal{N}) < \varepsilon$.

Proposition 2.6.2 *Let $T \in \mathrm{End}_A(H_A)$ (resp. $T \in \mathrm{End}^*_A(H_A)$). Then the following conditions are equivalent:*

(i) *$T \in \mathcal{BK}(H_A)$ (resp., $T \in \mathcal{K}(H_A)$);*

(ii) *the image $T(B_1(H_A))$ of the unit ball $B_1(H_A)$ is A-pre-compact.*

Proof: If the statement (i) holds, it is sufficient to prove that it is possible to find an approximating module $\mathcal{N} \cong L_n(A)$ for a finite set of the elements from H_A and it can be easily done by the Dupré – Fillmore method, as in the proof of Theorem 1.4.4. So suppose that (ii) is carried out. Then for any $\varepsilon > 0$ it is possible to find elements $b_1, \ldots, b_k \in H_A$ such that $\langle b_i, b_j\rangle = \delta_{ij}$, which generate the module

$\mathcal{N} \subset H_A$ and $\operatorname{dist}(T(B_1(H_A)), \mathcal{N}) < \varepsilon$, where $B_1(H_A)$ is the unit ball of the module H_A. Let us denote by $P_{\mathcal{N}}$ the projection onto $\mathcal{N}$ and let us consider the operator $P_{\mathcal{N}}T$. It can be decomposed as

$$P_{\mathcal{N}}Tx = b_1\langle f_1, x\rangle + \cdots + b_n\langle f_n, x\rangle \tag{14}$$

where $f_i \in H'_A$. Since $x \in B_1(H_A)$, we can find an element $b \in \mathcal{N}$ such that $\|Tx - b\| < \varepsilon$, therefore

$$\|Tx - P_{\mathcal{N}}Tx\| = \|Tx - b + b - P_{\mathcal{N}}Tx\| = \|Tx - b\| + \|P_{\mathcal{N}}(b - Tx)\| \leq \varepsilon + \|P_{\mathcal{N}}\|\,\varepsilon = 2\varepsilon, \tag{15}$$

therefore $\|T - P_{\mathcal{N}}T\| \leq 2\varepsilon$ and T lies in the norm closure of the set of operators of the form (14). If T admits an adjoint then $P_{\mathcal{N}}T$ admits an adjoint too, hence $f_i \in H_A$ and T is compact. $\square$

2.7 C^*-Fredholm operators. Index

The material of this section is taken mainly from [48]. Let us remind in the beginning the definition of K-groups.

Definition 2.7.1 [31, § II.1] Let M be an Abelian monoid. Let us consider the direct product $M \times M$ and its quotient-monoid with respect to the following equivalence relation

$$(m, n) \sim (m', n') \;\Leftrightarrow\; \exists\, p, q: \;(m, n) + (p, p) = (m', n') + (q, q).$$

This quotient monoid is a group, denoted by $S(M)$ and is called the *symmetrization* of M. Let us consider now the additive category $\mathcal{P}(A)$ of projective modules over a unital C^*-algebra A and let us denote by $[E]$ the isomorphism class of an object $\mathcal{M}$ from $\mathcal{P}(A)$, then the set $\Phi(\mathcal{P}(A))$ of these classes has a structure of an Abelian monoid with respect to the operation $[\mathcal{M}]+[\mathcal{N}] = [\mathcal{M}\oplus\mathcal{N}]$. In this case the group $S(\Phi(\mathcal{P}(A)))$ is denoted by $K(A)$ or $K_0(A)$ and is called the *K-group* of A or the *Grothendeick group* of the category $\mathcal{P}(A)$. If A has no unit then the natural map $A^+ \to \mathbf{C}$ induces a map of K-groups and let us put

$$K_0(A) := \operatorname{Ker}(K_0(A^+) \to K_0(\mathbf{C})).$$

The groups $K_n(A) := K_0(A \otimes C_0(\mathbf{R}^n))$ for natural n appear to be 2-periodic in n, and the definition can be extended to $n \in \mathbf{Z}$ by periodicity.

For unital algebras we could use classes of stable (after adding direct summand) homotopy of projections in A^n instead of classes of isomorphic projective modules. More precisely, projections $p : A^n \to A^n$ and $q : A^m \to A^m$ are equivalent, if there can be found m' and n' such that $n + n' = m + m' = s$ and projections

$$A^s = A^n \oplus A^{n'} \xrightarrow{\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}} A^n \oplus A^{n'} = A^s, \qquad A^s = A^m \oplus A^{m'} \xrightarrow{\begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}} A^m \oplus A^{m'} = A^s$$

can be connected by a norm continuous homotopy in the set of projectors from $\operatorname{End}(A^s) = \operatorname{End}^*(A^s)$ (A unital!). It is possible to consider also equivalence classes of projections in the algebraic sense. The details can be found in [9, 31, 50, 72].

Let us remind the following well known statement.

Lemma 2.7.2 (compare [59, Theorem 4.15]) *The set of epimorphisms is open in the space of bounded linear maps of a Banach space E to a Banach space G.*

Lemma 2.7.3 [48, 1.4] *Let $\mathcal{N}$ be a finitely generated A-module over a unital A, $a_1, \ldots a_s$ be its generators. Then there exists a number $\varepsilon > 0$ such that if for some elements $a'_1, \ldots a'_s \in \mathcal{N}$ the following inequalities hold*

$$\|a'_k - a_k\| < \varepsilon, \qquad (k = 1, \ldots s)$$

then the elements $a'_1, \ldots a'_s$ also generate $\mathcal{N}$.

Proof: The map

$$f : L_s(A) \to \mathcal{N}, \qquad (0, \ldots, 0, \underset{i}{1}, 0, \ldots, 0) \mapsto a_i$$

is an epimorphism. Hence, by Lemma 2.7.2, there exists $\varepsilon > 0$ such that if $\|g - f\| < s\varepsilon$, then g is an epimorphism. Let

$$g : L_s(A) \to \mathcal{N}, \qquad (0, \ldots, 0, \underset{i}{1}, 0, \ldots, 0) \mapsto a_i',$$

Then for any $\alpha = (\alpha_1, \ldots, \alpha_s) \in L_s(A)$ with norm $\|\alpha\| \leq 1$

$$\|(g - f)\alpha\| = \left\| \sum_{i=1}^{s} (a_i' - a_i)\alpha_i \right\| \leq s\varepsilon.$$

Hence g is an epimorphism and the elements $a_1', \ldots a's$ generate $\mathcal{N}$. $\square$

In this section we assume that the algebra A is unital. Let us remind the definition of a Fredholm operator [48].

Definition 2.7.4 A bounded A-operator $F : H_A \to H_A$, is called *Fredholm*, if

(i) operator F admits an adjoint;

(ii) there exists a decompositions of domain $H_A = \mathcal{M}_1 \widetilde{\oplus} \mathcal{N}_1$ and range $H_A = \mathcal{M}_2 \widetilde{\oplus} \mathcal{N}_2$ (where $\mathcal{M}_1, \mathcal{M}_2, \mathcal{N}_1, \mathcal{N}_2$ are closed A-modules, $\mathcal{N}_1, \mathcal{N}_2$ have finite number of generators), such that the operator F has the following matrix form $F = \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix}$ with respect to these decompositions, where $F_1 : \mathcal{M}_1 \to \mathcal{M}_2$ is an isomorphism.

Theorem 2.7.5 [48] *Let $H_A \cong \mathcal{M} \widetilde{\oplus} \mathcal{N}$, where $\mathcal{M}$ and $\mathcal{N}$ are closed A-modules, $\mathcal{N}$ has finite number of generators $a_1, \ldots, a_s$. Then $\mathcal{N}$ is a projective A-module of finite type.*

Proof: By Lemma 2.7.3 there exists $\varepsilon > 0$ such that if

$$\|a_k' - a_k\| < \varepsilon, \quad a_k' \in \mathcal{N}, \quad k = 1, \ldots, s,$$

then $\{a_k'\}$ generate $\mathcal{N}$. Let $P : H_A \to \mathcal{N}$ is the projection along the summand $\mathcal{M}$ in the module H_A. Then P is a bounded A-operator. Therefore there exists $\delta > 0$ such that if $\|x\| < \delta$ then $\|Px\| < \varepsilon$. Let us choose a number n_0 such that

$$\|a_k - \overline{a}_k\| < \delta, \qquad k = 1, \ldots, s,$$

where $\overline{a}_k$ is the projection of a_k onto L_{n_0} along $L_{n_0}^{\perp}$. Let us represent $\overline{a}_k$ in the correspondence with the decomposition $H_A = \mathcal{M} \widetilde{\oplus} \mathcal{N}$ as

$$\overline{a}_k = a_k' + a_k'', \quad a_k' \in \mathcal{N}, \quad a_k'' \in \mathcal{M}.$$

Then $a_k - a_k' = P(a_k - \overline{a}_k)$. Therefore $\|a_k - a_k'\| < \varepsilon$ and $\{a_k'\}$ generate $\mathcal{N}$. Let $\overline{\mathcal{N}}$ be the module generated by $\{\overline{a}_k\}$. Then H_A is equal to the sum $\mathcal{M} + \overline{\mathcal{N}}$. Indeed, if $x \in H_A$ then

$$x = x_{\mathcal{M}} + \sum \lambda^k a_k' = (x_{\mathcal{M}} - \sum \lambda^k a_k'') + \sum \lambda^k \overline{a}_k.$$

Let us consider the bounded A-operator Q of projection onto L_{n_0} along $L_{n_0}^{\perp}$. Then

$$\begin{array}{ll} Q(a_k) = \overline{a}_k, & Q(\mathcal{N}) = \overline{\mathcal{N}}, \\ P(\overline{a}_k) = a_k', & P(\overline{\mathcal{N}}) = \mathcal{N}. \end{array}$$

Since a_k are close to a_k', the composition of A-operators

$$\mathcal{N} \xrightarrow{Q} \overline{\mathcal{N}} \xrightarrow{P} \mathcal{N}$$

is an A-isomorphism. Therefore $Q : \mathcal{N} \to \overline{\mathcal{N}}$ and $P : \overline{\mathcal{N}} \to \mathcal{N}$ are A-isomorphisms. In particular, if $\sum_{k=1}^{s} \lambda_k a_k' = 0$ holds then $\sum_{k=1}^{s} \lambda_k \overline{a}_k = 0$. Therefore $\mathcal{M} \cap \mathcal{N} = 0$, i. e. $H_A = \mathcal{M} \widetilde{\oplus} \overline{\mathcal{N}}$. It is clear, that $\overline{\mathcal{N}}$ is a closed A-submodule in H_A and

$$L_{n_0} = (\mathcal{M} \cap L_{n_0}) \widetilde{\oplus} \overline{\mathcal{N}}.$$

Indeed, $(\mathcal{M} \cap L_{n_0}) \cap \overline{\mathcal{N}} = 0$; on the other hand, if $x \in L_{n_0}$ then $x = x' + x''$, $x' \in \overline{\mathcal{N}}$, $x'' \in \mathcal{M}$. Since $\overline{\mathcal{N}} \subset L_{n_0}$, then $x'' \in L_{n_0}$, i. e. $x'' \in \mathcal{M} \cap L_{n_0}$. Thus $\mathcal{N}$ is isomorphic to a direct summand in the free finitely generated A-module L_{n_0}. $\square$

Theorem 2.7.6 [67] *In the decomposition mentioned in the definition of a Fredholm operator,* (*see* 2.7.4) *it is possible to suppose always that the modules* $\mathcal{M}_0$ *and* $\mathcal{M}_1$ *are orthogonally complemented. More precisely, there exist such decompositions for* F

$$\begin{pmatrix} F_3 & 0 \\ 0 & F_4 \end{pmatrix} : H_A = V_0 \widetilde{\oplus} W_0 \to V_1 \widetilde{\oplus} W_1 = H_A,$$

that $V_0^\perp \oplus V_0 = H_A$, $V_1^\perp \oplus V_1 = H_A$, *or* (*which is the same by Lemma* 2.3.8) *that projections* $V_0 \widetilde{\oplus} W_0 \to V_0$ *and* $V_1 \widetilde{\oplus} W_1 \to V_1$ *admit an adjoint.*

Proof: Let $W_0 = \mathcal{N}_0$, $V_0 = W_0^\perp$. The orthogonal complement exists by Theorem 2.3.7. The restriction $F|_{W_0^\perp}$ is an isomorphism. Indeed, if $x_n \in W_0^\perp$, then let $x_n = x_1^n + x_2^n$, $x_1^n \in \mathcal{M}_0$, $x_2^n \in W_0$, $\|x_n\| = 1$. Let us assume that $\|Fx_n\| \to 0$, then $\|Fx_1^n + Fx_2^n\| \to 0$; and since $Fx_1^n \in \mathcal{M}_1$, $Fx_2^n \in \mathcal{N}_1$, $\mathcal{M}_1 \widetilde{\oplus} \mathcal{N}_1 = H_A$, it means that $\|Fx_1^n\| \to 0$ and $\|Fx_2^n\| \to 0$. The operator F_1 is an isomorphism, therefore $\|x_1^n\| \to 0$. If $a_1, \ldots, a_s$ are generators for $W_0 = \mathcal{N}_0$, then $0 = \langle x_n, a_j\rangle = \langle x_1^n, a_j\rangle + \langle x_2^n, a_j\rangle$,

$$\|\langle x_2^n, a_j\rangle\| = \|\langle x_1^n, a_j\rangle\| \le \|x_1^n\| \|a_j\| \to 0, \quad n \to \infty, \quad j = 1, \ldots, s.$$

Since $x_2^n \in \mathcal{N}_0$, $x_2^n \to 0$ $(n \to \infty)$ and $x_n = x_1^n + x_2^n \to 0$ — a contradiction to $\|x_n\| = 1$. This contradiction shows that $F|_{W_0^\perp}$ is an isomorphism.

Let $V_1 = F(V_0)$. Since $W_0 = \mathcal{N}_0$, it is possible to suppose that $W_1 = \mathcal{N}_1$. Indeed, any $y \in H_A$ has the form $y = m_1 + n_1 = F(m_0) + n_1$, where $m_1 \in \mathcal{M}_1$, $n_1 \in \mathcal{N}_1$, $m_0 \in \mathcal{M}_0$. In turn, $m_0 = v_0 + n_0$, where $v_0 \in V_0$, $n_0 \in W_0 = \mathcal{N}_0$, and

$$y = F(v_0 + n_0) + n_1 = F(v_0) + (F(n_0) + n_1) \in V_1 + \mathcal{N}_1.$$

Thus $H_A = V_1 + W_1$.

Let $y \in V_1 \cap W_1 = V_1 \cap \mathcal{N}_1$, i. e. $n_1 = y = F(v_0)$, $n_1 \in \mathcal{N}_1$, $v_0 \in V_0$. Let us decompose $v_0 = m_0 + n_0$, where $m_0 \in \mathcal{M}_0$, $n_0 \in \mathcal{N}_0$. Then

$$n_1 = F(m_0) + F(n_0),$$
$$F(m_0) = n_1 - F(n_0), \quad F(m_0) \in \mathcal{M}_1, \quad n_1 - F(n_0) \in \mathcal{N}_1,$$

whence we obtain that $F(m_0) = 0$, $\quad n_1 - F(n_0) = 0$. Since $F : \mathcal{M}_0 \cong \mathcal{M}_1$, $m_0 = 0$. Further, $v_0 \in V_0 = \mathcal{N}_0^\perp$ and therefore

$$0 = \langle v_0, n_0\rangle = \langle m_0 + n_0, n_0\rangle = \langle n_0, n_0\rangle, \quad n_0 = 0.$$

Thus $v_0 = m_0 + n_0 = 0$, $y = F(v_0) = 0$. Hence $V_1 \cap W_1 = 0$ and $H_A = V_1 \widetilde{\oplus} W_1$.

By Corollary 2.3.5 the module V_1 has the orthogonal complement $V_1^\perp$, $V_1 \oplus V_1^\perp = H_A$, and it completes the proof. □

Remark 2.7.7 If we do not require that the operator F is supposed to have an adjoint then it is possible to state that there exists a decomposition $F : \mathcal{N}_0^\perp \oplus \mathcal{N}_0 \to \mathcal{M}_1 \widetilde{\oplus} L_n$, where $L_n = \mathrm{span}_A(e_1, \ldots, e_n)$, but $\mathcal{M}_1$ does not necessarily have an orthogonal complement. This result was obtained in [27].

Definition 2.7.8 Let the conditions of Definition 2.7.4 hold. By Theorem 2.7.5, $\mathcal{N}_1$ and $\mathcal{N}_2$ are projective A-modules and we can define an *index*

$$\mathrm{index}\, F = [\mathcal{N}_1] - [\mathcal{N}_2] \in K(A).$$

Theorem 2.7.9 *The index is well-defined.*

Proof: It is necessary to check out that the index does not depend on decompositions of range and domain involved in the definition of index 2.7.4. Let p_m be the projection onto L_m along $L_m^\perp$. Let F be an A-Fredholm operator, $H_A = \mathcal{M}_1 \widetilde{\oplus} \mathcal{N}_1$ be a decomposition of domain, $H_A = \mathcal{M}_2 \widetilde{\oplus} \mathcal{N}_2$ be a decomposition of range,

$$F = \begin{pmatrix} F_1 & 0 \\ 0 & F_4 \end{pmatrix},$$

where $F_1 : \mathcal{M}_1 \to \mathcal{M}_2$ is an isomorphism. According to the proof of Theorem 2.7.5 and Theorem 2.7.6 it is possible to suppose that

$$\mathcal{N}_1 \subset L_n, \qquad L_n = \mathcal{N}_1 \widetilde{\oplus} \mathcal{P}_1, \qquad \mathcal{M}_1 = \mathcal{P}_1 \oplus L_n^\perp,$$

where $\mathcal{P}_1$ is a projective finitely generated A-module. Let anther decomposition of domain and range be given:

$$H_A = \mathcal{M}'_1 \widetilde{\oplus} \mathcal{N}'_1, \qquad H_A = \mathcal{M}'_2 \widetilde{\oplus} \mathcal{N}'_2.$$

Then there exists $m \geq n$ such that

$$L_m = \mathcal{P}'_1 \widetilde{\oplus} p_m(\mathcal{N}'_1), \qquad p_m(\mathcal{N}'_1) \cong \mathcal{N}'_1,$$

where $\mathcal{P}'_1$ is a projective finitely generated A-module. This is exactly the result of the proof of Theorem 2.7.5.

Let us show that there exists $m \geq n$ such that if

$$L'_m = F(L_m) + \mathcal{N}_2, \qquad \text{and} \quad Q'_m : H_A \to H_A$$

is the projection on L'_m along $L''_m = F(L_m^\perp)$ [1] then

$$L'_m = \mathcal{P}_2 \widetilde{\oplus} Q'_m(\mathcal{N}_2), \qquad Q'_m(\mathcal{N}_2) \cong \mathcal{N}_2,$$

where $\mathcal{P}_2$ is a projective finitely generated A-module; and

$$L'_m = \mathcal{P}'_2 \widetilde{\oplus} Q'_m(\mathcal{N}'_2), \qquad Q'_m(\mathcal{N}'_2) \cong \mathcal{N}'_2,$$

where $\mathcal{P}'_2$ is a projective finitely generated A-module. Indeed, $H_A = L'_m \widetilde{\oplus} L''_m$. If $a_1, \ldots, a_k$ are generators of the module $\mathcal{N}_2$ then

$$a_j = a'_j + a''_j, \quad a'_j \in L'_m, \quad a''_j \in L''_m, \quad j = 1, \ldots, k.$$

For $m \to \infty$ we have $\|a''_j\| \to 0$, as $a''_j = F(x_m^\perp)$, where x is arbitrary, $x_m^\perp$ is a projection of x onto $L_m^\perp$ and $\|x_m^\perp\| \to 0$ for $m \to \infty$. Then for big enough m we have

$$L'_m = (L'_m \cap \mathcal{M}_2) \widetilde{\oplus} Q'_m(\mathcal{N}_2), \qquad Q'_m(\mathcal{N}_2) \cong \mathcal{N}_2$$

(the proof of this fact repeats the proof of Theorem 2.7.5). Similarly

$$L'_m = (L'_m \cap \mathcal{M}'_2) \widetilde{\oplus} Q'_m(\mathcal{N}'_2), \qquad Q'_m(\mathcal{N}'_2) \cong \mathcal{N}'_2.$$

Since $m \geq n$, $L_m \cong \mathcal{N}_1 \widetilde{\oplus} \overline{\mathcal{P}}_1$, where $\overline{\mathcal{P}}_1$ is a finitely generated projective A-module. From the equalities

$$F(\overline{\mathcal{P}}_1) = F(L_m \cap \mathcal{M}_1) = \mathcal{P}_2, \qquad \overline{\mathcal{P}}_1 \subset \mathcal{M}_1,$$

we obtain that $F : \overline{\mathcal{P}}_1 \cong \mathcal{P}_2$, and it follows from relations $F(\mathcal{P}'_1) = \mathcal{P}'_2$, $\mathcal{P}'_1 \subset \mathcal{M}_1$ that $F : \mathcal{P}'_1 \cong \mathcal{P}'_2$. Therefore we have the following equalities in $K(A)$

$$\begin{array}{ll} [\mathcal{N}_1] + [\overline{\mathcal{P}}_1] = [\mathcal{N}'_1] + [\mathcal{P}'_1] = [L_m], & [\overline{\mathcal{P}}_1] = [\mathcal{P}_2], \\ [\mathcal{N}_2] + [\mathcal{P}_2] = [\mathcal{N}'_2] + [\mathcal{P}'_2] = [L'_m], & [\mathcal{P}'_1] = [\mathcal{P}'_2]. \end{array}$$

Thus $[\mathcal{N}_1] - [\mathcal{N}_2] = [\mathcal{N}'_1] - [\mathcal{N}'_2]$ and we have proved that index is well-defined. $\square$

Lemma 2.7.10 *Let an operator $F : H_A \to H_A$ be A-Fredholm. Then there exists a number $\varepsilon > 0$ such that any bounded A-operator D satisfying the condition $\|F - D\| < \varepsilon$ and admitting an adjoint is an A-Fredholm operator and* $\operatorname{index} D = \operatorname{index} F$.

Proof: By the definition of the Fredholm property

$$H_A = \mathcal{M}_1 \widetilde{\oplus} \mathcal{N}_1, \qquad H_A = \mathcal{M}_2 \widetilde{\oplus} \mathcal{N}_2, \qquad F = \begin{pmatrix} F_1 & 0 \\ 0 & F_4 \end{pmatrix},$$

[1] this projection is well-defined, since $L_m^\perp \subset \mathcal{M}_1$ for $m \geq n$ and hence $F|_{L_m^\perp}$ is an isomorphism; whence it follows that $L''_m \cong H_A$ is a closed A-module, $L'_m \cap L''_m = 0$, $L'_m + L''_m = H_A$; therefore $H_A = L'_m \widetilde{\oplus} L''_m$ is a direct sum of closed A-modules and Q'_m is a bounded A-operator

$F_1 : \mathcal{M}_1 \cong \mathcal{M}_2$. Then $\|F_1\| \leq \|F\|$; moreover, if $D : H_A \to H_A$ is an arbitrary bounded A-operator, then

$$D = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix},$$

then there exists a constant C such that $\|D_1\| \leq C\|D\|$ (cf. [48, p. 842]). Therefore, if D is an arbitrary A-operator satisfying the estimate $\|F - D\| < \varepsilon$ then $\|F_1 - D_1\| < C \cdot \varepsilon$. Since F_1 is an isomorphism, we can find $\delta > 0$ such that if $\|F_1 - D_1\| < \delta$ and D_1 is an A-operator then D_1 is also an A-isomorphism. By putting $\varepsilon = \delta/C$ we obtain that for the operator

$$D = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix}$$

the element D_1 is an isomorphism. Then

$$U_2 D U_1 = \begin{pmatrix} D_1 & 0 \\ 0 & D_4 - D_3 D_1^{-1} D_2 \end{pmatrix},$$

where

$$U_2 = \begin{pmatrix} 1 & 0 \\ -D_3 D_1^{-1} & 1 \end{pmatrix} : (\mathcal{M}_2 \tilde{\oplus} \mathcal{N}_2) \to (\mathcal{M}_2 \tilde{\oplus} \mathcal{N}_2),$$
$$U_1 = \begin{pmatrix} 1 & -D_1^{-1} D_2 \\ 0 & 1 \end{pmatrix} : (\mathcal{M}_1 \tilde{\oplus} \mathcal{N}_1) \to (\mathcal{M}_1 \tilde{\oplus} \mathcal{N}_1)$$

are A-isomorphisms. With the help of U_1 and U_2 we obtain a new decomposition of domain and range in direct sums

$$H_A = \mathcal{M}'_1 \tilde{\oplus} \mathcal{N}'_1, \quad \mathcal{M}'_1 = U_1(\mathcal{M}_1), \quad \mathcal{N}'_1 = U_1(\mathcal{N}_1),$$
$$H_A = \mathcal{M}'_2 \tilde{\oplus} \mathcal{N}'_2, \quad \mathcal{M}'_2 = U_2^{-1}(\mathcal{M}_2), \quad \mathcal{N}'_2 = U_2^{-1}(\mathcal{N}_2).$$

With respect to the new decomposition the matrix of the operator D is equal to $U_2 D U_1$. Thus the operator D is Fredholm with the index

$$[U_1(\mathcal{N}_1)] - [U_2^{-1}(\mathcal{N}_2)] = [\mathcal{N}_1] - [\mathcal{N}_2] = \operatorname{index} F. \qquad \square$$

Lemma 2.7.11 *Let F and D be A-Fredholm operators*

$$F : H_A \to H_A, \qquad D : H_A \to H_A.$$

Then $DF : H_A \to H_A$ is an A-Fredholm operator and $\operatorname{index} DF = \operatorname{index} D + \operatorname{index} F$.

Proof: Let us consider for F and D decompositions from the definition

$$H_A = \mathcal{M}_1 \tilde{\oplus} \mathcal{N}_1 \xrightarrow{F} \mathcal{M}_2 \tilde{\oplus} \mathcal{N}_2 \cong H_A,$$

$$H_A = \mathcal{M}'_1 \tilde{\oplus} \mathcal{N}'_1 \xrightarrow{D} \mathcal{M}'_2 \tilde{\oplus} \mathcal{N}'_2 \cong H_A,$$

where

$$F = \begin{pmatrix} F_1 & 0 \\ 0 & F_4 \end{pmatrix}, \qquad D = \begin{pmatrix} D_1 & 0 \\ 0 & D_4 \end{pmatrix},$$

F_1 and D_1 are isomorphisms, $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}'_1, \mathcal{N}'_2$ are projective finitely generated A-modules. As well as earlier, without loss of generality it is possible to suppose that

$$\mathcal{N}_2 \subset L_n, \qquad L_n = \mathcal{N}_2 \tilde{\oplus} \mathcal{P}, \qquad \mathcal{M}_2 = L_n^{\perp} \oplus \mathcal{P}.$$

Moreover, as $F_1 : \mathcal{M}_1 \to \mathcal{M}_2$ is an isomorphism, it is possible to change decomposition into direct sums, by putting

$$\overline{\mathcal{M}}_1 = F_1^{-1}(L_n^{\perp}),\ \overline{\mathcal{N}}_1 = F_1^{-1}(\mathcal{P}) \tilde{\oplus} \mathcal{N}_1,\ \overline{\mathcal{M}}_2 = L_n^{\perp},\ \overline{\mathcal{N}}_2 = L_n;$$

Thus a number n can be choosen as big as necessary. Let us choose n in such a way that

$$L_n = \mathcal{P}' \tilde{\oplus} p_n(\mathcal{N}'_1), \qquad \mathcal{P}' = \mathcal{M}'_1 \cap L_n, \qquad p_n(\mathcal{N}'_1) \cong \mathcal{N}'_1,$$

where, as well as earlier, $p_n : H_A \to H_A$ is the projection on L_n along $L_n^\perp$. Then

$$H_A = L_n^\perp \widetilde{\oplus} \mathcal{P}' \widetilde{\oplus} p_n(\mathcal{N}_1').$$

Let us put $\overline{\overline{\mathcal{M}}}_2 = L_n^\perp$, $\overline{\overline{\mathcal{N}}}_2 = \mathcal{P}' \widetilde{\oplus} \mathcal{N}_1'$. With respect to the new decomposition $H_A = \overline{\overline{\mathcal{M}}}_2 \widetilde{\oplus} \overline{\overline{\mathcal{N}}}_2$ the the matrix of the operator F has the form

$$F = \begin{pmatrix} F_1 & F_2 \\ 0 & F_4 \end{pmatrix},$$

and F_1 is an isomorphism. Then

$$\begin{pmatrix} F_1 & F_2 \\ 0 & F_4 \end{pmatrix} \begin{pmatrix} 1 & -F_1^{-1}F_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} F_1 & 0 \\ 0 & F_4 \end{pmatrix}.$$

Denoting by U the matrix $\begin{pmatrix} 1 & -F_1^{-1}F_2 \\ 0 & 1 \end{pmatrix}$, let us put $\overline{\overline{\mathcal{M}}}_1 = U(\overline{\mathcal{M}}_1)$, $\overline{\overline{\mathcal{N}}}_1 = U(\overline{\mathcal{N}}_1)$. We have obtained a new decomposition of space $H_A = \overline{\overline{\mathcal{M}}}_1 \widetilde{\oplus} \overline{\overline{\mathcal{N}}}_1$, and the matrix F for decompositions

$$H_A = \overline{\overline{\mathcal{M}}}_1 \widetilde{\oplus} \overline{\overline{\mathcal{N}}}_1 \xrightarrow{F} \overline{\overline{\mathcal{M}}}_2 \widetilde{\oplus} \overline{\overline{\mathcal{N}}}_2 = H_A$$

has the former diagonal form. Let us consider the projection

$$T : H_A = \overline{\overline{\mathcal{M}}}_2 \widetilde{\oplus} \mathcal{P}' \widetilde{\oplus} \mathcal{N}_1' \to \overline{\overline{\mathcal{M}}}_2 \widetilde{\oplus} \mathcal{P}'.$$

Since $H_A \cong \mathcal{M}_2' \widetilde{\oplus} \mathcal{N}_1'$, the restriction $T|_{\mathcal{M}_2'} : \mathcal{M}_2' \to \overline{\overline{\mathcal{M}}}_2 \widetilde{\oplus} \mathcal{P}'$ is an isomorphism. Let us consider the matrix D with respect to the decomposition

$$H_A = (\overline{\overline{\mathcal{M}}}_2 \widetilde{\oplus} \mathcal{P}') \widetilde{\oplus} \mathcal{N}_1' \xrightarrow{D} \mathcal{M}_2' \widetilde{\oplus} \mathcal{N}_2' = H_A.$$

This matrix has the form $D = \begin{pmatrix} D_1 & 0 \\ D_3 & D_4 \end{pmatrix}$, where D_1 is an isomorphism. Let us put

$$V\,D := \begin{pmatrix} 1 & 0 \\ -D_3 D_1^{-1} & 1 \end{pmatrix} \begin{pmatrix} D_1 & 0 \\ D_3 & D_4 \end{pmatrix} = \begin{pmatrix} D_1 & 0 \\ 0 & D_4 \end{pmatrix}.$$

Therefore it is possible to change decomposition in the range

$$H_A = \overline{\mathcal{M}}'_2 \widetilde{\oplus} \overline{\mathcal{N}}'_2, \qquad \overline{\mathcal{M}}'_2 = V(\mathcal{M}_2'), \qquad \overline{\mathcal{N}}'_2 = V(\mathcal{N}_2'),$$

in such a way that the matrix of the operator D with respect to the new decomposition

$$H_A = \overline{\mathcal{M}}'_2 \widetilde{\oplus} \overline{\mathcal{N}}'_2, \qquad \overline{\mathcal{M}}'_2 = V(\mathcal{M}_2'), \qquad \overline{\mathcal{N}}'_2 = V(\mathcal{N}_2'),$$

has diagonal form. Let us change decomposition in the range once again:

$$\overline{\overline{\mathcal{M}}}'_2 = D(\overline{\overline{\mathcal{M}}}_2), \qquad \overline{\overline{\mathcal{N}}}_2 = D(\mathcal{P}') \widetilde{\oplus} \overline{\mathcal{N}}'_2.$$

The matrix D with the respect to the new decomposition

$$H_A = \overline{\overline{\mathcal{M}}}_2 \widetilde{\oplus} \overline{\overline{\mathcal{N}}}_2 \xrightarrow{D} \overline{\overline{\mathcal{M}}}'_2 \widetilde{\oplus} \overline{\overline{\mathcal{N}}}'_2 = H_A$$

has the diagonal form. Then the composition DF with the respect to the decomposition

$$H_A = \overline{\overline{\mathcal{M}}}_1 \widetilde{\oplus} \overline{\overline{\mathcal{N}}}_1 \to \overline{\overline{\mathcal{M}}}'_2 \widetilde{\oplus} \overline{\overline{\mathcal{N}}}'_2 = H_A$$

has the form $DF = \begin{pmatrix} (DF)_1 & 0 \\ 0 & (DF)_4 \end{pmatrix}$, and $(DF)_1$ is an isomorphism. Taking into account the fact that $\operatorname{End}^* H_A$ is a C^*-algebra, we conclude that DF is an A-Fredholm operator and

$$\operatorname{index} F = [\overline{\overline{\mathcal{N}}}_1] - [\overline{\overline{\mathcal{N}}}_2], \quad \operatorname{index} D = [\overline{\overline{\mathcal{N}}}_2] - [\overline{\overline{\mathcal{N}}}'_2],$$

$$\operatorname{index} DF = [\overline{\overline{\mathcal{N}}}_1] - [\overline{\overline{\mathcal{N}}}'_2].$$

We obtain from it that $\operatorname{index} DF = \operatorname{index} D + \operatorname{index} F$. $\square$

Lemma 2.7.12 *Let $K : H_A \to H_A$ be a compact operator. Then $1+K$ is an A-Fredholm operator and* $\operatorname{index}(1+K)=0$.

Proof: It is obvious that $1+K$ admits an adjoint. Let us choose a number n such that the inequality $\|K|_{L_n^\perp}\|<1$ is fulfilled. With respect to the decomposition $H_A = L_n^\perp \oplus L_n$ we have the following matrix presentation:

$$K = \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix}, \quad (1+K) = \begin{pmatrix} 1+K_1 & K_2 \\ K_3 & 1+K_4 \end{pmatrix}.$$

By the estimate $\|K|_{L_n^\perp}\|<1$ the operator $1+K_1$ is invertible, hence, as well as earlier, there exist invertible operators U_1 and U_2 such that

$$U_2(1+K)U_1 = \begin{pmatrix} 1+K_1 & 0 \\ 0 & (1+K_4)-K_3(1+K_1)^{-1}K_2 \end{pmatrix}.$$

Then, with respect to the new decomposition $H_A = \mathcal{M}_1 \tilde{\oplus} \mathcal{N}_1 \to \mathcal{M}_2 \tilde{\oplus} \mathcal{N}_2 = H_A$, where $\mathcal{M}_1 = U_1(L_n^\perp)$, $\mathcal{N}_1 = U_1(L_n)$, $\mathcal{M}_2 = U_2^{-1}(L_n^\perp)$, $\mathcal{N}_2 = U_2^{-1}(L_n)$, the operator $(1+K)$ has the diagonal form and, therefore, is an A-Fredholm operator and

$$\operatorname{index}(1+K) = [U_1(L_n)] - [U_2^{-1}(L_n)] = 0. \qquad \square$$

Lemma 2.7.13 *Let us consider an A-Fredholm operator $F : H_A \to H_A$ and let $K \in \mathcal{K}_A$. Then the operator $F+K$ is A-Fredholm and* $\operatorname{index}(F+K) = \operatorname{index} F$.

Proof: Let us consider decompositions of the space H_A in direct sums such that the matrix F has the diagonal form:

$$H_A = \mathcal{M}_1 \tilde{\oplus} \mathcal{N}_1 \xrightarrow{F} \mathcal{M}_2 \tilde{\oplus} \mathcal{N}_2 = H_A.$$

Without loss of generality we can suppose that

$$L_n = \mathcal{N}_1 \tilde{\oplus} \mathcal{P}_1, \; \mathcal{M}_1 = L_n^\perp \tilde{\oplus} \mathcal{P}_1,$$

where $\mathcal{P}_1$ is a finitely generated closed A-module. Let us choose a big enough number n in such a way that $\|K|_{L_n^\perp}\| < \|F_1^{-1}\|^{-1}$. Let us consider the new decomposition of space H_A:

$$\overline{\mathcal{M}}_1 = L_n^\perp, \; \overline{\mathcal{N}}_1 = L_n, \; \overline{\mathcal{M}}_2 = FL_n^\perp, \; \overline{\mathcal{N}}_2 = F(\mathcal{P}_1)\tilde{\oplus}\mathcal{N}_2.$$

Let $F = \begin{pmatrix} F_1 & 0 \\ 0 & F_4 \end{pmatrix}$ и $K = \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix}$ is a matrix of F and K with respect to the decomposition $H_A = \overline{\mathcal{M}}_1 \tilde{\oplus} \overline{\mathcal{N}}_1 \to \overline{\mathcal{M}}_2 \tilde{\oplus} \overline{\mathcal{N}}_2 = H_A$. Then

$$F+K = \begin{pmatrix} F_1+K_1 & K_2 \\ K_3 & F_4+K_4 \end{pmatrix},$$

and the operator F_1+K_1 is invertible. By repeating the construction of Lemma 2.7.10 (about operators close to a Fredholm operator), we obtain

$$\begin{aligned} \operatorname{index}(F+K) &= [\overline{\mathcal{N}}_1] - [\overline{\mathcal{N}}_2] = [L_n] - [F(\mathcal{P}_1)+\mathcal{N}_2] = \\ &= [\mathcal{N}_1] + [\mathcal{P}_1] - [\mathcal{P}_1] - [\mathcal{N}_2] = \operatorname{index} F. \qquad \square \end{aligned}$$

Theorem 2.7.14 *Let*

$$F : H_A \to H_A, \quad D : H_A \to H_A, \quad D' : H_A \to H_A$$

be bounded A-operators admitting an adjoint and

$$FD = \operatorname{Id}_{H_A} + K_1, \quad D'F = \operatorname{Id}_{H_A} + K_2, \quad K_1,\, K_2 \in \mathcal{K}(H_A).$$

Then F is an A-Fredholm operator.

Proof: Let us consider a decomposition H_A, for which the operator $FD = 1_{H_A} + K_1$ has the diagonal form (Lemma 2.7.12)

$$H_A = \mathcal{M}_1 \widetilde{\oplus} \mathcal{N}_1 \xrightarrow{1+K_1} \mathcal{M}_2 \widetilde{\oplus} \mathcal{N}_2 = H_A,$$

and the decomposition of space H_A satisfies the conditions of Theorem 2.7.6. Let us consider the projection

$$P : H_A = \mathcal{M}_2 \widetilde{\oplus} \mathcal{N}_2 \to \mathcal{N}_2.$$

It is a compact operator, $P \in \operatorname{End}^* H_A$. The image of the operator $(1-P)(1+K_1) = (1-P)FD$ is exactly equal to $\mathcal{M}_2$. It is easy to see that up to an isomorphism

$$(1-P)FD = (1-P)(1+K_1) = 1 + (-P(1+K_1)) + K_1 = 1 + \widetilde{K}_1,$$

$$D'(1-P)F = D'F - D'PF = 1 + \widetilde{K}_2,$$

where $\widetilde{K}_1 \in \mathcal{K}_{\mathcal{A}}$, $\widetilde{K}_2 \in \mathcal{K}_{\mathcal{A}}$. By Lemma 2.7.13 it is possible to suppose without loss of generality that $F : H_A \to \mathcal{M}_2$ is an epimorphism. Otherwise, we will pass to the operator $(1-P)F$. Let us consider now the decomposition for $1 + K_2$:

$$H_A = \overline{\mathcal{M}}_1 \widetilde{\oplus} \overline{\mathcal{N}}_1 \xrightarrow{F} \mathcal{M}_2 \widetilde{\oplus} \mathcal{N}_2 \xrightarrow{D'} \overline{\mathcal{M}}_2 \widetilde{\oplus} \overline{\mathcal{N}}_2 = H_A.$$

The composition $D'F|_{\overline{\mathcal{M}}_1} : \overline{\mathcal{M}}_1 \to \overline{\mathcal{M}}_2$ is an isomorphism. Therefore, since $F : H_A \to \mathcal{M}_2$ is an epimorphism, $F : \overline{\mathcal{M}}_1 \widetilde{\oplus} \overline{\mathcal{N}}_1 \to \mathcal{M}_2$ maps $\overline{\mathcal{M}}_1$ isomorphically in $\mathcal{M}_2$ and $\operatorname{Ker} F \subset \overline{\mathcal{N}}_1$, $\mathcal{M}_2 = F(\overline{\mathcal{M}}_1) + F(\overline{\mathcal{N}}_1)$. Let us show that $F(\overline{\mathcal{M}}_1) \cap F(\overline{\mathcal{N}}_1) = 0$. Decompose for this purpose F into a composition

$$\overline{\mathcal{M}}_1 \widetilde{\oplus} \overline{\mathcal{N}}_1 \to (\overline{\mathcal{M}}_1 \widetilde{\oplus} \overline{\mathcal{N}}_1)/\operatorname{Ker} F = \overline{\mathcal{M}}_1 \widetilde{\oplus} (\overline{\mathcal{N}}_1/\operatorname{Ker} F) \xrightarrow{\widetilde{F}} \mathcal{M}_2,$$

where $\widetilde{F}$ is an isomorphism. Therefore

$$\mathcal{M}_2 = F(\overline{\mathcal{M}}_1) \widetilde{\oplus} \widetilde{F}(\overline{\mathcal{N}}_1/\operatorname{Ker} F) = F(\overline{\mathcal{M}}_1) \widetilde{\oplus} F(\overline{\mathcal{N}}_1).$$

Since the A-module $\overline{\mathcal{N}}_1$ is finitely generated, $F(\overline{\mathcal{N}}_1)$ is finitely generated too. We have obtained a decomposition

$$H_A = \overline{\mathcal{M}}_1 \widetilde{\oplus} \overline{\mathcal{N}}_1 \to F(\overline{\mathcal{M}}_1) \widetilde{\oplus} [F(\overline{\mathcal{N}}_1) \widetilde{\oplus} \mathcal{N}_2] = H_A,$$

where $F|_{\overline{\mathcal{M}}_1} : \overline{\mathcal{M}}_1 \to F(\overline{\mathcal{M}}_1)$ is an isomorphism. □

Lemma 2.7.15 *If bounded A-operators D, D' and F admitting an adjoint are such that FD and $D'F$ are A-Fredholm operators then F is an A-Fredholm operator.*

Proof: By the definition of Fredholm property of FD and $D'F$ we can find operators T and T' admitting an adjoint such that

$$\begin{aligned} (FD)T &= 1 + K, \\ T'(D'F) &= 1 + K'. \end{aligned}$$

By Theorem 2.7.14 the operator F is Fredholm. For T, for example, it is possible to take an operator with the matrix $\begin{pmatrix} (F_1)^{-1} & 0 \\ 0 & 0 \end{pmatrix}$, where FD has the form $\begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix}$ in the sense of Definition 2.7.4. □

Remark 2.7.16 For A-Fredholm operators over W^*-algebra A their properties are more similar to properties of usual Fredholm operators. This problem we will discuss in Proposition 3.6.8.

Remark 2.7.17 For applications to elliptic operators it is important to develop the theory for spaces of the form $l_2(\mathcal{P})$. It can be done similarly (see [61]).

2.8 Representations of groups on Hilbert modules

In this section we assume that G denotes a compact group. First of all, we prove an equivariant variant of the Kasparov stabilization theorem. Let us follow here the original proof [34]. For closely related problems see also [45].

Definition 2.8.1 For a C^*-algebra B put

$$\mathcal{H}_B := \sum_{i=1}^{\infty} (V_i \otimes_{\mathbf{C}} B),$$

where $\{V_i\}$ is a countable set of finite-dimensional spaces, in which all irreducible unitary representations G are realized (up to isomorphism) and each representation repeats an infinite number of times; the B-Hilbert completion of the algebraic sum is carried out with respect to the norm given by the following B-inner product on summands

$$(x_1 \otimes b_1, x_2 \otimes b_2) := \langle x_1, x_2 \rangle_{V_i} \cdot b_1^* b_2, \qquad x_1,\, x_2 \in V_i.$$

Theorem 2.8.2 [34] *Let B be a C^*-algebra with a continuous action of a group G and $\mathcal{E}$ be a countably generated Hilbert G-B-module. The action is assumed to be unitary and agrees with the module structure in the sense that*

$$g(xb) = g(x)g(b), \qquad \langle g(x), g(y) \rangle = g(\langle x, y \rangle), \qquad x,\, y \in \mathcal{E}, \quad b \in B, \quad g \in G.$$

Then there exists an equivariant B-isomorphism preserving the inner product

$$\mathcal{E} \oplus \mathcal{H}_B \cong \mathcal{H}_B.$$

Proof: Let us denote by $\mathcal{E}^+$ the module $\mathcal{E}$, considered as a B^+-module. Let us suppose that the action of G on B^+ is extended from B by the formula $g(1) = 1$. Let us assume that we know how to prove the theorem for unital algebras, so that

$$\mathcal{E}^+ \oplus \mathcal{H}_{B^+} \cong \mathcal{H}_{B^+},$$

whence

$$\mathcal{E} \oplus \mathcal{H}_B \cong ((\mathcal{E} \oplus \mathcal{H}_B)^+)B \cong (\mathcal{E}^+ \oplus \mathcal{H}_{B^+})B \cong (\mathcal{H}_{B^+})B = \mathcal{H}_B.$$

Thus, we can restrict ourselves to the case of unital B.

Let $\{x_k\}$ be a countable system of generators of $\mathcal{E}$ and $\{e_k\}$ be an orthonormal basis of $\mathcal{H}_B$ and each $e_k = v_k \otimes 1_B$, where $v_k \in V_{s(k)}$. In other words, if $\{w_k\}$ is a union of some orthonormal bases of all V_j then $e_k = w_k \otimes 1_B$. Let $\{y_i\}$ be a system of elements in $\mathcal{E} \oplus \mathcal{H}_B$, in which each element of the form $x_k \oplus 0$ and $0 \oplus e_k$ is repeated an infinite number of times. We can suppose that $y_1 = 0 \oplus e_1$ and put $W_1 = 0 \oplus V_1 \otimes B$. Let us assume that by induction we have already constructed subspaces $W_1, \ldots, W_n$ satisfying the following conditions

(i) W_i is a $\mathbf{C}$-finite dimensional G-invariant subspace in $\mathcal{E} \oplus \mathcal{H}_B$,

(ii) each W_i has a basis $(z_i^1, \ldots, z_i^{K(i)}) = (f_1, \ldots, f_p)$ such that

$$\langle z_i^j, z_i^j \rangle = 1_B, \qquad \langle z_i^j, z_r^s \rangle = 0 \text{ for } i \neq r \text{ or } j \neq s,$$

(iii) there exists $m = m(n)$ such that

$$W_1 + \ldots + W_n \subset \mathcal{E}_m := \mathcal{E} \oplus \left(\bigoplus_{i=1}^{m} V_i \otimes B \right),$$

and consequently $(W_1 + \ldots + W_n)B \subset \mathcal{E}_m$,

(iv) the distance between y_n and $(W_1 + \ldots + W_n)B$ does not exceed $1/n$.

Remark that it follows from items (i) and (ii) that the modules $W_i B$ are pairwise orthogonal and G-invariant, as well as $(W_1 + \ldots + W_n)B$. The last module is free, so by Lemma 2.3.7, it has an orthogonal complement, which is G-invariant due to the unitarity of the action.

Let us pass to the construction of W_{n+1}. Put

$$y'_{n+1} := \sum_{j=1}^{p} f_j \langle f_j, y_{n+1}\rangle, \qquad y''_{n+1} = y_{n+1} - y'_{n+1}.$$

Then for any $w \in (W_1 + \ldots + W_n)B$

$$\langle w, y''_{n+1}\rangle = \langle \sum_{j=1}^{p} f_j b_j, y_{n+1} - \sum_{j=1}^{p} f_j \langle f_j, y_{n+1}\rangle\rangle = \sum_{j=1}^{p} b_j^* [\langle f_j, y_{n+1}\rangle - \langle f_j, y_{n+1}\rangle] = 0.$$

As by the definition of the sequence y_j the element y_{n+1} lies either in $\mathcal{E}$ or in some $V_i \otimes B$, so we have $y''_{n+1} \in \mathcal{E}_{m'}$ for some $m' > m$. Let us consider the orthogonal complement $S_{n,m'}$ for $(W_1 + \ldots + W_n)B$ in $\mathcal{E}_{m'}$. It is an invariant module and $y''_{n+1} \in S_{n,m'}$. By the Mostow theorem about periodic vectors [49] the elements with $\mathbf{C}$-finite dimensional orbits are dense in $S_{n,m'}$. Hence one can find a vector $z \in S_{n,m'}$ such that $\|z - y''_{n+1}\| < \frac{1}{2n+2}$ and $R := Gz$ is an invariant finite-dimensional subspace of $S_{n,m'}$. As z is a totalizing vector, so R is an irreducible G-module. Therefore there exists $m'' > m'$ such that there exists an equivariant isomorphism $\Gamma : R \to V_{m''}$. Let $\{h_1, \ldots, h_k\}$ be an orthonormal basis of $V_{m''}$ and $r_i := \Gamma^{-1}(h_i)$, $i = 1, \ldots, k$. Then for the corresponding irreducible matrix representation $T : G \to U(k)$ we have

$$g(h_i) = \sum_{j=1}^{k} T_i^j(g) h_j, \qquad g(r_i) = \sum_{j=1}^{k} T_i^j(g) r_j, \qquad g \in G.$$

Since $R \subset \mathcal{E}_{m'}$ and $m'' > m'$, R is orthogonal to $V_{m''}$. More precisely, each element of R is orthogonal to $V_{m''} \otimes B$ in $\mathcal{E} \otimes \mathcal{H}_B$. Hence $\langle r_i, h_j\rangle = 0$ for any i and j. Let

$$z := \sum_{i=1}^{k} r_i \alpha_i, \quad \alpha_i \in \mathbf{C}, \qquad \alpha := \left(\sum_{i=1}^{k} |\alpha_i|^2\right)^{1/2} + 1, \qquad r'_i := r_i + (h_i \otimes 1_B) \cdot \frac{1}{(2n+2)\alpha}.$$

Then $\langle r'_i, r'_j\rangle = \langle r_i, r_j\rangle + \{(2n+2)\alpha\}^{-2}\delta_{ij}$ and the matrix $L := \|\langle r'_i, r'_j\rangle\|_{i,j=1}^{k}$ is positive and invertible in $M_k(\mathbf{C}) \subset M_k(B)$. Let $D = \|d_{ji}\| := L^{-1/2} \in M_k(B)$ and $r''_i := \sum_{j=1}^{k} r'_j d_{ji}$. Let us take W_{n+1} equal to the complex linear span of vectors r''_i, $i = 1, \ldots, k$, or, what is the same, to the span of r'_i, as D has complex coefficients. Then $W_{m+1} \subset \mathcal{E}_{m''}$ and

$$\langle r''_i, r''_j\rangle = \sum_{p,q=1}^{k} \langle r'_p d_{pi}, r'_q d_{pj}\rangle = (D^* L D)_{ij} = \delta_{ij}.$$

Since all h_i and r_i are orthogonal to $W_1 + \ldots + W_n$, then W_{n+1} is orthogonal to it too. Further, let $F := L^{1/2}$, so that $r'_i := \sum_{j=1}^{k} r''_j F_{ji}$. Then

$$\begin{aligned}
g(r''_i) = g\left(\sum_{j=1}^{k} r'_j d_{ji}\right) &= \sum_{j=1}^{k} (g r'_j) d_{ji} = \sum_{j=1}^{k} \left(g r_j + (g h_j \otimes 1_B) \cdot \frac{1}{(2n+2)\alpha}\right) d_{ji} \\
&= \sum_{j=1}^{k} \left(\sum_{s=1}^{k} T_j^s(g) r_s + \left(\sum_{s=1}^{k} T_j^s(g) h_s \otimes 1_B\right) \cdot \frac{1}{(2n+2)\alpha}\right) d_{ji} \\
&= \sum_{j=1}^{k}\sum_{s=1}^{k} T_j^s(g) \left(r_s + (h_s \otimes 1_B) \cdot \frac{1}{(2n+2)\alpha}\right) d_{ji} = \sum_{j=1}^{k}\sum_{s=1}^{k} T_j^s(g) r'_s d_{ji} \\
&= \sum_{j=1}^{k}\sum_{s=1}^{k} T_j^s(g) \left(\sum_{t=1}^{k} r''_t F_{ts}\right) d_{ji} = \sum_{t=1}^{k} r''_t \left(\sum_{j=1}^{k}\sum_{s=1}^{k} T_j^s(g) F_{ts} d_{ji}\right) \in W_{n+1}.
\end{aligned}$$

Thus W_{n+1} is G-invariant. Let us estimate the distance by putting $z' = \sum_{i=1}^{k} r'_i \alpha_i$, so that

$$\rho(z, W_{n+1}) \le \rho(z, z') = \left\| \sum_{i=1}^{k} (r_i - r'_i)\alpha_i \right\| = \left\| \sum_{i=1}^{k} (h_i \otimes 1_B) \cdot \frac{1}{(2n+2)\alpha} \alpha_i \right\| = \frac{1}{(2n+2)\alpha} \cdot \left\| \sum_{i=1}^{k} h_i \alpha_i \right\|$$

$$= \frac{1}{(2n+2)\alpha} \cdot \left(\sum_{i=1}^{k} |\alpha_i|^2 \right)^{1/2} = \frac{1}{(2n+2)} \cdot \frac{\left(\sum_{i=1}^{k} |\alpha_i|^2 \right)^{1/2}}{\left(\sum_{i=1}^{k} |\alpha_i|^2 \right)^{1/2} + 1} < \frac{1}{(2n+2)}.$$

Therefore

$$\rho(y_{n+1}, (W_1 + \ldots + W_n)B) \le \rho(y''_{n+1}, z) + \rho(z, W_{n+1}B) \le \frac{1}{2n+2} + \rho(z, z') \le \frac{1}{n+1} < \frac{1}{n}.$$

Thus, by induction, the $\mathbf{C}$-subspaces W_i with properties (i) – (iv) are defined for any i. From the explicit expression for r'_i we obtain that W_n is isomorphic to some V_i, i. e., is irreducible. Further, the B-Hilbert completion of $\mathcal{M}$ (i. e., the closure in $\mathcal{E} \oplus \mathcal{H}_B$) of the algebraic orthogonal sum of modules $W_n B$ gives the whole $\mathcal{E} \oplus \mathcal{H}_B$. Indeed, by the property (iv) the algebraic sum is dense in $\mathcal{E} \oplus \mathcal{H}_B$. So, $\mathcal{M} \cong \mathcal{E} \oplus \mathcal{H}_B$. Now it is proved that $\mathcal{M}$ is isomorphic to $\mathcal{H}_B$, i. e. that each irreducible representation repeats indefinitely many times among $W_n B$. Let us suppose the opposite, then $\mathcal{M} \oplus \mathcal{H}_B \cong \mathcal{H}_B$, or

$$\mathcal{E} \oplus \mathcal{H}_B = \mathcal{E} \oplus \mathcal{H}_B \oplus \mathcal{H}_B \cong \mathcal{M} \oplus \mathcal{H}_B \cong \mathcal{H}_B. \qquad \square$$

Let us prove now the theorem of decomposition of representations [47]. Let $\mathcal{M}$ be a Hilbert B-module with a strongly continuous unitary representation G

$$T : G \to U(\mathcal{M}) \subset \mathrm{End}^*_B(\mathcal{M}), \qquad g \mapsto T_g,$$

and suppose that the group acts trivially on B. Let now $\{V_s\}$ be a complete collection of pairwise nonequivalent unitary representations of G, d_s be their dimensions, and D^s_{pq} be their matrix elements, which are continuous functions on G. For an invariant normalized Haar measure dg on G we define an operator

$$P^s_{pq} : \mathcal{M} \to \mathcal{M}, \qquad P^s_{pq}(x) := d_s \int_G \overline{D^s_{pq}(g)} T_g(x)\, dg. \tag{16}$$

As for a fixed $x \in \mathcal{M}$ a product of continuous complex-valued function by a continuous module-valued function is integrated and as the group is compact, so the integral converges to some element $\mathcal{M}$. We obtain a bounded operator. Indeed,

$$\|P^s_{pq} x\| \le d_s \int_G \left| \overline{D^s_{pq}(g)} \right| \|T_g(x)\|\, dg \le d_s \sup_{g \in G} \left| \overline{D^s_{pq}(g)} \right| \|x\|.$$

Therefore

$$\|P^s_{pq}\| \le d_s \sup_{g \in G} \left| \overline{D^s_{pq}(g)} \right|.$$

It is well known [8, I,§7.1, Theorem 5], that

$$\int_G D^s_{ij}(g) \overline{D^{s'}_{mn}(g)} = \begin{cases} 0, & s \ne s', \\ \frac{1}{d_s} \delta_{im} \delta_{jn}, & s = s'. \end{cases} \tag{17}$$

We need the following Peter-Weyl theorem.

Theorem 2.8.3 [8, I,§7.2, Theorem 1] *The functions $\sqrt{d_s} D^s_{jk}(g)$ form a complete orthonormalized system in $L^2(G)$.*

Lemma 2.8.4 *The operators P^s_{pq} have the following properties*

(i) P^s_{pq} *admits an adjoint and*

$$(P^s_{pq})^* = P^s_{qp}; \tag{18}$$

(ii) *the following equality is fulfilled*

$$P^s_{pq} P^{s'}_{p'q'} = \delta^{ss'} \delta_{qp'} P^s_{pq'}. \tag{19}$$

(iii) *the following equalities are fulfilled*

$$T_g P^s_{jm} = \sum_{i=1}^{d_s} D^s_{ij}(g)\, P^s_{im}, \tag{20}$$

$$P^s_{jm} T_g = \sum_{i=1}^{d_s} D^s_{mi}(g)\, P^s_{ji}. \tag{21}$$

Proof: First of all we remark that for unitary operators in $\mathcal{M}$ the mapping $F \mapsto F^*$ is continuous in the strong operator topology. In other words for unitary operators the strong continuity implies the $*$-strong one. Indeed,

$$\|(F'^* - F^*)x\| = \|(F'^{-1} - F^{-1})x\| = \|F'(F'^{-1} - F^{-1})Fz\| = \|Fz - F'z\| \to 0.$$

Therefore it is possible to take T^*_g instead of T_g in (16), and then take it out of the integral. More precisely, the first equality in the following chain

$$(P^s_{pq})^* = d_s \int_G D^s_{pq}(g) T^*_g(x)\, dg = d_s \int_G D^s_{pq}(g^{-1}) T_g(x)\, d(g^{-1}) = d_s \int_G \overline{D^s_{qp}(g)} T_g(x)\, dg = P^s_{qp}$$

has to be verified at first at the level of integral sums, and passage to the limit is possible due to the indicated $*$-strong continuity. Remaining equalities in the chain above are obtained by the invariance of Haar measure and by the relations $T^*_g = T^{-1}_g = T_{g^{-1}}$. The item (i) is proved.

It follows from (16) that

$$P^s_{pq} P^{s'}_{p'q'} = d_s d_{s'} \int_G \int_G \overline{D^s_{pq}(g)}\; \overline{D^{s'}_{p'q'}(g')} T_g T_{g'}\, dg\, dg'.$$

Since $T_g T_{g'} = T_{gg'}$, by putting $\widetilde{g} := gg'$ we obtain from

$$D^s_{pq}(g) = D^s_{pq}(\widetilde{g}\, g'^{-1}) = D^s_{pr}(\widetilde{g}) D^s_{rq}(g'^{-1}) = D^s_{pr}(\widetilde{g}) \overline{D^s_{qr}(g')}$$

and relations (17) that

$$P^s_{pq} P^{s'}_{p'q'} = d_s d_{s'} \int_G D^s_{qr}(g')\, \overline{D^{s'}_{p'q'}(g')}\, dg' \cdot \int_G \overline{D^s_{pr}(\widetilde{g})} T_{\widetilde{g}}\, d\widetilde{g} = d_{s'} \delta^{ss'} \frac{1}{d_{s'}} \delta_{qp'} \delta_{rq'} P^s_{pr} = \delta^{ss'} \delta_{qp'} P^s_{pq'}.$$

To prove the item (iii) let us remark that

$$T_g P^s_{jm}(x) = d_s \int_G \overline{D^s_{jm}(h)} T_{gh}(x)\, dh = d_s \int_G \overline{D^s_{jm}(g^{-1}h)} T_h(x)\, dh = d_s \int_G \sum_{i=1}^{d_s} \overline{D^s_{ji}(g^{-1})}\; \overline{D^s_{im}(h)} T_h(x)\, dh$$

$$= \sum_{i=1}^{d_s} \overline{D^s_{ji}(g^{-1})}\, d_s \int_G \sum_{i=1}^{d_s} \overline{D^s_{im}(h)} T_h(x)\, dh = \sum_{i=1}^{d_s} \overline{D^s_{ji}(g^{-1})}\, P^s_{im}(x) = \sum_{i=1}^{d_s} D^s_{ij}(g)\, P^s_{im}(x).$$

The second equality of this item can be proved similarly. □

Lemma 2.8.5 *The operators $P^s_p := P^s_{pp}$ are selfadjoint pairwise orthogonal projections.*

Proof: If we will rewrite the statement of the Lemma as

$$(P^s_p)^* = P^s_p, \qquad P^s_p P^{s'}_{p'} = \delta^{ss'} \delta_{pp'} P^s_p, \tag{22}$$

then the proof can be immediately obtained from (18) and (19). □

Lemma 2.8.6 *Let us put*

$$P^s := \sum_{p=1}^{d_s} P^s_p = \sum_{p=1}^{d_s} P^s_{pp}.$$

The operators P^s have the following properties

$$(P^s)^* = P^s, \tag{23}$$

$$P^s P^{s'} = \delta_{ss'} P^s, \tag{24}$$

$$T_g P^s = P^s T_g. \tag{25}$$

In other words, P^s are selfadjoint invariant pairwise orthogonal projections in $\mathcal{M}$.

Proof: By the definition of P^s the formulas (23) and (24) follow from (22) at once. To verify the third relation let us consider the character of the representation V_s

$$\chi^s(g) := \sum_{p=1}^{d_s} D^s_{pp}(g),$$

which, like the trace, satisfies the relation $\chi^s(g) = \chi^s(hgh^{-1})$. One has also

$$P^s = d_s \int_G \chi^s(g) T_g \, dg,$$

$$T_g P^s = d_s T_g \int_G \chi^s(g') T_{g'} \, dg' = d_s \int_G \chi^s(g') T_{gg'g^{-1}} T_g \, dg' = d_s \int_G \chi^s(gg'g^{-1}) T_{gg'g^{-1}} \, dg' T_g = P^s T_g. \qquad \square$$

Lemma 2.8.7 *Let us define*

$$\mathcal{M}^s := P^s(\mathcal{M}), \qquad \mathcal{M}^\bullet := \bigoplus_{s=1}^{\infty} \mathcal{M}^s, \tag{26}$$

where the sum is supposed to be completed either as a Hilbert sum or (that is the same) as a closure in $\mathcal{M}$ of the algebraic sum. Then

$$\mathcal{M}^\bullet = \mathcal{M}. \tag{27}$$

Proof: Let us assume that a $\mathbf{C}$-linear functional f on $\mathcal{M}$ vanishes on $\mathcal{M}^\bullet$ and that $x \in \mathcal{M}$ is an arbitrary vector. Then for any set of indices we have $P^s_{ij}(x) \in \mathcal{M}^\bullet$, so that

$$0 = f(P^s_{ij}(x)) = d_s \int_G \overline{D^s_{ij}(g)} f(T_g(x)) \, dg.$$

Therefore by the Peter-Weyl theorem 2.8.3 $f(T_g(x)) = 0$ holds almost everywhere, and by continuity it vanishes everywhere. In particular, $f(T_e(x)) = f(x) = 0$. Hence by the Hahn-Banach theorem $\mathcal{M}^\bullet = \mathcal{M}$.
$\square$

Theorem 2.8.8 [47] *Let $\mathcal{M}$ be a Hilbert B-module with a strongly continuous unitary representation G and let the group acts trivially on B. Let now $\{V_s\}$ be a complete collection of pairwise nonequivalent unitary representations of G and*

$$\mathcal{M}_s := \mathrm{Hom}_{G,\mathbf{C}}(V_s, \mathcal{M}) \subset \mathrm{Hom}_{\mathbf{C}}(V_s, \mathcal{M}) \cong V_s^* \otimes \mathcal{M}$$

is a Hilbert B-module so that B-product is defined by the formula

$$\langle \varphi, \psi \rangle := \sum_{i,j=1}^{\dim V_s} \langle \varphi(h^s_i), \psi(h^s_j) \rangle_{\mathcal{M}}, \qquad h^s_1, \dots, h^s_{\dim V_s} \text{ — orthobasis } V_s.$$

Then for the Hilbert sum we have G-B-isomorphism

$$\Gamma = \bigoplus_{s=1}^{\infty} \Gamma_s : \bigoplus_{s=1}^{\infty} V_s \otimes \mathcal{M}_s \cong \mathcal{M}, \qquad \Gamma_s : v \otimes \varphi \mapsto \varphi(v), \quad v \in V_s,\ \varphi \in \mathcal{M}_s,$$

and

$$\Gamma(V_s \otimes \mathcal{M}_s) = \mathcal{M}^s,$$

where $\mathcal{M}^s$ *is introduced in* (26).

Proof: Let us remark first of all, that Γ_s are algebraically injective. Indeed, let

$$0 = \Gamma_s \left(\sum_{j=1}^{d_s} h_j^s \alpha_j \otimes \varphi \right) = \varphi \left(\sum_{j=1}^{d_s} h_j^s \alpha_j \right).$$

Since by the Schur lemma φ is either isomorphism, or 0, this equality can be true only if $\sum_{j=1}^{d_s} h_j^s \alpha_j = 0$ or $\varphi = 0$. But then $\sum_{j=1}^{d_s} h_j^s \alpha_j \otimes \varphi = 0$.

By Lemma 2.8.7 it is sufficient to prove only that Γ_s maps bijectively $V_s \otimes \mathcal{M}_s$ to $\mathcal{M}^s$.

Let us remark that by putting $\mathcal{M}_i^s := P_i^s(\mathcal{M}) = P_{ii}^s(\mathcal{M})$, we obtain by relations (19) that the operators P_{ij}^s realize isomorphisms

$$P_{ij}^s : \mathcal{M}_j^s \to \mathcal{M}_i^s.$$

Thus $\mathcal{M}^s = \bigoplus_{j=1}^{d_s} \mathcal{M}_j^s$ is a sum of isomorphic modules.

Let $\{h_1^s, \dots, h_{d_s}^s\}$ be that orthobasis of V_s, with respect to which the matrix elements D_{ij}^s were defined. Let us define a homomorphism

$$\Phi^s : V_s \otimes [\mathcal{M}_1^s] \to \mathcal{M}^s, \qquad \Phi^s(h_j^s \otimes x) = P_{j1}^s(x), \tag{28}$$

where we have taken $\mathcal{M}_1^s$ in square brackets to underline, that there is no action of G on it. By the properties of the operators P_{j1}^s the map Φ^s is an isomorphism. Since by (20)

$$T_g \Phi^s(h_j^s \otimes x) = T_g P_{j1}^s(x) = \sum_{i=1}^{d_s} D_{ij}^s(g)\ P_{i1}^s(x),$$

and

$$\Phi^s(g(h_j^s) \otimes x) = \Phi^s \left(\sum_{i=1}^{d_s} D_{ij}^s(g) h_i^s \otimes x \right) = \sum_{i=1}^{d_s} D_{ij}^s(g) P_{i1}^s(x),$$

the map Φ^s is equivariant. Further, there is a map

$$\Psi^s : \mathcal{M}_1^s \to \mathcal{M}_s, \qquad \Psi^s(x)(v) := \Phi^s(v \otimes x).$$

Then

$$\Gamma_s \circ (\mathrm{Id}_{V_s} \otimes \Psi^s)(v \otimes x) = \Phi^s(v \otimes x).$$

As we have an isomorphism on the right and as Γ_s is algebraically injective, so Γ_s is an isomorphism (see Lemma 2.8.10), whence Ψ^s is an isomorphism. In particular, the images of Γ_s coincide with $\mathcal{M}^s$ and are orthogonal to each other. Hence Γ is topologically injective and its image coincides with $\mathcal{M}$. □

Remark 2.8.9 Let G—A-module $\mathcal{M}$ belong to the class $\mathcal{P}(A)$ of projective finitely generated modules. Then obviously $\mathcal{M}_s = \mathrm{Hom}_G(V_s, \mathcal{M}) \in \mathcal{P}(A)$. Let us show that in the sum $\bigoplus_s$ only finite number of summands does not vanish. Let us denote by $a_1, \dots, a_s$ generators of $\mathcal{M}$. Let us choose by the Mostow lemma [49] **C**-periodic vectors $b_1, \dots, b_s$ so close to $a_1, \dots, a_s$, that they generate $\mathcal{M}$ as an A-module (see Lemma 2.7.3). By decomposing the finite-dimensional G—**C**-module equal to the linear span of the orbit Gb_j, into irreducible modules, let us discover a new system of generators $c_1, \dots, c_N$, now lying in irreducible G—**C**-modules. From here it is evident that the number of nonzero summands does not exceed N.

Lemma 2.8.10 *Let $F : L \to M$, $T : N \to L$ be continuous maps of Banach spaces, $S = FT$ be an isomorphism and* $\operatorname{Ker} F = 0$. *Then F is an isomorphism.*

Proof: Since S is an isomorphism, and F is bounded, T is topologically injective and its image $T(N)$ is closed in L. Let it not coincide with L. Let us choose a vector $0 \neq x \in L \backslash T(N)$. Then $0 \neq F(x) \notin FT(N)$. Really, let $F(x) = FT(y)$ for some $y \in N$. Since $z = Ty \in T(N)$, $z - x \neq 0$, while $F(z - x) = FT(y) - F(z) = 0$. We have got a contradiction with $\operatorname{Ker} F = 0$. Hence, T is a topologically injective epimorphism, i. e., isomorphism, as well as $F = ST^{-1}$. $\square$

Let us remind some facts about integrating operator-valued functions [32]. Let X be a compact space, A be a C^*-algebra, $\varphi : C(X) \to A$ be an involutive homomorphism of unital algebras, $F : X \to A$ be a continuous map and for each $x \in X$ the element $F(x)$ commutes with the image of φ. In this case an integral

$$\int_X F(x)\, d\varphi \quad \in \quad A$$

can be defined as follows. Let $X = \cup_{i=1}^n U_i$ be an open covering and $\sum_{i=1}^n \alpha_i(x) = 1$ be a subordinate partition of unit. Let us choose points $\xi_i \in U_i$ and form an integral sum

$$\sum (F, \{U_i\}, \{\alpha_i\}, \{\xi_i\}) = \sum_{i=1}^n F(\xi_i)\varphi(\alpha_i).$$

If there exists the limit of such integral sums then it is called an integral.

If X is a Lie group G, it is natural to take as φ a Haar measure $\varphi : C(X) \to \mathbf{C}$, $\varphi(\alpha) = \int_G \alpha(g)\, dg$ and to define for a norm continuous map $Q : G \to \mathcal{B}(H)$

$$\int_G Q(g)\, dg := \lim \sum_i Q(\xi_i) \int_G \alpha_i(g)\, dg,$$

where the algebra A is realized as a subalgebra in the algebra $\mathcal{B}(H)$ of bounded operators on a Hilbert space H. If $Q : G \to P^+(A) \subset \mathcal{B}(H)$, then since $\int_G \alpha_i(g)\, dg \geq 0$, we obtain that

$$\sum_i Q(\xi_i) \cdot \int_G \alpha_i(g)\, dg \ \in \ P^+(A) \quad \text{and} \quad \int_G Q(g)\, dg \ \in \ P^+(A)$$

(the positive cone $P^+(A)$ is convex and closed). Hence we have proved the following lemma.

Lemma 2.8.11 *Let $Q : G \to P^+(A)$ be a continuous function. Then for the integral in the sense of* [32] *the following inequality holds*

$$\int_G Q(g)\, dg \geq 0. \qquad \square$$

Theorem 2.8.12 [66] *Let* $\mathrm{GL} = \mathrm{GL}(A)$ *be the complete general linear group, i. e. the group of invertible operators from* $\operatorname{End} l_2(A)$, *and suppose that for the group G a representation $g \mapsto T_g$ $(g \in G, T_g \in \mathrm{GL})$ is given, and that the map*

$$G \times l_2(A) \to l_2(A), \quad (g, u) \mapsto T_g u$$

is continuous.

Then there exists an A-inner product on $l_2(A)$ equivalent to the initial one (i. e. generating an equivalent norm) and such that the representation $g \mapsto T_g$ is unitary with respect to this new product.

Proof: Let $\langle\, , \rangle'$ be the initial inner product. For any u and v from $l_2(A)$ there exists a continuous map $G \to A, \quad x \mapsto \langle T_x u, T_x v\rangle'$. Let us define a new product by the formula

$$\langle u, v\rangle \ = \ \int_G \langle T_x u, T_x v\rangle'\, dx,$$

where the integral can be considered in the sense of any of two definitions in [32, p. 810], since the map is continuous with respect to the C^*-algebra norm. It is easy to see that this new product sets an A-Hermitian map $l_2(A) \times l_2(A) \to A$ and that by Lemma 2.8.11 $\langle u, u\rangle \geq 0$. Let us show that this map is continuous. Let us fix an arbitrary $u \in l_2(A)$. Then $x \mapsto T_x(u)$, $\quad G \to l_2(A)$ is a continuous map defined on a compact space, thus the set $\{T_x(u) \mid x \in G\}$ is bounded. Therefore, by the principle of uniform boundedness [8, v. 2]

$$\lim_{v \to 0} T_x(v) = 0 \tag{29}$$

is uniform on $x \in G$. If u is fixed, then

$$||T_x(u)|| \leq M_u = const$$

and by the equality (29) one has

$$\begin{aligned} ||\langle u, v \rangle|| &= || \int_G \langle T_x(u), T_x(v) \rangle' \, dx|| \\ &\leq M_u \cdot vol\, G \cdot \sup_{x \in G} ||T_x(v)|| \to 0 \quad (v \to 0). \end{aligned}$$

We have obtained continuity at the point 0, hence on the whole space $l_2(A) \times l_2(A)$. For $T_x u = (a_1(x), a_2(x), \ldots) \in l_2(A)$ the equality $\langle u, u \rangle = 0$ takes the form

$$\int_G \sum_{i=1}^{\infty} a_i(x) a_i^*(x) \, dx = 0.$$

Let A be realized as a subalgebra of the algebra of bounded operators on a Hilbert space L with an inner product $(\ ,\ )_L$. For each $p \in L$ we have

$$\begin{aligned} 0 &= \left(\left(\int_G \sum_{i=1}^{\infty} a_i(x) a_i^*(x) \, dx \right) p, p \right)_L \\ &= \int_G \left(\sum_{i=1}^{\infty} a_i(x) a_i^*(x) p, p \right)_L dx = \int_G \left(\sum_{i=1}^{\infty} (a_i^*(x) p, a_i^*(x) p)_L \right) dx \end{aligned}$$

(cf. [32]). Therefore $a_i(x) = 0$ almost everywhere, therefore $a_i(x) = 0$ for all x by continuity and $T_x u = 0$. In particular, $u = 0$.

Since each operator T_y is an automorphism, we obtain

$$\langle T_y u, T_y v \rangle = \int_G \langle T_{xy} u, T_{xy} v \rangle' \, dx = \int_G \langle T_z u, T_z v \rangle' \, dz = \langle u, v \rangle.$$

Now we show the equivalence of two norms, which, in particular, imply continuity of the representation. There is a number $N > 0$ such that $||T_x||' \leq N$ for any $x \in G$. Hence by [32] we have

$$\begin{aligned} ||u||^2 &= ||\langle u, u \rangle||_A = || \int_G \langle T_x u, T_x u \rangle' \, dx||_A \\ &\leq \left(\sup_{x \in G} ||T_x u||' \right)^2 \leq N^2 (||u||')^2. \end{aligned}$$

On the other hand, applying Theorem 2.1.4 and Lemma 2.8.11, we obtain that

$$\begin{aligned} \langle u, u \rangle' &= \int_G \langle T_{g^{-1}} T_g u, T_{g^{-1}} T_g u \rangle' \, dg \leq \int_G ||T_{g^{-1}}||^2 \langle T_g u, T_g u \rangle' \, dg \\ &\leq \int_G N^2 \langle T_g u, T_g u \rangle' \, dg = N^2 \int_G \langle T_g u, T_g u \rangle' \, dg = N^2 \langle u, u \rangle. \end{aligned}$$

Then $(||u||')^2 = ||\langle u, u \rangle'||_A \leq N^2 ||\langle u, u \rangle||_A = N^2 ||u||^2$. $\quad\square$

Remark 2.8.13 Since $l_2(P)$ is a direct summand in $l_2(A)$, the previous theorem remains valid for $l_2(P)$ and any other countably generated module $\mathcal{M}$.

Remark 2.8.14 Before averaging we had had operators, which, in general, had not admitted an adjoint, and after averaging we have obtained unitary operators out of them. In relation with this remark the following problem arises. Is it true, that if a given operator represents an element of compact group, then it admits an adjoint? The negative answer to this problem is contained in Example 2.3.2, as a decomposition into direct (topological) sum defines a representation of the group $\mathbf{Z}/2\mathbf{Z}$.

Corollary 2.8.15 [67] *Let $\mathcal{M} = \mathcal{M}_1 \widetilde{\oplus} \mathcal{M}_2$ be a topological decomposition into a direct sum of closed Hilbert modules (not necessarily orthogonal). Then there exists a new inner product on the module $\mathcal{M}$ equivalent to the initial one, with the respect to which the indicated decomposition is orthogonal.*

Proof: Let us define an operator $J : \mathcal{M} \longrightarrow \mathcal{M}$ by the equality

$$Jx = \begin{cases} x, & \text{if } x \in \mathcal{M}_1, \\ -x, & \text{if } x \in \mathcal{M}_2. \end{cases}$$

It is possible to consider the operator J as a representation of the group $\mathbf{Z}/2\mathbf{Z}$ on the module $\mathcal{M}$, and by Theorem 2.8.12 the inner product $\langle x, y \rangle_\beta = \langle x, y \rangle + \langle Jx, Jy \rangle$ is equivalent to the initial one. Orthogonality of $\mathcal{M}_1$ and $\mathcal{M}_2$ with the respect to this inner product is evident. $\square$

In Theorem 2.3.13 [68] we will show how the averaging theorem 2.8.12 can be generalized from the case of compact group to the case of amenable group, but only for Hilbert W^*-modules.

3 Hilbert modules over W^*-algebras

3.1 W^*-algebras

Detailed information about W^*-algebras can be found in the books [62, 12, 18, 60, 29]. We recommend also the original papers of Murray and von Neumann [51] which are are still actual. We list here the basic definitions and necessary facts.

Topologies on $\mathcal{B}(H)$. Besides the norm topology we will consider on the algebra $\mathcal{B}(H)$ of bounded linear operators on a Hilbert space H also a number of other locally convex topologies, which we define by sets of seminorms on $\mathcal{B}(H)$. Let $a \in \mathcal{B}(H)$, $\xi, \xi_i, \eta, \eta_i \in H$.

1. The *σ-weak topology* is defined by the seminorms

$$p(a) = \left| \sum_{i=1}^{\infty} (a\xi_i, \eta_i) \right|, \qquad \sum_{i=1}^{\infty} \|\xi_i\|^2 < \infty, \quad \sum_{i=1}^{\infty} \|\eta_i\|^2 < \infty.$$

2. The *σ-strong topology* is defined by the seminorms

$$p(a) = \sum_{i=1}^{\infty} \|a\xi_i\|, \qquad \sum_{i=1}^{\infty} \|\xi_i\|^2 < \infty.$$

3. The *σ-strong* topology* is defined by the seminorms

$$p(a) = \sum_{i=1}^{\infty} (\|a\xi_i\| + \|a^*\xi_i\|)^{1/2}, \qquad \sum_{i=1}^{\infty} \|\xi_i\|^2 < \infty.$$

4. The *weak topology* is defined by the seminorms $p(a) = |(a\xi, \eta)|$.

5. The *strong topology* is defined by the seminorms $p(a) = \|a\xi\|$.

6. The *strong* topology* is defined by the seminorms $p(a) = (\|a\xi\| + \|a^*\xi\|)^{1/2}$.

On bounded subsets in $\mathcal{B}(H)$ the σ-weak topology coincides with the weak topology, the σ-strong topology coincides with the strong, and the σ-strong* topology coincides with the strong* topology.

A *commutant* of a subset $R \subset \mathcal{B}(H)$ is the set $R^! := \{a \in \mathcal{B}(H) : ar = ra \text{ for each } r \in R\}$. A *bicommutant* of set R is the set $R^{!!} = (R^!)^!$.

Theorem 3.1.1 (von Neumann bicommutant theorem) *Let $\mathcal{A} \subset \mathcal{B}(H)$ be an involutive subalgebra. Then the following conditions are equivalent:*

(i) *the algebra $\mathcal{A}$ contains the identity operator and is closed with respect to the σ-weak topology;*

(ii) *the algebra $\mathcal{A}$ contains the identity operator and is closed with respect to the σ-strong topology;*

(iii) *the algebra $\mathcal{A}$ coincides with its bicommutant, $\mathcal{A}^{!!} = \mathcal{A}$.*

In particular, it follows from here that if $A \subset B \subset \mathcal{B}(H)$ are two subsets then $A^{!!} \subset B^{!!}$.

Definition 3.1.2 An involutive subalgebra in $\mathcal{B}(H)$ is called a *von Neumann algebra* if it satisfies the conditions of Theorem 3.1.1.

In the von Neumann algebras there exists the polar decomposition: any element $a \in \mathcal{A}$ can be be represented in a unique way as $a = uh$, where u is a partial isometry, and h is a positive element of the algebra $\mathcal{A}$, and $\operatorname{Ker} u = \operatorname{Ker} h$.

Universal enveloping von Neumann algebra.

Let ω be a positive linear functional on a C^*-algebra A, (π_ω, H_ω) be a cyclic representation of algebra A on a Hilbert space H_ω constructed with the help of the GNS-constructions. Let us put

$$(\pi, H) = \bigoplus_{\omega \in A^*_+} (\pi_\omega, H_\omega),$$

where A^*_+ denotes the set of all positive linear functionals on the C^*-algebra A. The representation (π, H) is called *universal*. The universal representation of the C^*-algebra contains any representation of this algebra as a subrepresentation.

Theorem 3.1.3 *The second dual space A^{**} for a C^*-algebra A equipped with the $\sigma(A^{**}, A^*)$-topology is homeomorphic to the bicommutant $A^{!!}$ of algebra $A \subset \mathcal{B}(H)$ with respect to the universal representation equipped with the σ-weak topology.*

The von Neumann algebra $A^{!!}_u$, where bicommutant is taken with respect to the universal representation, is called a *universal enveloping vn Neumann algebra* for the C^*-algebra A and is denoted by A^{**}. Any homomorphism of C^*-algebras $A \longrightarrow B$ admits a natural extension up to a homomorphism of the second dual spaces $A^{**} \longrightarrow B^{**}$. If $A \subset \mathcal{B}(H_0)$ and $A \longrightarrow \mathcal{B}(H)$ is the universal representation then $H_0 \subset H$ and $A^{!!} \subset A^{!!}_u \cong A^{**}$.

W^*-algebras. The notion of W^*-algebra allows to speak about von Neumann algebras without relation with a concrete Hilbert space where they act.

Definition 3.1.4 A C^*-algebra $\mathcal{A}$, which, as a Banach space, is dual to some Banach space F, $\mathcal{A} = F^*$, is called a *W^*-algebra*.

A Banach space F is called *pre-dual* for $\mathcal{A}$.

Definition 3.1.5 A linear functional φ on the von Neumann algebra $\mathcal{A}$ is called *normal*, if for any increasing net $a_\lambda \in \mathcal{A}$, $\lambda \in \Lambda$, with the least upper bound $a \in \mathcal{A}$ the value $\varphi(a)$ is the least upper bound of the set $\varphi(a_\lambda)$.

Theorem 3.1.6 *Let $\mathcal{A}$ be a W^*-algebra. Then there exists a unique (up to an isomorphism) pre-dual space for $\mathcal{A}$, which coincides with the space of all normal linear functionals on $\mathcal{A}$.*

The pre-dual space of a W^*-algebra $\mathcal{A}$ we will denote by $\mathcal{A}_*$. By P we shall denote the set of normal positive functionals on $\mathcal{A}$, $P \subset \mathcal{A}_*$. The pre-dual space $\mathcal{A}_*$ is the linear span of the set P.

3.2 Inner product on dual modules

Hilbert modules over W^*-algebras we shall call Hilbert W^*-modules. Some aspects of the theory of Hilbert C^*-modules become more simple in the W^*-case.

Theorem 3.2.1 ([52]) *Let $\mathcal{M}$ be a Hilbert $\mathcal{A}$-module. An $\mathcal{A}$-valued inner product $\langle \cdot, \cdot \rangle$ admits an extension to the Banach module $\mathcal{M}'$, making it a self-dual Hilbert $\mathcal{A}$-module. In particular, the extended inner product satisfies the equality $\langle f, \widehat{x} \rangle = f(x)$ for all $x \in \mathcal{M}$, $f \in \mathcal{M}'$.*

Proof: Let $f, g \in \mathcal{M}'$. Our task is to define an inner product of these functionals $\langle f, g\rangle$. Let us define for this purpose a map $\Gamma : P \longrightarrow \mathbf{C}$ by the formula $\Gamma(\varphi) = (f_\varphi, g_\varphi)_\varphi$, where $\varphi \in P$ is a normal positive functional on $\mathcal{A}$, and let us show that the map Γ admits an extension to the set $\mathcal{A}_*$ of all normal functionals on $\mathcal{A}$. For this purpose the following two technical Lemmas will be necessary.

Lemma 3.2.2 *Let $\lambda_1, \ldots, \lambda_n \in \mathbf{C}$ and $\varphi_1 \ldots, \varphi_n \in P$ be such that $\sum_{i=1}^n \lambda_i \varphi_i = 0$. Then $\sum_{i=1}^n \lambda_i \Gamma(\varphi_i) = 0$.*

Proof: Let us consider a normal positive functional $\varphi = \sum_{i=1}^n \varphi_i$. Then $\varphi \geq \varphi_i$, $i = 1, \ldots, n$. If $x, y \in \mathcal{M}$ then by the assumption

$$\sum_{i=1}^n \lambda_i (V^*_{\varphi,\varphi_i} V_{\varphi,\varphi_i}(x + N_\varphi), y + N_\varphi)_\varphi = \sum_{i=1}^n \lambda_i (x + N_{\varphi_i}, y + N_{\varphi_i})_{\varphi_i} = \sum_{i=1}^n \lambda_i \varphi_i(\langle x, y\rangle) = 0,$$

therefore $\sum_{i=1}^n \lambda_i V^*_{\varphi,\varphi_i} V_{\varphi,\varphi_i} = 0$. Remark that by (2.5.7)

$$\sum_{i=1}^n \lambda_i \Gamma(\varphi_i) = \sum_{i=1}^n \lambda_i (f_{\varphi_i}, g_{\varphi_i})_{\varphi_i} = \sum_{i=1}^n \lambda_i (V_{\varphi,\varphi_i} f_\varphi, V_{\varphi,\varphi_i} g_\varphi)_{\varphi_i} = \sum_{i=1}^n \lambda_i (V^*_{\varphi,\varphi_i} V_{\varphi,\varphi_i} f_\varphi, g_\varphi)_\varphi = 0.$$

It means, that the map Γ can be extended to $\mathcal{A}_*$. $\square$

Lemma 3.2.3 *The map Γ is bounded.*

Proof: It is possible to present an arbitrary normal functional $\psi \in \mathcal{A}_*$ as $\psi = \varphi_1 - \varphi_2 + i(\varphi_3 - \varphi_4)$, where $\varphi_i \in P$ and $\sum_{i=1}^4 \|\varphi_i\| \leq 2\|\psi\|$. Then

$$|\Gamma(\psi)| \leq \sum_{i=1}^4 |(f_{\varphi_i}, g_{\varphi_i})_{\varphi_i}| \leq \sum_{i=1}^4 \|f_{\varphi_i}\|_{\varphi_i} \|g_{\varphi_i}\|_{\varphi_i} \leq \sum_{i=1}^4 \|\varphi_i\| \|f\| \|g\| \leq 2\|f\| \|g\| \|\psi\|$$

as required. $\square$

Let us continue the proof of the theorem. We have defined a linear functional on $\mathcal{A}_*$, which is also denoted by Γ. Since $\mathcal{A}$ is isomorphic to the space of linear functionals on the pre-dual space $\mathcal{A}_*$ [62], there exists a unique element $\langle f, g\rangle \in \mathcal{A}$ such that $\Gamma(\psi) = \psi(\langle f, g\rangle)$ for all $\psi \in \mathcal{A}_*$, in particular, $(f_\varphi, g_\varphi)_\varphi = \varphi(\langle f, g\rangle)$ for all $\varphi \in P$. Sesquilinearty of the defined map $\langle\cdot,\cdot\rangle : \mathcal{M}' \times \mathcal{M}' \longrightarrow \mathcal{A}$ follows from linearity of the map $f \longmapsto f_\varphi$ from $\mathcal{M}'$ to H_φ for $\varphi \in P$. Let us show that $\langle\cdot,\cdot\rangle$ satisfies the properties (i) - (iv) of Definition 1.2.1.

(i) The inequality $\langle f, f\rangle \geq 0$ follows from the fact, that for all $\varphi \in P$ we have $\varphi(\langle f, f\rangle) = (f_\varphi, f_\varphi)_\varphi \geq 0$.

(ii) Let $\langle f, f\rangle = 0$ for $f \in \mathcal{M}'$. Then $f_\varphi = 0$ for all $\varphi \in P$, hence $\varphi(f(x)) = 0$ for all $x \in \mathcal{M}$, whence it follows that $f = 0$.

(iii) Since for any $\varphi \in P$

$$\varphi(\langle f, g\rangle) = (f_\varphi, g_\varphi)_\varphi = \overline{(g_\varphi, f_\varphi)_\varphi} = \overline{\varphi(\langle g, f\rangle)} = \varphi(\langle g, f\rangle^*),$$

we conclude that $\langle f, g\rangle = \langle g, f\rangle^*$.

(iv) Let $a \in \mathcal{A}$, $\varphi \in P$. Let us define a functional φ_a on the algebra $\mathcal{A}$ by the equality $\varphi_a(b) = \varphi(a^* b)$, $b \in \mathcal{A}$. Then $\varphi_a \in \mathcal{A}_*$ and $\varphi_a = \sum_{i=1}^4 \lambda_i \varphi_i$, where $\lambda_i \in \mathbf{C}$, $\varphi_i \in P$. Let us put $\psi = \varphi + \sum_{i=1}^4 \varphi_i$, then ψ is a positive functional and $\psi \geq \varphi, \varphi_1, \ldots, \varphi_4$. It follows from Proposition 2.5.7 that

$$\varphi(a^* \langle f, g\rangle) = \sum_{i=1}^4 \lambda_i \varphi_i(\langle f, g\rangle) = \sum_{i=1}^4 \lambda_i (f_{\varphi_i}, g_{\varphi_i})_{\varphi_i} = \sum_{i=1}^4 \lambda_i (f_{\varphi_i}, V_{\psi,\varphi_i} g_\psi)_{\varphi_i}.$$

But for each $x \in \mathcal{M}$

$$\begin{aligned}\sum_{i=1}^4 \lambda_i (f_{\varphi_i}, V_{\psi,\varphi_i}(x + N_\psi))_{\varphi_i} &= \sum_{i=1}^4 \lambda_i (f_{\varphi_i}, x + N_{\varphi_i})_{\varphi_i} = \sum_{i=1}^4 \lambda_i \varphi_i(f(x)) = \varphi_a(f(x)) = \varphi(a^* f(x)) \\ &= \varphi((f \cdot a)(x)) = ((f \cdot a)_\varphi, x + N_\varphi)_\varphi = ((f \cdot a)_\varphi, V_{\psi,\varphi}(x + N_\psi))_\varphi.\end{aligned}$$

Since the subspace $\mathcal{M}/N_\psi$ is dense in H_ψ, for any $\varphi \in P$ the following equality holds

$$\varphi(a^*\langle f,g\rangle) = \sum_{i=1}^{4} \lambda_i (f_{\varphi_i}, V_{\psi,\varphi_i} g_\psi)_{\varphi_i} = ((f\cdot a)_\varphi, V_{\psi,\varphi} g_\psi)_\varphi = ((f\cdot a)_\varphi, g_\varphi)_\varphi = \varphi(\langle f\cdot a, g\rangle),$$

hence $a^*\langle f,g\rangle = \langle f\cdot a, g\rangle$. Passing to adjoints, we obtain also $\langle f,g\rangle a = \langle f, g\cdot a\rangle$.

The thus obtained inner product on $\mathcal{M}'$ is an extension of the inner product from $\mathcal{M}$. Indeed, if $x,y \in \mathcal{M}$, $\varphi \in P$ then

$$\varphi(\langle \widehat{x}, \widehat{y}\rangle) = (\widehat{x}_\varphi, \widehat{y}_\varphi)_\varphi = (x+N_\varphi, y+N_\varphi)_\varphi = \varphi(\langle x,y\rangle).$$

Hence $\langle \widehat{x}, \widehat{y}\rangle = \langle x,y\rangle$. Further, if $f \in \mathcal{M}'$, then

$$\varphi(\langle f, \widehat{x}\rangle) = (f_\varphi, \widehat{x}_\varphi)_\varphi = \varphi(f(x)),$$

therefore $\langle f, \widehat{x}\rangle = f(x)$. Let us show that $\mathcal{M}'$ is complete with respect to the norm $\|\cdot\|_{\mathcal{M}'}$ defined by the constructed inner product. On $\mathcal{M}'$ there exists also the norm $\|\cdot\|$ defined as the norm of linear maps from $\mathcal{M}$ to $\mathcal{A}$, with the respect to which the space $\mathcal{M}'$ is complete. Let us prove that $\|\cdot\|_{\mathcal{M}'} = \|\cdot\|$. Since

$$f(x)^* f(x) = \langle \widehat{x}, f\rangle\langle f, \widehat{x}\rangle \le \|f\|^2_{\mathcal{M}'} \cdot \langle x,x\rangle,$$

we obtain that $\|f\| \le \|f\|_{\mathcal{M}'}$. But, since $\|f_\varphi\| \le \|f\|\,\|\varphi\|^{1/2}$ for each $\varphi \in P$,

$$\|f\|^2_{\mathcal{M}'} = \|\langle f,f\rangle\| = \sup\{\|f_\varphi\|^2_\varphi : \varphi \in P, \|\varphi\| = 1\} \le \|f\|^2,$$

and $\|\cdot\|_{\mathcal{M}'} = \|\cdot\|$.

So, it is proved that $\mathcal{M}'$ is a Hilbert $\mathcal{A}$-module. It remains to verify that it is self-dual. Let $F \in (\mathcal{M}')'$. The restriction of F to $\mathcal{M} \subset \mathcal{M}'$ is an element of the module $\mathcal{M}'$, therefore it is possible to find a functional $f \in \mathcal{M}'$ such that $F(\widehat{x}) = f(x)$ for all $x \in \mathcal{M}$. Let us define a functional $F_0 \in (\mathcal{M}')'$ by the equality

$$F_0(g) = F(g) - \langle f, g\rangle, \qquad g \in \mathcal{M}'.$$

It is obvious that $F_0(\widehat{x}) = 0$ for all $x \in \mathcal{M}$. We have to verify that $F_0(g) = 0$ for all $g \in \mathcal{M}'$. Let $\varphi \in P$. Choose a sequence $\{y_n + N_\varphi\}$ in $\mathcal{M}/N_\varphi$, converging to g_φ. Since F_0 is bounded, we can find a number K such that $F_0(h)^* F_0(h) \le K\langle h,h\rangle$ for all $h \in \mathcal{M}'$. For all $n = 1,2,\ldots$

$$\varphi(F_0(g)^* F_0(g)) = \varphi(F_0(g-\widehat{y}_n)^* F_0(g-\widehat{y}_n)) \le K\varphi(\langle g-\widehat{y}_n, g-\widehat{y}_n\rangle).$$

But since

$$\begin{aligned}\varphi(\langle g-\widehat{y}_n, g-\widehat{y}_n\rangle) &= (g_\varphi, g_\varphi)_\varphi - (y_n+N_\varphi, g_\varphi)_\varphi - (g_\varphi, y_n+N_\varphi)_\varphi + (y_n+N_\varphi, y_n+N_\varphi)_\varphi \\ &= \|g_\varphi - (y_n+N_\varphi)\|^2_\varphi,\end{aligned}$$

$\varphi(\langle g-\widehat{y}_n, g-\widehat{y}_n\rangle) \to 0$, therefore

$$\varphi(F_0(g)^* F_0(g)) = 0. \tag{1}$$

Since the equality (1) is true for any normal functional $\varphi \in P$, we obtain that $F_0(g) = 0$. □

3.3 Hilbert W^*-modules and dual Banach spaces

Proposition 3.3.1 *Let $\mathcal{M}, \mathcal{N}$ be Hilbert C^*-modules over a W^*-algebra $\mathcal{A}$, $T : \mathcal{M} \longrightarrow \mathcal{N}$ be a bounded operator, $T \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$. Then there exists a unique extension of the operator T up to an operator $\widetilde{T} : \mathcal{M}' \longrightarrow \mathcal{N}'$.*

Proof: Let us define an operator $T^\star : \mathcal{N} \longrightarrow \mathcal{M}'$ by the equality $(T^\# y)(x) := \langle y, Tx\rangle$, $x \in \mathcal{M}$, $y \in \mathcal{N}$. Since $\left\|(T^\# y)(x)\right\| \le \|T\|\,\|x\|\,\|y\|$, the operator $T^\#$ is bounded, $\left\|T^\# y\right\| \le \|T\|\,\|y\|$. For any $a \in \mathcal{A}$

$$(T^\#(y\cdot a))(x) = \langle y\cdot a, Tx\rangle = a^*\langle y, Tx\rangle = ((T^\# y)\cdot a)(x),$$

Therefore, the map $T^{\#}$ is $\mathcal{A}$-linear. Let us define a map $\widetilde{T}:\mathcal{M}'\longrightarrow\mathcal{N}'$ by the equality $(\widetilde{T}f)(y)=\langle f,T^{\#}y\rangle$ for $y\in\mathcal{N}$, $f\in\mathcal{M}'$. Since $\widetilde{T}=(T^{\#})^{\#}$, the map $\widetilde{T}$ is also a bounded $\mathcal{A}$-module map. The equality $(\widetilde{T}\widehat{x})(y)=\widehat{(Tx)}(y)$ demonstrates that the operator $\widetilde{T}$ is an extension of the operator T.

Let us show uniqueness of this extension. Let $S:\mathcal{M}'\longrightarrow\mathcal{N}'$ be a bounded $\mathcal{A}$-module map coinciding with $\widetilde{T}$ on the submodule $\widehat{\mathcal{M}}=\{\widehat{x}:x\in\mathcal{M}\}\subset\mathcal{M}'$. Then their difference $V=\widetilde{T}-S$ vanishes on $\widehat{\mathcal{M}}$. Since the module $\mathcal{M}'$ is self-dual, the operator V has an adjoint operator $V^*:\mathcal{N}'\longrightarrow\mathcal{M}'$. If $g\in\mathcal{N}'$, $x\in\mathcal{M}$ then

$$(V^*g)(x)=\langle V^*g,\widehat{x}\rangle=\langle g,V\widehat{x}\rangle=0,$$

i. e. $V^*=0$, therefore $V=0$, hence $S=\widetilde{T}$. $\square$

Corollary 3.3.2 *Let $\mathcal{M}$ be a Hilbert $\mathcal{A}$-module. Then the map $T\longmapsto\widetilde{T}$ defines a monomorphism* $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})\subseteq\mathrm{End}^*_{\mathcal{A}}(\mathcal{M}')=\mathrm{End}_{\mathcal{A}}(\mathcal{M}')$. $\square$

Let us show that self-dual Hilbert W^*-modules are dual Banach spaces, as well as the C^*-algebras of operators acting on them.

Proposition 3.3.3 ([52]) *Let $\mathcal{M}$ be a self-dual Hilbert W^*-module. Then $\mathcal{M}$ is a dual Banach space.*

Proof: Let us introduce the denotation $\mathcal{M}^o$ for the Hilbert module $\mathcal{M}$ considered as a Banach space with multiplication by scalars given by the formula $\lambda\cdot x:=\bar{l}x$, $x\in\mathcal{M}^o$. Let us consider an algebraic tensor product $\mathcal{A}_*\otimes\mathcal{M}^o$ over the field $\mathbf{C}$, where $\mathcal{A}_*$ is a pre-dual space of normal functionals on $\mathcal{A}$. Let us equip the space $\mathcal{A}_*\otimes\mathcal{M}^o$ by the maximal cross-norm and for $x\in\mathcal{M}$ let us define a linear functional $\breve{x}$ on $\mathcal{A}_*\otimes\mathcal{M}^o$ by the formula

$$\breve{x}\left(\sum_{i=1}^n\varphi_i\otimes y_i\right)=\sum_{i=1}^n\varphi_i(\langle y_i,x\rangle),$$

where $\varphi_1,\dots,\varphi_n\in\mathcal{A}_*$, $y_1,\dots,y_n\in\mathcal{M}^o$. This functional is well-defined. Since

$$\left|\breve{x}\left(\sum_{i=1}^n\varphi_i\otimes y_i\right)\right|\le\|x\|\sum_{i=1}^n\|\varphi_i\|\,\|y_i\|\,,$$

it follows from the definition of the maximal cross-norm [3] that $\|\breve{x}\|\le\|x\|$. Let us show that actually $\|\breve{x}\|=\|x\|$. Let $\{\psi_n\}$ be a sequence of functionals of the norm 1 in $\mathcal{A}_*$ such that $|\psi_n(\langle x,x\rangle)|\to\|x\|^2$. For each element of the form $\psi_n\otimes x\in\mathcal{A}_*\otimes\mathcal{M}^o$ we have $\|\psi_n\|\,\|x\|=\|x\|$ and $|\breve{x}(\psi_n\otimes x)|\to\|x\|^2$, therefore $\|x\|\le\|\breve{x}\|$. Hence it is shown that the map $x\longmapsto\breve{x}$ defines an isometric inclusion $\mathcal{M}\subset(\mathcal{A}_*\otimes\mathcal{M}^o)^*$. To prove the statement it is sufficient to demonstrate that the set $\breve{\mathcal{M}}=\{\breve{x}:x\in\mathcal{M}\}$ is closed in $(\mathcal{A}_*\otimes\mathcal{M}^o)^*$ with respect to the weak* topology, because it would mean that $\mathcal{M}$ is isomeric to the dual space of some quotient space of $\mathcal{A}_*\otimes\mathcal{M}^o$. Let $\{\breve{x}_\alpha\}$ be a net in $\mathcal{M}$, converging to some element $F\in(\mathcal{A}_*\otimes\mathcal{M}^o)^*$ with respect to the weak* topology. For $y\in\mathcal{M}$ let us define a linear functional on $\mathcal{A}_*$ by the formula $\Phi_y(\psi)=F(\psi\otimes y)$, where $\psi\in\mathcal{A}_*$. The functional Φ is bounded, $\|\Phi\|\le\|F\|\,\|y\|$, therefore there exists a unique element $f(y)\in\mathcal{A}$ satisfying the properties $\|f(y)\|\le\|F\|\,\|y\|$ and $F(\psi\otimes y)=\psi(f(y)^*)$ for all $\psi\in\mathcal{A}_*$. The map f is linear. Let us show that it is $\mathcal{A}$-linear as well. Let $y\in\mathcal{M}$, $a,b\in\mathcal{A}$, $\varphi\in\mathcal{A}_*$. Let us define a normal functional $\psi\in\mathcal{A}_*$ by the equality $\psi(b)=\varphi(a^*b)$. Then it follows from the equalities

$$\begin{aligned}\varphi(f(y\cdot a)^*)&=F(\varphi\otimes(y\cdot a))=\lim_\alpha\breve{x}_\alpha(\varphi\otimes(y\cdot a))=\lim_\alpha\varphi(\langle y\cdot a,x_\alpha\rangle)\\&=\lim_\alpha\varphi(a^*\langle y,x_\alpha\rangle)=\lim_\alpha\psi(\langle y,x_\alpha\rangle)=F(\psi\otimes y)\\&=\psi(f(y)^*)=\varphi(a^*f(y)^*)=\varphi((f(y)a)^*),\end{aligned}$$

which hold for any $\varphi\in\mathcal{A}$, that $f(y\cdot a)=f(y)a$. Since the module $\mathcal{M}$ is self-dual, we can find an element $x_0\in\mathcal{M}$ such that $f(y)=\langle x_0,y\rangle$, therefore $F=\breve{x}_0$, hence $\breve{\mathcal{M}}$ is closed in $(\mathcal{A}_*\otimes\mathcal{M}^o)^*$. $\square$

Consider the weak* topology on the dual Banach space $\mathcal{M}$. Obviously a net $\{x_\alpha\}$ in $\mathcal{M}$ converges to an element $x\in\mathcal{M}$ with respect to this topology iff $\varphi(\langle y,x_\alpha\rangle)\longrightarrow\varphi(\langle y,x\rangle)$ for every $\varphi\in\mathcal{A}_*$ and for every $y\in\mathcal{M}$.

Some modification of the previous reasoning allows to obtain also the following

Proposition 3.3.4 ([52]) *Let $\mathcal{M}$ be a self-dual Hilbert W^*-module. Then the C^*-algebra $\mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$ is a W^*-algebra.* $\square$

3.4 Properties of Hilbert W^*-modules

The elements of self-dual Hilbert W^*-modules admit the following convenient representation (an analog of polar decomposition).

Proposition 3.4.1 ([52]) *Let $\mathcal{M}$ be a self-dual Hilbert W^*-module. Any element $x \in \mathcal{M}$ can be represented as $x = z \cdot \langle x, x\rangle^{1/2}$, where $z \in \mathcal{M}$ is such that $\langle z, z\rangle$ is the projection onto the image of $\langle x, x\rangle^{1/2}$. Such a decomposition is unique in the sense that if $x = z' \cdot a$, where $a \geq 0$, and if $\langle z', z'\rangle$ is the projection onto the image of a then $z' = z$ and $a = \langle x, x\rangle^{1/2}$.*

Proof: For $x \in \mathcal{M}$, $n \in \mathbf{N}$ let us put

$$a_n = (\langle x, x\rangle + 1/n)^{1/2}, \qquad x_n = x \cdot a_n^{-1}.$$

Since $\langle x_n, x_n\rangle = \langle x, x\rangle(\langle x, x\rangle + 1/n)^{-1}$, $\|x_n\| \leq 1$. Let $y \in \mathcal{M}$ be a point of accumulation of the sequence $\{x_n\}$ in the weak* topology (which exists due to compactness of the unit ball). Since $\left\|a_n - \langle x, x\rangle^{1/2}\right\| \to 0$ and $x_n \cdot a_n = x$, then $x = y \cdot \langle x, x\rangle^{1/2}$. Let p be the projection onto the image of $\langle x, x\rangle^{1/2}$. Then

$$p\langle x, x\rangle^{1/2} = \langle x, x\rangle^{1/2} p = \langle x, x\rangle^{1/2},$$

therefore $x = y \cdot p\langle x, x\rangle^{1/2}$ and

$$\langle x, x\rangle = \langle x, x\rangle^{1/2} p\langle y, y\rangle p\langle x, x\rangle^{1/2}.$$

Hence,

$$\langle x, x\rangle^{1/2}(p - p\langle y, y\rangle p)\langle x, x\rangle^{1/2} = 0.$$

Since $\|y\| \leq 1$, we have $p - p\langle y, y\rangle p \geq 0$, therefore

$$\langle x, x\rangle^{1/2}(p - p\langle y, y\rangle p)^{1/2} = 0,$$

whence it follows that $p(p - p\langle y, y\rangle p)^{1/2} = 0$, hence $p = p\langle y, y\rangle p$. Let us put $z = y \cdot p$. Then $z \cdot \langle x, x\rangle^{1/2} = y \cdot p\langle x, x\rangle^{1/2} = x$, $\langle z, z\rangle = p\langle y, y\rangle p = p$ and $z \cdot p = z$.

To prove the uniqueness of the decomposition suppose that $x = z' \cdot a$, where $a \geq 0$, and that $\langle z', z'\rangle$ is the projection onto the image of a. Then $\langle x, x\rangle = a\langle z', z'\rangle a = a^2$, therefore $a = \langle x, x\rangle^{1/2}$ and $\langle z', z'\rangle = p$. Since $\langle z' - z' \cdot p, z' - z' \cdot p\rangle = 0$, we obtain $z' = z'p$. Also one has

$$\langle z, x\rangle = \langle x, x\rangle^{1/2} = \langle z, z'\rangle\langle x, x\rangle^{1/2},$$

i.e. $(p - \langle z, z'\rangle)\langle x, x\rangle^{1/2} = 0$, whence we obtain that $(p - \langle z, z'\rangle)p = p - \langle z, z' \cdot p\rangle = p - \langle z, z'\rangle = 0$. Now it can be easily seen that $\langle z - z', z - z'\rangle = 0$, hence $z' = z$, and it completes the proof. $\square$

Let $\{p_\alpha\}$ be some set of projections in a W^*-algebra $\mathcal{A}$. For each of them the set $\mathcal{M}_\alpha = p_\alpha\mathcal{A} \subset \mathcal{A}$ has a natural structure of one-generated projective Hilbert $\mathcal{A}$-module. Similarly to the definition of the standard Hilbert module we can define the module $\oplus_\alpha\mathcal{M}_\alpha$ as the set of sequences (m_α), $m_\alpha \in \mathcal{M}_\alpha \subset \mathcal{A}$ such that the series $\sum_\alpha m_\alpha^* m_\alpha$ converges with respect to the norm in $\mathcal{A}$. The dual Hilbert module $(\oplus_\alpha\mathcal{M}_\alpha)'$ is called an *ultra weak direct sum* of the modules $\mathcal{M}_\alpha$. For self-dual Hilbert W^*-modules we have the following structural

Theorem 3.4.2 ([52]) *Let $\mathcal{M}$ be a self-dual Hilbert W^*-module over $\mathcal{A}$. Then there exists a set $\{p_\alpha\}$ of projections in $\mathcal{A}$ such that the module $\mathcal{M}$ is isomorphic to the ultra weak direct sum of the modules $p_\alpha\mathcal{A}$.* $\square$

Proposition 3.4.3 *Let $\mathcal{N} \subset H_\mathcal{A}$ be a Hilbert submodule over W^*-algebra $\mathcal{A}$. If $\mathcal{N}^\perp = 0$ then the dual module $\mathcal{N}'$ coincides with $H'_\mathcal{A}$.*

Proof: Let $j : \mathcal{N} \longrightarrow H_{\mathcal{A}}$ be an inclusion of modules. The restriction of functionals $f \longmapsto f|_{\mathcal{N}}$, $f \in H'_{\mathcal{A}}$, defines a map $j' : H'_{\mathcal{A}} \longrightarrow \mathcal{N}'$ dual to j. If $f \in H'_{\mathcal{A}}$ is such that $f|_{\mathcal{N}} = 0$ then $f \perp \mathcal{N}$, and by assumption one has $f = 0$, therefore the map j' is injective. Let us consider the composition of maps

$$i = j' \circ \widehat{\ } \circ j : \mathcal{N} \hookrightarrow H_{\mathcal{A}} \hookrightarrow H'_{\mathcal{A}} \longrightarrow \mathcal{N}'.$$

If $n \in \mathcal{N}$ then $i(n) = j'(\widehat{j(n)}) = \widehat{j(n)}|_{\mathcal{N}} = \widehat{n}$, therefore the map i coincides with the inclusion map $\widehat{\ } : \mathcal{N} \hookrightarrow \mathcal{N}'$. The dual map (after the identification of the first and second dual modules)

$$(i' \circ i)' = i' \circ i'' : \mathcal{N}' = \mathcal{N}'' \longrightarrow H'_{\mathcal{A}} \longrightarrow \mathcal{N}'$$

is an isomorphism, therefore the map i' should be surjective, therefore the map j' is surjective. $\square$

Proposition 3.4.4 *Let $\mathcal{A}$ be a W^*-algebra, $\mathcal{R} \subset H_{\mathcal{A}}$ be an $\mathcal{A}$-submodule without orthogonal complement, i.e. $\mathcal{R}^{\perp} = 0$ in $H_{\mathcal{A}}$. Then $\mathcal{R}' = H'_{\mathcal{A}}$.*

Proof: It is sufficient to demonstrate that if orthogonal complement to a submodule $\mathcal{R}$ in $H_{\mathcal{A}}$ is equal to zero then the orthogonal complement to $\mathcal{R}$ in the module $H'_{\mathcal{A}}$ is equal to zero too. Let us assume the contrary. Suppose that it is possible to find a functional $f \in H'_{\mathcal{A}}$ such that $f(r) = \langle f, r \rangle \neq 0$ for some $r \in \mathcal{R}$. But the series $\sum_{i=1}^{\infty} f_i^* r_i$ is norm convergent in $\mathcal{A}$, therefore there is a number n such that $f^{(n)}(z) \neq 0$ for $f^{(n)} = (f_1, \ldots, f_n, 0, \ldots)$. But, as $f^{(n)} \in H_{\mathcal{A}}$, so we get a contradiction. $\square$

3.5 Topological characterization of self-dual Hilbert W^*-modules

Let $\mathcal{A}$ be a W^*-algebra, $\mathcal{M}$ be a Hilbert $\mathcal{A}$-module, $P \in \mathcal{A}_*$ be the set of normal states on $\mathcal{A}$. Let us define (see [22]) two topologies on $\mathcal{M}$ with the help of sets of seminorms. A topology given by the system of seminorms $\varphi(\langle \cdot, \cdot \rangle)^{1/2}$, $\varphi \in P$, we denote by τ_1, and a topology given by the system of seminorms $\varphi(\langle y, \cdot \rangle)$, $y \in \mathcal{M}$, $\varphi \in P$, we denote by τ_2. In the case, when $\mathcal{A} = \mathbf{C}$ and $\mathcal{M}$ is a Hilbert space, the topology τ_1 is the norm topology and the topology τ_2 coincides with the weak topology, therefore in general these two topologies do not coincide.

Theorem 3.5.1 ([22]) *Let $\mathcal{M}$ be a Hilbert W^*-module. Then the following conditions are equivalent*

(i) *the module $\mathcal{M}$ is self-dual;*

(ii) *the unit ball $B_1(\mathcal{M})$ is complete with respect to the topology τ_1;*

(iii) *the unit ball $B_1(\mathcal{M})$ is complete with respect to the topology τ_2.*

Proof: Let us prove the implication (i)$\Rightarrow$ (ii). Assume for this purpose that the unit ball $B_1(\mathcal{M})$ is not complete with respect to the topology τ_1. Let us denote by L the linear span of the completion of $B_1(\mathcal{M})$ with respect to the topology τ_1. For the extensions of seminorms from $\mathcal{M}$ to L we use the same notation. By the assumption there exists an element $r \in L \setminus \mathcal{M}$ and a net $\{y_\alpha\}$, $\alpha \in \Lambda$, bounded with respect to the norm, such that for any $\varphi \in P$ and for any $\varepsilon > 0$ there exists some $\alpha \in \Lambda$, for which $\varphi(\langle r - y_\beta, r - y_\beta \rangle) < \varepsilon$ for all $\beta \in \Lambda$, $\beta \geq \alpha$. For arbitrary $x \in \mathcal{M}$ we have

$$|\varphi(\langle y_\beta, x \rangle) - \varphi(\langle y_\gamma, x \rangle)| = |\varphi(\langle y_\beta - y_\gamma, x \rangle)| \leq \varphi(\langle x, x \rangle)^{1/2} \varphi(\langle y_\beta - y_\gamma, y_\beta - y_\gamma \rangle)^{1/2} \leq (2\varepsilon\, \varphi(\langle x, x \rangle))^{1/2}$$

for all $\beta, \gamma \geq \alpha$. Therefore there exists in the W^*-algebra $\mathcal{A}$ the limit (with respect to the $\sigma(\mathcal{A}, \mathcal{A}_*)$-topology)

$$R(x) = \lim_{\alpha} \langle y_\alpha, x \rangle \in \mathcal{A}$$

for each $x \in \mathcal{M}$. The inequality

$$|\varphi(\langle y_\beta, x \rangle)| \leq \|x\| \sup_{\alpha} \{\|y_\alpha\| : \alpha \in \Lambda\}$$

shows the continuity of the map $R : \mathcal{M} \longrightarrow \mathcal{A}$, $x \longmapsto R(x)$. It is obvious that the map R is an $\mathcal{A}$-module map, therefore R is a functional on $\mathcal{M}$. By assumption the module $\mathcal{M}$ is self-dual, therefore there exists an element $z \in \mathcal{M}$ such that $R(x) = \langle z, x \rangle$. Then $\lim_\alpha \langle y_\alpha, x \rangle = \langle z, x \rangle$ (where the limit is taken in

$\sigma(\mathcal{A},\mathcal{A}_*)$-topology), therefore the net $\{y_\alpha\}$ converges to the element $z \in \mathcal{M}$ in the topology τ_1 and $r = z$, — a contradiction to our assumption.

Let us prove now the implication (ii) $\Rightarrow$ (i). The extension of the inner product to the dual module $\mathcal{M}'$ we still denote by $\langle\cdot,\cdot\rangle$. It is easy to see that the ideal $\langle\mathcal{M},\mathcal{M}\rangle \in \mathcal{A}$ is norm dense in the ideal $\langle\mathcal{M}',\mathcal{M}'\rangle \in \mathcal{A}$, therefore $\langle\mathcal{M},\mathcal{M}\rangle \in \mathcal{A} = \langle\mathcal{M}',\mathcal{M}'\rangle \in \mathcal{A}$. Let us consider at first the case, when a W^*-algebra $\mathcal{A}$ is σ-unital. Then there exists an exact normal state $\psi \in \mathcal{A}_*$ (see [12], Prop. 2.5.6). Let $\{H,\pi,\xi\}$ be the cyclic representation associated with ψ. The vector $\xi \in H$ is simultaneously cyclic and separating. The linear space $\mathcal{M}$ with the inner product $(\cdot,\cdot) = \psi(\langle\cdot,\cdot\rangle)$ becomes a pre-Hilbert space and the map $\psi(f(\cdot)) : \mathcal{M}\longrightarrow\mathbf{C}$, where $f \in \mathcal{M}'$, becomes a linear functional on $\mathcal{M}$. Then one can find an element f_ψ in the completion of the space $\mathcal{M}$ with respect to the norm $\psi(\langle\cdot,\cdot\rangle)^{1/2}$ such that $(f_\psi, x) = \psi(f(x))$ for all $x \in \mathcal{M}$. It means that there exists a sequence (x_i), $x_i \in \mathcal{M}$, $i \in \mathbf{N}$, such that

$$0 = \lim_{i\to\infty}(x_i - f_\psi, x_i - f_\psi) = \lim_{i\to\infty}\psi(\langle\widehat{x}_i - f, \widehat{x}_i - f\rangle) = \lim_{i\to\infty}\left\|\left(\pi(\langle\widehat{x}_i - f, \widehat{x}_i - f\rangle)\right)^{1/2}\xi\right\|^2,$$

where by $\widehat{x}$ the image of the element x under the canonical inclusion $\mathcal{M} \subset \mathcal{M}'$ is denoted. Since the vector ξ is cyclic and separating, there exists the limit (in the $\sigma(\mathcal{A},\mathcal{A}_*)$-topology on $\mathcal{A}$) $\lim_i \langle\widehat{x}_i - f, \widehat{x}_i - f\rangle = 0$ (see [12], Lemmas 2.5.38, 2.5.39), Therefore $f \in \mathcal{M}$ and the module $\mathcal{M}$ is self-dual. Let us pass to the general case. If a W^*-algebra $\mathcal{A}$ is not σ-unital then it is possible to choose a directed set of projections $\{p_\alpha\}$, $\alpha \in \Lambda$, $p_\alpha \in \mathcal{A}$ such that for each $\alpha \in \Lambda$ the algebra $p_\alpha\mathcal{A}p_\alpha$ is a σ-unital W^*-algebra and $\lim_\alpha p_\alpha = 1$ where the limit is taken in the $\sigma(\mathcal{A},\mathcal{A}_*)$-topology. As it was proved earlier, the functional fp_α on Hilbert $p_\alpha\mathcal{A}p_\alpha$-module $p_\alpha\mathcal{M}$ is an element of the module $p_\alpha\mathcal{M}$ for all $\alpha \in \Lambda$. But then there exists the limit (in the τ_1 topology) of the net $\{fp_\alpha\}$, it belongs to $\mathcal{M}$, and is equal to f, so self-duality of the module $\mathcal{M}$ is proved.

It remains to show equivalence of conditions (ii) and (iii). If $B_1(\mathcal{M})$ is complete with respect to the topology τ_1 then $\mathcal{M}$ is self-dual, therefore it is a dual Banach space with respect to the topology τ_2 (see Prop. 3.3.3), hence $B_1(\mathcal{M})$ is complete with respect to the topology τ_2. Let us assume now that $B_1(\mathcal{M})$ is complete with respect to the topology τ_2 and $\{x_\alpha\}$, $\alpha \in \Lambda$ is a bounded Cauchy τ_1-net. For all $y \in \mathcal{M}$, $\beta,\gamma \in \Lambda$, $\varphi \in P$ we have

$$|\varphi(\langle y, x_\beta\rangle) - \varphi(\langle y, x_\gamma\rangle)|^2 \le \varphi(\langle y, y\rangle)\varphi(\langle x_\beta - x_\gamma, x_\beta - x_\gamma\rangle). \tag{2}$$

As well as earlier, L denotes the linear span of τ_1-completion of $B_1(\mathcal{M})$. There exists the limit in L (with respect to the topology τ_1) $\lim_\alpha x_\alpha = t \in L$. It follows from the inequality (2) that $\{x_\alpha\}$ is a Cauchy net with respect to the topology τ_2 as well. But as the topology τ_2 is weaker than the topology τ_1, so one has $L \subseteq \mathcal{M}$, therefore limits $\lim_i x_i$ with respect to the topologies τ_1 and τ_2 coincide and are equal to $t \in \mathcal{M}$, whence one gets the completeness of $\mathcal{M}$ with respect to the topology τ_1. $\square$

3.6 Fredholm operators over W^*-algebras

In this section we denote by $\mathcal{A}$ an arbitrary W^*-algebra. Let us prove that the properties of Fredholm operators in this case are more similar to the properties of the $\mathbf{C}$-Fredholm operators than in the general C^*-case. We present here slightly modified results of [26]

Lemma 3.6.1 *Let $\mathcal{M}$ be a self-dual Hilbert C^*-module over a W^*-algebra $\mathcal{A}$. For each closed submodule $\mathcal{N} \subseteq \mathcal{M}$ the biorthogonal set $\mathcal{N}^{\perp\perp} \subseteq \mathcal{M}$ is a Hilbert $\mathcal{A}$-submodule and a direct summand of $\mathcal{M}$, as well as its orthogonal complement $\mathcal{N}^\perp$.*

Proof: The fact, that $\mathcal{N}^{\perp\perp} \subseteq \mathcal{M}$ is an $\mathcal{A}$-submodule, is obvious by the definition of the orthogonal complement. Let us consider the inclusion $i : \mathcal{N} \to \mathcal{M}$ and its adjoint map $i^* : \mathcal{M} = \mathcal{M}' \to \mathcal{N}'$. Since dual W^*-modules are self-dual, i^* admits an adjoint, and consequently its kernel is an image of a selfadjoint projection, i. e. it has orthogonal complement in $\mathcal{M}$. But

$$i^*(m) = 0 \Leftrightarrow i^*(m)(n) = 0\,\forall n \in \mathcal{N} \Leftrightarrow \langle i(n), m\rangle = 0\,\forall n \in \mathcal{N} \Leftrightarrow m \in \mathcal{N}^\perp. \quad \square$$

The example 3.6.3 below demonstrates that the situation, which differes from described in Lemma 3.6.1, can arise, for example, for Hilbert C^*-modules over the C^*-algebra $A = C([0,1])$.

Lemma 3.6.2 *Let ϕ be a bounded module morphism of a self-dual module $\mathcal{M}$. Then the kernel* $\mathrm{Ker}(\phi)$ *of the map ϕ is a direct summand in $\mathcal{M}$ and satisfies the equality* $\mathrm{Ker}(\phi) = \mathrm{Ker}(\phi)^{\perp\perp}$.

Proof: By Propositon 3.3.4 the algebra $\mathrm{End}_{\mathcal{A}}(\mathcal{M}) = \mathrm{End}^*_{\mathcal{A}}(\mathcal{M})$ is a W^*-algebra, therefore there exists a polar decomposition in it

$$\phi = US, \quad S \geq 0, \quad U \text{ is a partial isometry, } \mathrm{Ker}\,\phi = \mathrm{Ker}\,U,$$

$$\text{exists } p = p^* = p^2, \quad U(1-p) = 0, \quad pU^*Up = p,$$

$\mathrm{Ker}\,\phi = \mathrm{Ker}\,U$ is the image of a selfadjoint projection $1-p$. $\square$

Example 3.6.3 Notice that the kernel of bounded A-linear operators on Hilbert A-modules over arbitrary C^*-algebra A is not a direct summand. For example, consider the C^*-algebra $A = C([0,1])$ of all continuous functions on the interval [0,1] as a Hilbert A-module over itself equipped with the standard inner product $\langle a, b\rangle_A = a^*b$. Define the mapping ϕ_g by the formula $\phi_g(f) = g \cdot f$ for the fixed function

$$g(x) = \begin{cases} -2x+1 & : \quad x \leq 1/2 \\ 0 & : \quad x \geq 1/2 \end{cases}$$

and for every $f \in A$. Then $\mathrm{Ker}(\phi_g)$ equals the Hilbert A-submodule and (left) ideal $\{f \in A : f(x) = 0 \text{ for } x \in [0,1/2]\}$, being not a direct summand of A, but nevertheless, it coincides with the bi-orthogonal complement to itself in A.

Corollary 3.6.4 *Let $\phi : \mathcal{M} \to \mathcal{N}$ be a bounded A-linear mapping. Then the kernel* $\mathrm{Ker}(\phi)$ *of ϕ is a direct summand of $\mathcal{M}$ and has the property* $\mathrm{Ker}(\phi) = \mathrm{Ker}(\phi)^{\perp\perp}$.

Proof: Consider the self-dual Hilbert A-module $\mathcal{L}$ formed as the direct sum $\mathcal{L} = \mathcal{M} \oplus \mathcal{N}$ equipped with the A-valued inner product $\langle .,. \rangle_{\mathcal{M}} + \langle .,. \rangle_{\mathcal{N}}$. The mapping ϕ can be identified with a bounded A-linear mapping ϕ' on $\mathcal{L}$ acting on the direct summand $\mathcal{M}$ as ϕ and on the direct summand $\mathcal{N}$ as the zero operator. Since the kernel of ϕ' is a direct summand of $\mathcal{L}$ which contains $\mathcal{N}$ by Lemma 3.6.2, its orthogonal complement is a direct summand of $\mathcal{M}$. $\square$

Example 3.6.5 Let $\mathcal{A}$ be the set of all bounded linear operators $\mathcal{B}(H)$ on a separable Hilbert space H with the basis $\{e_i : i \in \mathbf{N}\}$. Denote by k the operator $k(e_i) = \lambda_i e_i$ for a sequence $\{\lambda_i : i \in \mathbf{N}\} \in c_o(\mathbf{R})$. Then the mapping $\phi_k : \mathcal{A} \to \mathcal{A}$, $\phi_k : a \to a \cdot k$ is a bounded $\mathcal{A}$-linear mapping on the left projective Hilbert $\mathcal{A}$-module $\mathcal{A}$. But the image is not a direct summand of this $\mathcal{A}$-module and is not even Hilbert because direct summands of $\mathcal{A}$ are of the form $\mathcal{A}p$ for some projection p of $\mathcal{A}$, and $1_{\mathcal{A}} \cdot k$ should equal p. The image of ϕ_k is a subset of the set of all compact operators on H. Notice that the mapping ϕ_k is not injective.

The following statement under some restrictions can be proved in the C^*-case as well [38, 41].

Proposition 3.6.6 *Let $\mathcal{M}$ be a self-dual Hilbert module and $\{\mathcal{N}, \langle .,. \rangle\}$ be arbitrary. Suppose there exists an injective bounded module mapping $\alpha : \mathcal{M} \to \mathcal{N}$ with the range property $\alpha(\mathcal{M})^{\perp\perp} = \mathcal{N}$. Then the operator $\alpha(\alpha^*\alpha)^{-1/2}$ is a bounded module isomorphism of $\mathcal{M}$ and $\mathcal{N}$. In particular, they are isomorphic as Hilbert $\mathcal{A}$-modules.*

Proof: The mapping α possesses an adjoint bounded module mapping $\alpha^* : \mathcal{N} \to \mathcal{M}$ due to self-duality of $\mathcal{M}$. As $\alpha^*\alpha$ is a positive element of the C^*-algebra $\mathrm{End}_A(\mathcal{M})$ of all bounded (adjointable) module mappings on the Hilbert $\mathcal{A}$-module $\mathcal{M}$, so its square root $(\alpha^*\alpha)^{1/2}$ is well-defined by the series

$$(\alpha^*\alpha)^{1/2} = \|.\| - \lim_{n\to\infty} \|(\alpha^*\alpha)\|^{1/2} \left(\mathrm{id}_{\mathcal{M}} - \sum_{k=1}^{n} \lambda_k \left(\mathrm{id}_{\mathcal{M}} - \frac{(\alpha^*\alpha)}{\|(\alpha^*\alpha)\|} \right)^k \right)$$

with coefficients $\{\lambda_k\}$ taken from the Taylor series at zero of the complex-valued function $f(x) = \sqrt{1-x}$ on the interval [0,1]. Moreover, because

$$\langle (\alpha^*\alpha)^{1/2}(x), (\alpha^*\alpha)^{1/2}(x) \rangle = \langle \alpha(x), \alpha(x) \rangle$$

and due to injectivity of α the mapping $(\alpha^*\alpha)^{1/2}$ has trivial kernel. Notice that the range of $(\alpha^*\alpha)^{1/2}$ is τ_1-dense in $\mathcal{M}$. Indeed, for every $\mathcal{A}$-linear bounded functional $r(\cdot) = \langle ., y\rangle$ on the self-dual Hilbert $\mathcal{A}$-module $\mathcal{M}$ mapping the range of $(\alpha^*\alpha)^{1/2}$ into the origin one has

$$0 = \langle (\alpha^*\alpha)^{1/2}(x), y\rangle = \langle x, (\alpha^*\alpha)^{1/2}(y)\rangle$$

for every $x \in \mathcal{M}$. Hence $y = 0$ as $(\alpha^*\alpha)^{1/2}$ is injective and $x \in \mathcal{M}$ was arbitrarily chosen.
Now consider the mapping $\alpha(\alpha^*\alpha)^{-1/2}$ defined on $\mathcal{M}$. Since $(\alpha^*\alpha)^{1/2}$ has both τ_1-dense range and trivial kernel by the assumptions on α, its inverse unbounded module operator $(\alpha^*\alpha)^{-1/2}$ is τ_1-densely defined. One gets

$$\langle \alpha(\alpha^*\alpha)^{-1/2}(x), \alpha(\alpha^*\alpha)^{-1/2}(y)\rangle = \langle x, y\rangle$$

for every x, y from the (τ_1-dense) domain of $(\alpha^*\alpha)^{-1/2}$. Consequently the operator $\alpha(\alpha^*\alpha)^{-1/2}$ can be extended to a bounded isometric module operator on $\mathcal{M}$ by τ_1-continuity. Its range is τ_1-closed (i.e. a self-dual direct summand of $\mathcal{N}$), hence it equals $\mathcal{N}$ by assumption. $\square$

Corollary 3.6.7 *Let $\mathcal{M}$ be a self-dual Hilbert module and $\{\mathcal{N}, \langle .,. \rangle\}$ be arbitrary. Every injective module mapping from $\mathcal{M}$ into $\mathcal{N}$ is a Hilbert $\mathcal{A}$-module isomorphism of $\mathcal{M}$ and of a direct summand of $\mathcal{N}$.*

Proposition 3.6.8 *Let $\mathcal{M}$ and $\mathcal{N}$ be countably generated Hilbert modules and $F : \mathcal{M} \to \mathcal{N}$ be a Fredholm operator. Then* $\operatorname{Ker} F$ *and* $(\operatorname{Im} F)^\perp$ *are projective finitely generated $\mathcal{A}$-submodules, and* $\operatorname{index} F = [\operatorname{Ker} F] - [(\operatorname{Im} F)^\perp]$ *in $K_0(\mathcal{A})$.*

Proof: Let $\mathcal{M} = \mathcal{M}_0 \widehat{\bigoplus} \mathcal{M}_1$, $\mathcal{N} = \mathcal{N}_0 \oplus \mathcal{N}_1$ be the decompositions from the definition of $\mathcal{A}$-Fredholm operator:

$$F = \begin{pmatrix} F_0 & 0 \\ 0 & F_1 \end{pmatrix} : \mathcal{M}_0 \widehat{\bigoplus} \mathcal{M}_1 \to \mathcal{N}_0 \oplus \mathcal{N}_1,$$

$F_0 : \mathcal{M}_0 \cong \mathcal{N}_0$, $F_1 : \mathcal{M}_1 \to \mathcal{N}_1$, $\mathcal{M}_1$ and $\mathcal{N}_1$ are the projective finitely generated modules. Let $x = x_0 + x_1$, $x_0 \in \mathcal{M}_0$, $x_1 \in \mathcal{M}_1$ and $F(x) = 0$, so $0 = F_0(x_0) + F_1(x_1) \in \mathcal{N}_0 \oplus \mathcal{N}_1$. Thus $F_0(x_0) = 0$, $F_1(x_1) = 0$, so $x_0 = 0$ and $x \in \mathcal{M}_1$. Thus $\operatorname{Ker} F = \operatorname{Ker} F_1 \subset \mathcal{M}_1$. By Lemma 3.6.2 $\operatorname{Ker} F$ is a projective finitely generated $\mathcal{A}$-module and has an orthogonal complement. So, by Corollary 3.6.7

$$F = \begin{pmatrix} F_0 & 0 & 0 \\ 0 & F_1' & 0 \\ 0 & 0 & 0 \end{pmatrix} : \mathcal{M}_0 \widehat{\bigoplus} \mathcal{M}_1' \oplus \operatorname{Ker} F \to \left(\mathcal{N}_0 \oplus \overline{F(\mathcal{M}_1')}\right) \widehat{\bigoplus} (\operatorname{Im} F)^\perp$$

and $\operatorname{index} F = [\operatorname{Ker} F] - [(\operatorname{Im} F)^\perp]$. $\square$

The following example shows that the situations may be quite different for general Hilbert C^*-modules and injective mappings between them:
Example 3.6.9 Consider the C^*-algebra $A = C([0,1])$ of all continuous functions on the interval [0,1] as a self-dual Hilbert A-module over itself equipped with the standard A-valued inner product $\langle a, b\rangle_A = a^*b$. The mapping $\phi : f(x) \to x \cdot f(x)$, $(x \in [0,1])$, is an injective bounded module mapping. Its range has trivial orthogonal complement, but it is not norm closed and, consequently, not a direct summand of A. Nevertheless, the bi-orthogonal complement of the range of ϕ with respect to A equals A.

Lemma 3.6.10 *Let $\mathcal{P}$ and $\mathcal{Q}$ be self-dual Hilbert $\mathcal{A}$-submodules of $\mathcal{M}$. Then $\mathcal{P} \cap \mathcal{Q}$ is a self-dual Hilbert $\mathcal{A}$-module and a direct summand of $\mathcal{M}$. Moreover, $\mathcal{P} + \mathcal{Q} \subseteq \mathcal{M}$ is a self-dual Hilbert $\mathcal{A}$-submodule.*
If $\mathcal{P}$ is projective and finitely generated then the intersection $\mathcal{P} \cap \mathcal{Q}$ is projective and finitely generated too. If both $\mathcal{P}$ and $\mathcal{Q}$ are projective and finitely generated then the sum $\mathcal{P} + \mathcal{Q}$ is projective and finitely generated too.

Proof: Let $p : \mathcal{M} = \mathcal{P} \oplus \mathcal{P}^\perp \to \mathcal{P}^\perp$ be the canonical orthogonal projection existing by Proposition 2.5.4. Let $p_Q = p : \mathcal{Q} \to \mathcal{P}^\perp$. Since $\mathcal{Q}$ is a self-dual Hilbert $\mathcal{A}$-module p_Q admits an adjoint operator and $\operatorname{Ker} p_Q \subseteq \mathcal{Q}$ is a direct summand by Lemma 3.6.2. Consequently it is a self-dual Hilbert $\mathcal{A}$-submodule of $\mathcal{Q} \subseteq \mathcal{M}$. But $\operatorname{Ker} p_Q = \mathcal{P} \cap \mathcal{Q}$. To obtain the second assertion one has to apply again the fact that every self-dual Hilbert $\mathcal{A}$-submodule is a direct summand by Proposition 2.5.4.
If $\mathcal{P}$ is projective and finitely generated then every its direct summand is projective and finitely generated, what proves the remaining assertion. $\square$

For C^*-algebras it is possible to prove the following analog of Lemma 3.6.2.

Proposition 3.6.11 *Let A be a C^*-algebra, $\mathcal{M}$ and $\mathcal{N}$ be self-dual Hilbert A-modules, $\phi : \mathcal{M} \to \mathcal{N}$ be a bounded A-linear mapping. Then the kernel* $\mathrm{Ker}(\phi)$ *of ϕ coincides with its bi-orthogonal complement inside $\mathcal{M}$. In general, it is not a direct summand.*

Proof: Let us assume that $\mathrm{Ker}(\phi) \neq \mathrm{Ker}(\phi)^{\perp\perp}$ with respect to the A-valued inner product on $\mathcal{M}$. Form the direct sum $\mathcal{L} = \mathcal{M} \oplus \mathcal{N}$. The mapping ϕ can be extended to a bounded A-linear mapping ψ on $\mathcal{L}$ if we set

$$\psi(x) = \begin{cases} \phi(x) & : \quad x \in \mathcal{M} \\ 0 & : \quad x \in \mathcal{N} \end{cases} .$$

Extend ψ further to a bounded A^{**}-linear operator on the corresponding Hilbert A^{**}-module $\mathcal{L}^{\#}$. By Lemma 3.6.2 both sets $\mathrm{Ker}(\phi)^{\#}$ and $(\mathrm{Ker}(\phi)^{\perp\perp})^{\#}$ are contained in the kernel $\mathrm{Ker}(\psi)$ of ψ, which is a direct summand of $\mathcal{L}^{\#}$ and $\mathrm{Ker}(\psi) = \mathrm{Ker}(\psi)^{\perp\perp}$ holds. This contradicts the assumption. The second assertion follows from Example 3.6.3. $\square$

4 Reflexive Hilbert C^*-modules

4.1 Inner product on bidual modules

For Hilbert C^*-module $\mathcal{M}$ over C^*-algebra A we shall define the *bidual* Banach right A-module $\mathcal{M}''$ as a set of bounded A-homomorphisms from the dual module $\mathcal{M}'$ into A. It turns out that an inner product on $\mathcal{M}$ can be extended to the bidual module for C^*-algebra A unlike the dual module, which admits an extension of an inner product only in the case of W^*-algebras.

Let $x \in \mathcal{M}$, $f \in \mathcal{M}'$. Put

$$\dot{x}(f) := f(x)^*.$$

The map $x \longmapsto \dot{x}$ is an isometric map from the A-module $\mathcal{M}$ into the A-module $\mathcal{M}''$:

$$\begin{aligned} \|\dot{x}\| &= \sup\{\|f(x)\| : f \in \mathcal{M}',\ \|f\| \leq 1\} \leq \|f\|\,\|x\| \leq \|x\|\,; \\ \|\dot{x}\| &\geq \frac{1}{\|\widehat{x}\|}\,\|\widehat{x}(x)\| = \frac{1}{\|\widehat{x}\|}\,\|\langle x, x\rangle\| = \|x\|\,. \end{aligned}$$

For a functional $F \in \mathcal{M}''$ we define a functional $\widetilde{F} \in \mathcal{M}'$ by the formula

$$\widetilde{F}(x) := F(\widehat{x}).$$

Identifying $\mathcal{M}$ and $\widehat{\mathcal{M}} = \{\widehat{x} : x \in \mathcal{M}\} \subset \mathcal{M}'$ we obtain that $\widetilde{F}$ is the restriction of F to $\mathcal{M} \subset \mathcal{M}'$. Remark that $(\dot{x})\widetilde{\ }= \widehat{x}$ for all $x \in \mathcal{M}$. It is clear that the map $F \longmapsto \widetilde{F}$ is an A-module map from $\mathcal{M}''$ to $\mathcal{M}'$ and $\left\|\widetilde{F}\right\| \leq \|F\|$. We will check soon that this map is an isometry.

Let us define an inner product $\langle \cdot, \cdot \rangle : \mathcal{M}'' \times \mathcal{M}'' \longrightarrow A$ by the equality

$$\langle F, G\rangle := F(\widetilde{G}), \qquad F, G \in \mathcal{M}''. \tag{1}$$

It can be directly checked that $\langle F \cdot a, G\rangle = a^* \langle F, G\rangle$ for $a \in A$. Besides, for $x, y \in \mathcal{M}$ one has

$$\langle \dot{x}, \dot{y}\rangle = \dot{x}((\dot{y})\widetilde{\ }) = \dot{x}(\widehat{y}) = (\widehat{y}(x))^* = \langle y, x\rangle^* = \langle x, y\rangle,$$

therefore the inner product defined by equality (1) is an extension of the inner product on $\mathcal{M}$. To check out the properties of an inner product we need the following construction.

Consider the right A-module $A \times \mathcal{M}$. Besides the natural inner product $\langle \cdot, \cdot \rangle_0$, defined by the formula $\langle (A, x), (b, y)\rangle_0 = a^* b + \langle x, y\rangle$, where $a, b \in A$, $x, y \in \mathcal{M}$, we consider another inner product on the module $A \times \mathcal{M}$. Let us take $f \in \mathcal{M}'$, $f \neq 0$, and a number $t > \|f\|$ and put

$$\langle (a, x), (b, y)\rangle_{f,t} := t^2 a^* b + a^* f(y) + f(x)^* b + \langle x, y\rangle. \tag{2}$$

Properties (iii) and (iv) of the definition 1.2.1 hold obviously. Let us check the properties (i) and (ii). The first one is valid due to the inequality

$$
\begin{aligned}
\langle (a,x),(a,x)\rangle_{f,t} &= t^2a^*a + a^*f(x) + f(x)^*a + \langle x,x\rangle \geq \\
&\geq t^2a^*a + a^*f(x) + f(x)^*a + \frac{1}{\|f\|^2} f(x)^*f(x) \qquad (3)\\
&\geq t^2a^*a + a^*f(x) + f(x)^*a + \frac{1}{t^2} f(x)^*f(x) \qquad (4)\\
&= \left(ta + \frac{1}{t}f(x)\right)^* \left(ta + \frac{1}{t}f(x)\right) \geq 0.
\end{aligned}
$$

Suppose that $\langle (a,x),(a,x)\rangle_{f,t} = 0$. Then equality should be reached at each step in (3) – (4). Subtracting the line (3) from the line (4), we obtain

$$(\|f\|^{-2} - t^{-2}) f(x)^*f(x) = 0,$$

therefore $f(x) = 0$, hence $t^2a^*a + \langle x,x\rangle = 0$, and we can conclude that $a = 0$ and $x = 0$, so we have checked validity of the property (ii). Thus, the module $A \times \mathcal{M}$ with the inner product defined by the formula (2), is a Hilbert A-module. The norm on this module corresponding to this inner product we denote by $\|\cdot\|_{f,t}$ and the Hilbert module $A\times\mathcal{M}$ equipped with this norm we denote by $(A\times\mathcal{M})_{f,t}$. Notice that $\|(0,x)\|_{f,t} = \|x\|$. For $x, y \in \mathcal{M}$, $a \in A$ we have

$$\|(f\cdot a + \widehat{x})(y)\| = \|a^*f(y) + \langle x,y\rangle\| = \left\|\langle (a,x),(0,y)\rangle_{f,t}\right\| \leq \|(a,x)\|_{f,t}\cdot\|(0,y)\|_{f,t} = \|y\|\cdot\|(a,x)\|_{f,t}\,,$$

Therefore

$$\|(f\cdot a + \widehat{x})\| \leq \|(a,x)\|_{f,t}\,. \qquad (5)$$

Proposition 4.1.1 ([53]) *Let $\mathcal{N} \subset \mathcal{M}'$ be a submodule, containing the module $\widehat{\mathcal{M}}$. Then the norm of any functional $\psi \in \mathcal{N}'$ satisfies the equality $\|\psi\| = \left\|\psi|_{\widehat{\mathcal{M}}}\right\|$.*

Proof: Without loss of generality we assume that $\|\psi\| = 1$. Define a functional $f \in \mathcal{M}'$ by the formula $f(x) := \psi(\widehat{x})$, $x \in \mathcal{M}$. Then $\|f\| \leq 1$. It is necessary to prove the inverce inequality $\|f\| \geq 1$. Take $g \in \mathcal{N}$ such that $\|g\| < 1$, and put $c = \psi(g) \in A$. For $a \in A$, $x \in \mathcal{M}$ we have

$$\|ca + f(x)\| = \|\psi(g\cdot a + \widehat{x})\| \leq \|g\cdot a + \widehat{x}\| \leq \|(a,x)\|_{g,1}$$

(the last inequality follows from (5)), hence the map

$$f_c : A\times\mathcal{M} \longrightarrow A; \qquad (a,x) \longmapsto ca + f(x)$$

is a bounded modular map, $\|f_c\|_{(A\times\mathcal{M})'_{g,1}} \leq 1$. Therefore,

$$F_c(a,x)^*f_c(a,x) \leq \langle (a,x),(a,x)\rangle_{g,1} \qquad (6)$$

for all $a \in A$, $x \in \mathcal{M}$. From the estimate (6) we get

$$A^*c^*ca + a^*c^*f(x) + f(x)^*ca + f(x)^*f(x) \leq a^*a + a^*g(x) + g(x)^*a + \langle x,x\rangle.$$

Taking $a = -2g(x)$ we obtain

$$4g(x)^*c^*cg(x) + f(x)^*f(x) \leq \langle x,x\rangle + 2(g(x)^*c^*f(x) + f(x)^*cg(x)).$$

But as

$$G(x)^*c^*f(x) + f(x)^*cg(x) \leq g(x)^*c^*cg(x) + f(x)^*f(x),$$

so

$$2g(x)^*c^*cg(x) \leq \langle x,x\rangle + f(x)^*f(x) \leq (1+\|f\|^2)\langle x,x\rangle,$$

hence, $\|g \cdot c^*\| \leq \frac{1}{\sqrt{2}}(1+\|f\|^2)^{1/2}$, and, therefore,

$$\|\psi(g \cdot c^*)\| = \|cc^*\| = \|c\|^2 \leq \frac{1}{\sqrt{2}}(1+\|f\|^2)^{1/2}.$$

The last inequality is valid for all $g \in \mathcal{N}$ for which $\|g\| < 1$, and as $\|\psi\| = 1$, so the inequality $1 \leq \frac{1}{\sqrt{2}}(1+\|f\|^2)^{1/2}$ should be valid too, whence it follows that $\|f\| \geq 1$. Thus, $\|f\| = \left\|\psi|_{\widehat{\mathcal{M}}}\right\| = 1$. $\square$

Remark that in a case when $\mathcal{N} = \mathcal{M}'$ the proposition 4.1.1 means that the map $F \longmapsto \widetilde{F}$ is an isometric inclusion $\mathcal{M}'' \subset \mathcal{M}'$.

Proposition 4.1.2 ([53]) *For all $F \in \mathcal{M}''$ one has $\langle F, F\rangle \geq 0$ and $\|F, F\rangle\| = \|F\|^2$.*

Proof: Let $F \in \mathcal{M}''$, $F \neq 0$. Put $c = F(\widetilde{F})$, $D = \|F\|$. Let us show at first that $D^2 \in \mathrm{Sp}(c)$. For $t > D$ consider the inner product $\langle \cdot, \cdot \rangle_{\widetilde{F},t}$ on the module $A \times \mathcal{M}$. Since $\left\|\widetilde{F} \cdot a + \widehat{x}\right\| \leq \|(a,x)\|_{\widetilde{F},t}$ for all $(a,x) \in A \times \mathcal{M}$, the map

$$f_c : A \times \mathcal{M} \longrightarrow A; \qquad (a,x) \longmapsto F(\widetilde{F} \cdot a + \widehat{x}) = ca + \widetilde{F}(x)$$

is bounded with a norm not exceeding D (we mean here the norm defined by the inner product $\langle \cdot, \cdot \rangle_{\widetilde{F},t}$). Therefore

$$(ca + \widetilde{F}(x))^*(ca + \widetilde{F}(x)) \leq D^2 \langle (a,x),(a,x)\rangle_{\widetilde{F},t}. \tag{7}$$

The inequality (7) holds for all $t > D$ and taking the limit $t \to D$ we obtain

$$(ca + \widetilde{F}(x))^*(ca + \widetilde{F}(x)) \leq D^2(D^2 a^* a + \widetilde{F}(x)^* a + a^* \widetilde{F}(x) + \langle x, x\rangle).$$

Taking $a = -D^{-2}\widetilde{F}(x)$ we get

$$\widetilde{F}(x)^*(D^{-2}c - 1)^*(D^{-2}c - 1)\widetilde{F}(x) \leq D^2(-D^{-2}\widetilde{F}(x)^*\widetilde{F}(x) + \langle x, x\rangle),$$

hence,

$$\widetilde{F}(x)^*((D^{-2}c - 1)^*(D^{-2}c - 1) + 1)\widetilde{F}(x) \leq D^2\langle x, x\rangle.$$

Suppose that $D^2 \notin \mathrm{Sp}(c)$. Then it is possible to find number $\delta > 0$ such that

$$\widetilde{F}(x)^*(D^{-2}c - 1)^*(D^{-2}c - 1)\widetilde{F}(x) \geq \delta \widetilde{F}(x)^*\widetilde{F}(x)$$

for all $x \in \mathcal{M}$. But then

$$\widetilde{F}(x)^*\widetilde{F}(x) \leq \frac{D^2}{1+\delta}\langle x, x\rangle,$$

whence we have

$$D^2 = \left\|\widetilde{F}\right\|^2 \leq \frac{D^2}{1+\delta} < D^2.$$

Obtained contradiction shows that $D^2 \in \mathrm{Sp}(c)$. But as

$$\|c\| = \left\|F(\widetilde{F})\right\| \leq \|F\| \left\|\widetilde{F}\right\| = \|F\|^2 = D^2,$$

so $\|c\| = D^2$, hence $\|\langle F, F\rangle\| = \|F\|^2$ and $\|\langle F, F\rangle\| \in \mathrm{Sp}(\langle F, F\rangle)$. For an arbitrary element $a \in A$ we have

$$\|aca\| = \left\|a^* F(\widetilde{F} \cdot a)\right\| = \|\langle F \cdot a, F \cdot a\rangle\| \in \mathrm{Sp}(\langle F \cdot a, F \cdot a\rangle) = \mathrm{Sp}(aca).$$

The following lemma concludes the proof.

Lemma 4.1.3 ([53]) *Let an element $c \in A$ be such that the inclusion $\|aca\| \in \mathrm{Sp}(aca)$ holds for any $a \in A$, $a \geq 0$. Then $c \geq 0$.* $\square$

The proposition 4.1.2 shows that the inner product defined on $\mathcal{M}''$ satisfies the conditions (i) and (ii) of the definition 4.1.2. It remains to check the condition (iii). Notice that

$$\langle F+G, F+G\rangle \geq 0; \qquad \langle F+iG, F+iG\rangle \geq 0,$$

hence these expressions are selfadjoint. Then

$$(\langle F,G\rangle + \langle G,F\rangle)^* = \langle F,G\rangle + \langle G,F\rangle; \qquad -i(\langle F,G\rangle - \langle G,F\rangle)^* = i(\langle F,G\rangle - \langle G,F\rangle),$$

therefore $\langle F,G\rangle = \langle G,F\rangle^*$. The module $\mathcal{M}''$ is a Hilbert A-module, as the operator norm on $\mathcal{M}''$ coincides (by the proposition 4.1.2) with the norm defined by the inner product. Thus, we have proved the following theorem.

Theorem 4.1.4 ([53]) *The map $\langle\cdot,\cdot\rangle : \mathcal{M}'' \times \mathcal{M}'' \longrightarrow A$ defined by the formula $\langle F,G\rangle = F(\widetilde{G})$, $F,G \in \mathcal{M}''$, is an A-valued inner product on $\mathcal{M}''$. The norm defined by this inner product coincides with the operator norm on $\mathcal{M}''$. The map $F \longmapsto \widetilde{F}$ is an isometric inclusion $\mathcal{M}'' \subset \mathcal{M}'$.* □

Let us pass now to dual modules of the higher order. Let $\Phi \in (\mathcal{M}'')'$. Define then a functional $f_\Phi \in \mathcal{M}'$ by the formula

$$F_\Phi(x) := \Phi(\dot{x}), \quad x \in \mathcal{M}.$$

Let further $f \in \mathcal{M}'$. Define $\Phi_f \in (\mathcal{M}'')'$ by the formula

$$\Phi_f(F) := (F(f))^*, \quad F \in \mathcal{M}''.$$

The maps $\Phi \longmapsto f_\Phi$ and $f \longmapsto \Phi_f$ are A-module morphisms. Consider their composition

$$\mathcal{M}' \longrightarrow \mathcal{M}''' \longrightarrow \mathcal{M}', \qquad F \longmapsto \Phi_f \longmapsto f_{\Phi_f}. \tag{8}$$

As for any $x \in \mathcal{M}$ we have

$$F_{\Phi_f}(x) = \Phi_f(\dot{x}) = (\dot{x}(f))^* = f(x),$$

so the composition (8) is identical map, whence it follows that the map $\mathcal{M}' \longrightarrow \mathcal{M}'''$ is an isometric inclusion and the map $\mathcal{M}''' \longrightarrow \mathcal{M}'$ is an epimorphism. Let us show, that the last map is also monomorphic. Apply for this purpose the proposition 4.1.1 for the case $\mathcal{N} = \mathcal{M}''$. Let $\Phi \in \mathcal{N}' = \mathcal{M}'''$. Then the functional $f_\Phi \in \mathcal{M}'$ is a restriction on $\widehat{\mathcal{M}}$ of the functional Φ, $f_\Phi = \Phi|_{\widehat{\mathcal{M}}}$. Suppose that $f_\Phi = 0$. Then by the proposition 4.1.1 we have $\|\Phi\| = \left\|\Phi|_{\widehat{\mathcal{M}}}\right\| = 0$, therefore the map $\mathcal{M}''' \longrightarrow \mathcal{M}'$, $\Phi \longmapsto f_\Phi$, is monomorphic. Thus this map is an isometric isomorphism.

Corollary 4.1.5 *For a Hilbert C^*-module $\mathcal{M}$ one has $(\mathcal{M}'')' = \mathcal{M}'$ and $(\mathcal{M}'')'' = \mathcal{M}''$.* □

So, the series of dual modules $\mathcal{M}$, $\mathcal{M}', \ldots$ stabilizes on the third entry and the inclusions

$$\mathcal{M} \subseteq \mathcal{M}'' = \mathcal{M}'''' \subseteq \mathcal{M}' = \mathcal{M}'''$$

are isometric. Thus the modules $\mathcal{M}$ and $\mathcal{M}''$ are Hilbert unlike the module $\mathcal{M}'$, which is, generally speaking, only Banach. Let us illustrate by examples that all possible variants can be realized:

$$\mathcal{M} = \mathcal{M}'' = \mathcal{M}'; \qquad \mathcal{M} \neq \mathcal{M}'' = \mathcal{M}'; \qquad \mathcal{M} = \mathcal{M}'' \neq \mathcal{M}'; \qquad \mathcal{M} \neq \mathcal{M}'' \neq \mathcal{M}'.$$

(i) Let A be a unital C^*-algebra and let $\mathcal{M} = L_n(A)$ be a free A-module with n generators. Then the module $\mathcal{M}$ is autodual, therefore, $\mathcal{M} = \mathcal{M}'' = \mathcal{M}'$.

(ii) Let A be a W^*-algebra. By the theorem 3.2.1 for any Hilbert A-module $\mathcal{M}$ its dual module $\mathcal{M}'$ is a self-dual Hilbert module, hence $\mathcal{M} \neq \mathcal{M}'' = \mathcal{M}'$.

(iii) [23] Let $A = C_0(0,1]$ be the C^*-algebra (without unit) of functions on a segment $[0,1]$ vanishing at zero, $\mathcal{M} = A$. Then $\mathcal{M}' = C[0,1]$, $\mathcal{M}'' = C_0(0,1]$ and $\mathcal{M} = \mathcal{M}'' \neq \mathcal{M}'$.

(iv) [23] Consider the module $C_0(0,1)$ of functions on the segment $[0,1]$ vanishing at the end points, over the C^*-algebra $A = C_0(0,1]$. In this case one has $\mathcal{M}' = C[0,1]$, $\mathcal{M}'' = C_0(0,1]$, that is, $\mathcal{M} \neq \mathcal{M}'' \neq \mathcal{M}'$.

Definition 4.1.6 A Hilbert C^*-module $\mathcal{M}$ is called *reflexive,* if $\mathcal{M}'' = \mathcal{M}$.

In the following we will encounter with other examples of reflexive Hilbert modules.

4.2 Reflexivity of Hilbert modules over $\mathcal{K}^+$

In this section describe results of the paper [63]. Let $\mathcal{K}$ be the C^*-algebra of compact operators acting on a separable Hilbert space H and let $\mathcal{K}^+$ be the C^*-algebra of operators of the form $a = \lambda + K$, where $\lambda \in \mathbf{C}$, $K \in \mathcal{K}$.

Theorem 4.2.1 ([63]) *Any countably generated Hilbert $\mathcal{K}^+$-module is reflexive.*

Proof: According to the stabilization theorem 1.4.1 any countably generated Hilbert module is a direct summand in the standard module $l_2(\mathcal{K}^+)$, therefore it is sufficient to prove reflexivity of the module $l_2(\mathcal{K}^+)$. The proposition 2.5.5 gives a description of the dual module:

$$l_2(\mathcal{K}^+)' = \{f = (f_i) : f_i \in \mathcal{K}^+, \sup_N \left\| \sum_{i=1}^{N} F_i^* f_i \right\| < \infty\}.$$

Lemma 4.2.2 *If $f = (f_i) \in l_2(\mathcal{K}^+)'$, $K \in \mathcal{K}$, $f \cdot K = (f_i K) \in l_2(\mathcal{K}^+)$.*

Proof: Since the operator K can be approximated by finite-dimensional operators, it is sufficient to prove the lemma in the case when K is finite-dimensional. Notice that the operator $(f_i K)^*(f_i K) = K^* f_i^* f_i K$ is a positive operator whose kernel contains $\operatorname{Ker} K$ and whose image is contained in $\operatorname{Im} K^*$. As $\dim \operatorname{Im} K^*$ and $\operatorname{codim} \operatorname{Ker} K$ are finite, the norm convergence of the series $\sum_{i=1}^{\infty} (f_i K)^*(f_i K)$ follows from its weak convergence. $\square$

Let $F \in l_2(\mathcal{K}^+)''$. Put $F_i = F(\widehat{e_i})^* \in \mathcal{K}^+$, where $\{e_i\}$ is the standard basis of $l_2(\mathcal{K}^+)$, $\widehat{e_i} \in l_2(\mathcal{K}^+)'$. Since $\mathcal{M}'' \subset \mathcal{M}'$ for any Hilbert module $\mathcal{M}$, the sequence $\widetilde{F} = (F_i)$ is an element of the module $l_2(\mathcal{K}^+)'$. Let us prove that the series $\sum_{i=1}^{\infty} F_i^* F_i$ converges in the C^*-algebra $\mathcal{K}^+$ to the element $F(\widetilde{F}) = \langle F, F \rangle$.

Let $K \in \mathcal{K}$ be a finite-dimensional operator in H. By the lemma 4.2.2

$$\langle F, F \rangle K = \langle F, F \cdot K \rangle = \sum_{i=1}^{\infty} F_i^* F_i K,$$

therefore,

$$K^* \langle F, F \rangle K = \sum_{i=1}^{\infty} K^* F_i^* F_i K,$$

and for any $\xi \in H$

$$\sum_{i=1}^{\infty} (K^* F_i^* F_i K \xi, \xi) = (\langle F, F \rangle K\xi, K\xi),$$

where $(\cdot,\cdot)$ is a (scalar) inner product on H. Let $\eta \in B_1(H)$, where $B_1(H)$ is the unit ball in H. Choose an element $\xi \in H$ and a finite-dimensional operator K' so that $\eta = K'\xi$. Then

$$\sum_{i=1}^{\infty} (F_i \eta, F_i \eta) = (\langle F, F \rangle \eta, \eta).$$

Therefore for any η

$$\left\| \sum_{i=1}^{\infty} (F_i^* F_i \eta, \eta) \right\| \leq \|\langle F, F \rangle\| \cdot \|\eta\|^2 . \tag{9}$$

Lemma 4.2.3 *Let $f = (k_i) \in l_2(\mathcal{K}^+)'$, and $k_i \in \mathcal{K}$ for all $i \in \mathbf{N}$. Then $F(f) \in \mathcal{K}$.*

Proof: Due to continuity of F and the closedness of the algebra $\mathcal{K}$ it is possible to assume that all k_i are finite-dimensional operators. Denote by $V_i \subset H$ the image of the operator k_i^*, $\dim V_i < \infty$. Let $H = H_1 \oplus H_2$ be an orthogonal decomposition of H into a sum of two closed infinite-dimensional subspaces. Assume at first, that each of the subspaces V_i lays in one of subspaces H_1 or H_2. Let $F(f) = K + \lambda$, $K \in \mathcal{K}$, $\lambda \in \mathbf{C}$. Choose a compact operator k with such image $L \subset H_2$ that $\dim L = \infty$. Replace by zeroes those terms in the sequence $(k_1, k_2, \ldots)$ for which $V_i \subset H_1$ and denote the obtained sequence by $(k'_1, k'_2, \ldots)$. Then $(k_1 k, k_2 k, \ldots) = (k'_1 k, k'_2 k, \ldots)$ because the condition $k_i k = 0$ is equivalent to the condition $\mathrm{Im}\, k_i^* \perp \mathrm{Im}\, k$. Thus

$$F(k'_1, k'_2 \ldots)k = F(k'_1 k, k'_2 k, \ldots) = F(k_1 k, k_2 k, \ldots) = F(k_1, k_2, \ldots)k = (K + \lambda)k,$$

i.e.

$$F(k'_1, k'_2, \ldots)|_L = (K + \lambda)|_L.$$

As $\dim L = \infty$, so $F(k'_1, k'_2, \ldots) = K' + \lambda$ with some $K' \in \mathcal{K}$. Interchanging subspaces H_1 and H_2, it is possible to construct a sequence $(k''_1, k''_2, \ldots)$ such that $F(k''_1, k''_2, \ldots) = K'' + \lambda$ with some $K'' \in \mathcal{K}$. Thus

$$(K_1, k_2, \ldots) = (k'_1, k'_2, \ldots) + (k''_1, k''_2, \ldots),$$

hence,

$$F(k_1, k_2, \ldots) = K' + K'' + 2\lambda,$$

therefore $\lambda = 0$. In case of arbitrary subspaces V_i is possible to find such finite-dimensional operators l_i, m_i, n_i that

$$K_i + l_i = m_i + n_i, \quad (l_1, l_2, \ldots) \in l_2(\mathcal{K}^+); \quad (M_1, m_2, \ldots), (n_1, n_2, \ldots) \in l_2(\mathcal{K}^+)',$$

and the image of each of operators m_i^* and n_i^* lays in one of the subspaces H_1, H_2. $\square$

Put $F_i = K_i + \lambda_i$, $\langle F, F \rangle = K + \lambda$, where $K_i, K \in \mathcal{K}$, $\lambda_i, \lambda \in \mathbf{C}$. Then

$$F_i^* F_i = K_i^* K_i + \lambda_i K_i^* + \overline{\lambda}_i K_i + |\lambda_i|^2.$$

Lemma 4.2.4 $\sum_{i=1}^{\infty} |\lambda_i|^2 = \lambda$.

Proof: At first we show that the series $\sum_{i=1}^{\infty} |\lambda_i|^2$ converges. Suppose that it is not so. Put $M = \|\langle F, F \rangle\|$ and find a number N such that $\sum_{i=1}^{N} |\lambda_i|^2 > M + 1$. Choose $\varepsilon > 0$ to satisfy the estimate

$$\sum_{i=1}^{N} (1 + 2|\lambda_i|)\varepsilon \le \frac{1}{2}.$$

Choose, further, a vector $\xi \in H$ with $\|\xi\| = 1$ to satisfy the inequalities

$$\|K_i^* K_i \xi\| \le \varepsilon, \quad \|K_i \xi\| \le \varepsilon, \quad \|K_i^* \xi\| \le \varepsilon, \quad i = 1, \ldots, N.$$

Then the inequalities

$$\sum_{i=1}^{\infty} (F_i^* F_i \xi, \xi) \ge \sum_{i=1}^{N} (F_i^* F_i \xi, \xi) \ge M + \frac{1}{2}$$

contradict (9). So, $\sum_{i=1}^{\infty} |\lambda_i|^2 < \infty$. But it means that $(\lambda_1, \lambda_2, \ldots) \in l_2(\mathcal{K}^+)$. Then

$$\begin{aligned} K + \lambda &= \langle F, F \rangle = F(F_1, F_2, \ldots) = F(K_1, K_2, \ldots) + F(\lambda_1, \lambda_2, \ldots) \\ &= F(K_1, K_2, \ldots) + \sum_{i=1}^{\infty} F_i^* \lambda_i = F(K_1, K_2, \ldots) + \sum_{i=1}^{\infty} (K_i^* + \overline{\lambda}_i)\lambda_i \\ &= \left(F(K_1, K_2, \ldots) + \sum_{i=1}^{\infty} K_i^* \lambda_i \right) + \sum_{i=1}^{\infty} |\lambda_i|^2. \end{aligned}$$

But, as $F(K_1, K_2, \ldots) + \sum_{i=1}^{\infty} K_i^* \lambda_i \in \mathcal{K}$, we conclude that $\sum_{i=1}^{\infty} |\lambda_i|^2 = \lambda$. $\square$

Lemma 4.2.5 ([63]) *Let X be a compact Hausdorff space and let f_n, f, g_n, g be real-valued functions on X, $n \in \mathbf{N}$. Assume that the functions f_n, f are continuous, that the functions $f_n + g_n$ and g_n are nonnegative, that the function g is bounded, that the series $\sum_{n=1}^{\infty} g_n$ uniformly converges to the function g, and that the series $\sum_{n=1}^{\infty} (f_n + g_n)$ converges pointwise to the function $f + g$. Then the series $\sum_{n=1}^{\infty} (f_n + g_n)$ uniformly converges to the function $f + g$.* $\square$

Let $X = B_1(H)$ be the unit ball of H with the weak topology, $\xi \in X$. Put

$$f_n(\xi) = ((K_n^* K_n + \lambda_n K_n^* + \overline{\lambda}_n K_n)\xi, \xi), \quad g_n(\xi) = |\lambda_n|^2 \cdot \|\xi\|^2, \quad f(\xi) = (K\xi, \xi), \quad g(\xi) = \lambda \cdot \|\xi\|^2 .$$

The conditions of the lemma 4.2.5 are satisfied, so the series $\sum_{i=1}^{\infty}(F_i^* F_i \xi, \xi)$ uniformly converges on X to the function $(\langle F, F\rangle \xi, \xi)$, therefore the series $\sum_{i=1}^{\infty} F_i^* F_i$ converges to $\langle F, F\rangle$ in the algebra $\mathcal{K}^+$ and, therefore, $F \in l_2(\mathcal{K}^+)$. $\square$

4.3 Reflexivity of modules over $C(X)$

In this section we describe results of the papers [47, 64].

Definition 4.3.1 ([64]) A compact Hausdorff space X is called an L-space if for an arbitrary sequence $f_1, f_2, \dots$ of continuous functions on X converging pointwise to some bounded function f the set of continuity points of the function f is dense in X.

Examples of L-spaces are any compact subsets of finite-dimensional Euclidean space [1]. Infinite Stonean spaces are not L-spaces.

Definition of L-spaces allows us to give a description of bidual Hilbert modules over algebras of functions on such spaces.

Theorem 4.3.2 ([47, 64]) *Let $A = C(X)$, where X is an L-space. Then any countably generated Hilbert A-module is reflexive.*

Proof: According to the stabilization theorem 1.4.1 any countably generated Hilbert module is a direct summand in the standard module H_A, therefore it is sufficient to prove reflexivity of the module H_A. The proposition 2.5.5 gives a description of the dual module:

$$H_A' = \{f = (f_i(t)) : f_i(t) \in C(X), \sup_N \left\| \sum_{i=1}^{N} |f_i(t)|^2 \right\| < \infty\}.$$

Since the sequence of partial sums $\sum_{i=1}^{N} |f_i(t)|^2$ is monotone and bounded at each point $t \in X$, the corresponding series converges pointwise to a bounded function. By the supposition the set of points of continuity of the limit function $\sum_{i=1}^{\infty} |f_i(t)|^2$ is dense in X. Let us fix a point of continuity $t_0 \in X$. Let $\lambda(t)$ be a continuous function on X, equal to 1 at the point t_0. For $F \in H_A''$ we have

$$F(f\lambda) = F\left(\sum_{i=1}^{N} e_i f_i \cdot \lambda\right) + F\left(\left(f - \sum_{i=1}^{N} e_i f_i\right)\lambda\right), \tag{10}$$

where $\{e_i\}$ is the standard basis in $H_A \subset H_A'$. The element $\sum_{i=1}^{N} e_i f_i \, \lambda$ belongs to the module H_A, therefore

$$F\left(\sum_{i=1}^{N} e_i f_i \cdot \lambda\right) = \sum_{i=1}^{N} F_i^* f_i \cdot \lambda,$$

where by $F_i = F_i(t)$ we denote $F(\widehat{e}_i)^*$. Let $\omega(f)$ denote the least upper bound of oscillation of the function $\sum_{i=1}^{N} |f_i(t)|^2$. Then, obviously,

$$\omega(f) = \lim_{N\to\infty} \sup_{t\in X} \sum_{i=N+1}^{\infty} |f_i(t)|^2 = \lim_{N\to\infty} \left\| f - \sum_{i=1}^{N} e_i f_i \right\|^2 .$$

Let us choose an arbitrary $\varepsilon > 0$ and a function $\lambda(t)$ such that on the support $\operatorname{Supp}\lambda(t)$ the oscillation $\omega(f)$ is less than ε^2. Then find such N that for all $N' > N$ the inequality

$$\left\| \left(f - \sum_{i=1}^{N'} e_i f_i\right)\lambda \right\|^2 < 2\varepsilon^2 \tag{11}$$

holds. It follows from inequalities (10),(11) that at the point t_0 (where $\lambda(t_0) = 1$) one has

$$|F(f)(t_0) - \sum_{i=1}^{N'} F_i^*(t_0) f_i(t_0)| \le \|F\| \cdot 2\varepsilon.$$

for all $N' > N$. Thus, the series $\sum_{i=1}^{\infty} F_i^*(t) f_i(t)$ converges to a continuous function $F(f)(t)$ at all points t_0 of continuity of the sum of the series $\sum_{i=1}^{\infty} |f_i(t)|^2$. Since $H_A'' \subset H_A'$, the series

$$\sum_{i=1}^{\infty} F_i^*(t) F_i(t) \tag{12}$$

is also convergent at each point and coincides with the continuous function $\langle F, F\rangle(t) = F(\widetilde{F})(t)$ at each point of continuity. Denote by $E \subset X$ the set of continuity points of the series (12). Let now t_0 be some point of discontinuity of the series (12). Without loss of generality (multiplying by some continuous function, if necessary) it is possible to assume that there is the sequence of points $t_n \in E$, converging to the point t_0 such that $\langle F, F\rangle(t_n) = 1$ and $\sum_{i=1}^{\infty} |F_i(t_0)|^2 < 1$. Choose functions $h_i(t) \in C(X)$ to satisfy conditions

(i) $h_i(t_0) = F_i(t_0)$; $\quad |h_i(t)| \le |F_i(t)|$;

(ii) $h = (h_i) \in H_A$, i.e. the series $\sum_{i=1}^{\infty} |h_i(t)|^2$ is uniformly convergent.

Define a function $\lambda(t)$ on X by conditions:

(i) $0 \le \lambda(t) \le 1$; $\quad \lambda(t_{2n}) = 0$, $\lambda(t_{2n+1}) = 1$,

(ii) Outside the point t_0 the function $\lambda(t)$ is continuous.

Consider the element

$$F_i(t) = h_i(t) + \lambda(t)(F_i(t) - h_i(t)).$$

It can be easily checked that the functions $f_i(t)$ are continuous on the whole X. Moreover, as $|f_i(t)| \le |F_i(t)|$, so we have $f = (f_i) \in H_A'$. Then

$$F(f)(t) = \langle F, f\rangle(t) = \sum_{i=1}^{\infty} F_i^*(t) h_i(t) + \lambda(t) \sum_{i=1}^{\infty} F_i^*(t)(F_i(t) - h_i(t)).$$

The series

$$\sum_{i=1}^{\infty} F_i^*(t) h_i(t) \tag{13}$$

is uniformly convergent because $h \in H_A$. On the set $E \subset X$ the series

$$\sum_{i=1}^{\infty} F_i^*(t)(F_i(t) - h_i(t))$$

converges uniformly to the continuous function $\langle F, F - h\rangle(t)$ which is continuous on the whole X. Then for $t \in E$ one has

$$\langle F, f\rangle(t) = \langle F, h\rangle(t) + \lambda(t)(\langle F, F\rangle(t) - \langle F, h\rangle(t)),$$

and for each t_n

$$\langle F, f\rangle(t_n) = \langle F, h\rangle(t_n) + \lambda(t_n)(1 - \langle F, h\rangle(t_n)). \tag{14}$$

The functions $\langle F, f\rangle(t)$ and $\langle F, h\rangle(t)$ are continuous, and as the series (13) is uniformly convergent, so

$$\langle F, h\rangle(t_0) = \sum_{i=1}^{\infty} |F_i(t_0)|^2 < 1.$$

But it contradicts continuity of the function $\langle F, f\rangle(t)$, as due to our choice of the function $\lambda(t)$ the limits of the right and the left part of the equality (14) are different for even and for odd n. $\square$

4.4 Hilbert modules related to conditional expectations of finite index

In this section we describe results of the paper [25]. Let $E : A \to B \subseteq A$ be an exact conditional expectation on a C^*-algebra A, i.e. a projection of norm one onto a C^*-algebra B such that the condition $E(x^*x) = 0$, $x \in A$, implies $x = 0$.

A conditional expectation $E : A \to B \subseteq A$ is called a conditional expectation of *algebraically finite index* if there exists a set of elements $\{u_1, ..., u_n\} \subset A$ such that for any $x \in A$ $x = \sum_{i=1}^{n} u_i E(u_i^* x)$. Then the element $\mathrm{Ind}(E) = \sum_{i=1}^{n} u_i u_i^*$ of the center of the C^*-algebra A is called the *шэфхъёюь* of E. $\mathrm{Ind}(E)$ does not depend on a choice of the elements $\{u_1, ..., u_n\} \subset A$, is positive, and $\mathrm{Ind}(E) \geq 1_A$ [69, 7]. It is shown in [7] that the algebraic finiteness of the index is equivalent to the property of A to be a projective finitely generated C^*-Hilbert module over the C^*-algebra B.

It is also interesting to consider another class of conditional expectations $E : A \to B \subseteq A$ for which there exists a number $K \geq 1$ such that the map $(K \cdot E - \mathrm{id}_A)$ is a positive element of the C^*-algebra A. Such conditional expectations are called conditional expectations of *finite index* and they have the following property:

Proposition 4.4.1 *Let A be a C^*-algebra and let $E : A \to B \subseteq A$ be a conditional expectation for which the set of fixed points coincides with B. Then there exists a finite number $K \geq 1$ such that the map $(K \cdot E - \mathrm{Id}_A)$ is positive iff E is exact and the (right) pre-Hilbert B-module $\{A, E(\langle \cdot, \cdot \rangle_A)\}$ is complete with respect to the norm $\|E(\langle \cdot, \cdot \rangle_A)\|_B^{1/2}$ (where $\langle a, b \rangle_A = a^* b$ for $a, b \in A$).*

Proof: If the algebra A is complete with respect to the norm $\|E(\langle \cdot, \cdot \rangle_A)\|_B^{1/2}$ then there exists such number K that the inequality $K\|E(x^*x)\| \geq \|x^*x\|$ holds for any $x \in A$. For $a \in A$, $\varepsilon > 0$ put $x = a(\varepsilon + E(a^*a))^{-1/2}$. Remark that

$$(\varepsilon + E(a^*a))^{-1/2} \cdot E(a^*a) \cdot (\varepsilon + E(a^*a))^{-1/2} \leq 1_A \, ,$$

whence the inequality $K \cdot 1_A \geq (\varepsilon + E(a^*a))^{-1/2} \cdot a^*a \cdot (\varepsilon + E(a^*a))^{-1/2}$ follows. Multiplying both parts of it by $(\varepsilon + E(a^*a))^{1/2}$, we conclude that $K \cdot (\varepsilon + E(a^*a)) \geq a^*a$ for all $a \in A$, $\varepsilon > 0$. The inverse statement obviously follows from the inequality $\|E(x)\| \geq K^{-1/2}\|x\|$ valid for any $x \in A$. $\square$

Notice that unlike algebraic finiteness of the index, in the case of conditional expectation of a finite index the Hilbert module $\{A, E(\langle \cdot, \cdot \rangle_A)\}$ can be infinitely generated.

Define

$$K(E) = \inf\{K : (K \cdot E - \mathrm{id}_A) \text{positively in} A\} \, .$$

Let us call $K(E)$ the *characteristic number* of the conditional expectation E.

Let X be a compact Hausdorff space with an action of a group G. Denote the C^*-algebra of G-invariant continuous functions on X by $C^G(X)$, and stabilizer of a point $x \in X$ denote by $G_x = \{g \in G : gx = x\}$.

Definition 4.4.2 A continuous action of group G on X is called *uniformly continuous* if for each point $x \in X$ and for each its neighbourhood U_x there is the neighbourhood V_x of the point x such that $g(V_x) \subseteq U_x$ for each $g \in G_x$.

Remark that the continuous action of compact group satisfies this definition.

Definition 4.4.3 *Let a group G acts uniformly continuously on a compact Hausdorff space X in such a manner that the length of each orbit $\#Gx$ does not exceed some number $k \in \mathbf{N}$. Define a conditional expectation $E_G : C(X) \to E_G(C(X)) \subseteq C(X)$ by the formula*

$$E_G(f)(x) = \frac{1}{\#(G/G_x)} \cdot \sum_{g_a \in G/G_x} f(g_a x) \, , (x \in X) \, .$$

Lemma 4.4.4 ([25]) *The conditional expectation E_G is well-defined.*

Theorem 4.4.5 ([25]) *Let a group G uniformly continuously acts on a locally compact Hausdorff space X so that $k := \max\{\#(Gx) : x \in X\} < +\infty$. Then the characteristic number of the conditional expectation E_G satisfies the equality*

$$K(E_G) = k := \max_{x \in X} \#(Gx).$$

Proof: Let $x \in X$ be an arbitrary point and let $k_x = \#Gx$. Then

$$KE_G(f)x = k\frac{1}{k_x} \cdot \sum_{g_a \in G/G_x} f(g_a x) \geq f(x),$$

where f is an arbitrary non-negative function in $C(X)$. Let us assume that $K(E_G) < k$ and choose such point x that $k_x > K(E_G)$. Then it is possible to choose a small enough neighbourhood U_x of the point x so that $g_i U_x \cap g_j U_x = \emptyset$ $(i \neq j)$ for the set $\{g_1 = 1, g_2, \ldots, g_m\} = G/G_x$. Let f be a continuous non-negative function with the support lying inside U_x. Then

$$K(E_G)E_G(f)x = K(E_G)\frac{1}{k_x} \cdot \sum_{g_a \in G/G_x} F(g_a x) < f(x).$$

Contradiction with the definition of $K(E_G)$ completes the proof. $\square$

Theorem 4.4.6 *Let X be a compact Hausdorff space and let G be a group uniformly continuously acting on X. If all orbits of the action of G have the same finite number of points then the conditional expectation*

$$E(f)(x) = \frac{1}{\#(Gx)} \sum_{g_i \in (G/G_x)_i} f(g_i x)\,,$$

is well-defined on $C(X)$, and the Hilbert $C^G(X)$-module $\{C(X), E(\langle \cdot, \cdot \rangle)\}$ is finitely generated and projective.

Proof: The idea of the proof is contained in [69], but beforehand we require two technical lemmas.

Lemma 4.4.7 *Let X be a compact Hausdorff space with a uniformly continuous action of a group G and let all orbits contain equal finite number of points. Then for any point $x \in X$ and for any element $g \in G_x$ one can find an open neighbourhood U_x of a point x on which g acts identically.*

Proof: Let us denote the length of orbits $\#(Gx)$ by n. Let $x_1, \ldots, x_n \in X$ be the orbit of the point x and let $h_i \in G$ be such elements that $h_i x = x_i$. Choose $g_0 \in G_x$ and assume that each neighbourhood U_x of the point x contains some point $y \in U_x$ such that $g_0 y \neq y$. Fix neighbourhoods U_{x_i} of the points x_i satisfying condition $U_{x_i} \cap U_{x_j} = \emptyset$ for $i \neq j$. Then we can find a neighbourhood $V_x \subseteq U_x$ of the point x such that $h_i(V_x) \subseteq U_{x_i}$. Since the group G acts uniformly continuously, it is possible to find a neighbourhood $W_x \subseteq V_x$ of the point x such that $g(W_x) \subseteq V_x$ for each $g \in G_x$. If $y \in W_x$ and $g_0 y \neq y$ then the orbit Gy of this point includes not less than $n+1$ different points $\{h_i y \in U_{x_i} : i = 1, \ldots, n\} \cup \{y, g_0 y \in V_x\}$. The obtained contradiction proves the lemma. $\square$

Lemma 4.4.8 *Under the suppositions of the lemma 4.4.7 for any point $x \in X$ one can find a neighbourhood V_x of this point such that the action of the subgroup G_x on V_x is the identity mapping.*

Proof: We should show that a neighbourhood U_x of the lemma 4.4.7 can be choosen for all $g \in G_x$ simultaneously. For each $g \in G_x$ put

$$U_x(g) = \{y \in X : gy = y\}\,.$$

Suppose the contrary, i.e. that the set $\cap_{g \in G_x} U_x(g)$ does not contain any neighbourhood of the point x. It means that any neighbourhood U_x of the point x contains some point z such that for some $g_z \in G_x$ we have $g_z z \neq z$. Consider a neighbourhood V_x of the point x and neighbourhoods $\{U_{h_i x}\}$ for fixed representatives $\{h_i \neq e\} \in G$ of cosets in G/G_x such that their intersections are empty and $h_i(V_x) \subseteq U_{h_i x}$. As in the proof of the lemma 4.4.7 we find a neighbourhood $W_x \subseteq V_x$ of the point x such that $g(W_x) \subseteq V_x$ for each $g \in G_x$. Put $U_x = \cup_{g \in G_x} G(W_x) \subseteq V_x$. It is a G_x-invariant open neighbourhood of the point $x \in X$. The supposition $g_z z \neq z$ for some $z \in U_x$, $g_z \in G_x$ means, that the orbit of the point z consists of not less than $n+1$ points. $\square$

Let $U_x \subset V_x$ be a neighbourhood of the point x such that the action of G_x on V_x is an identity mapping. Then one can find a function $f_x \in C(X)$ such that $\operatorname{supp} f_x \subset V_x$ and $f_x|_{U_x} = 1$. For each $g \in G$ one has either $(gV_x) \cap V_x = \emptyset$ or $gV_x \equiv V_x$, therefore

$$\alpha_g(f_x) \cdot f_x = \begin{cases} \alpha_g(f_x)^2 & \text{if } gx = x, \\ 0 & \text{if } gx \neq x, \end{cases}$$

where α denotes the action of G on functions, $\alpha_g(f)(x) = f(g^{-1}x)$. Let $\{U_{x_1}, \dots, U_{x_k}\}$ be a finite covering of the space X by sets of the above form. Put

$$v = \sum_{i=1}^{k} f_{x_i} \geq 1 \quad , \quad U_i = v^{-1/2}(f_{x_i})^{1/2} \in C(X)\,.$$

Notice that if we take one element g_i in each coset G/G_x then by the lemma 4.4.4 the map

$$E_G(f)(x) = \frac{1}{\#(Gx)} \sum_{i=1}^{n} f(g_i x)\,,$$

is well-defined for all $x \in X$, $f \in C(X)$ and it is a conditional expectation on $C(X)$. Moreover, for each function $f \in \mathrm{C}(X)$ one has

$$\sum_{i=1}^{k} u_i \cdot E_G(u_i^* f) = \sum_{i=1}^{k} \left(\frac{1}{n} \sum_{j=1}^{n} U_i(x) u_i^*(g_j x) f(g_j x) \right) = f \quad , \quad E_G(u_i^* u_j) = \delta_{i,j}\,,$$

hence the set $\{u_1, ..., u_n\}$ is a basis of the Hilbert $C^G(X)$-module $\{C(X), E_G(\langle\cdot,\cdot\rangle)\}$. Therefore, this Hilbert module is finitely generated and projective. $\square$

The theorem 4.4.6 generalizes results of [69] and shows that if all orbits consist of equal finite number of points then the corresponding conditional expectation is of algebraically finite index. From finite generatedness and projectivity it follows that the Hilbert module $A = \{C(X), E(\langle\cdot,\cdot\rangle)\}$ is autodual. In the case of a finite index (when $K(E_G) < \infty$ and the pre-Hilbert module A is complete) we can not expect that this module should be self-dual. However sometimes this module is reflexive, i.e. $A'' = A$, where A' is the dual Banach $C^G(X)$-module of bounded $C^G(X)$-homomorphisms from A into $C^G(X)$.

Theorem 4.4.9 ([25]) *Let the group G uniformly continuously acts on a compact Hausdorff space X. Suppose that all orbits consist of not more than n points, and that the number of points, for which the length of their orbit is less than n, is finite. Then the Hilbert $C^G(X)$-module $\{C(X), E_G(\langle\cdot,\cdot\rangle)\}$ is reflexive.*

Proof: Describe at first the dual Banach $C^G(X)$-module A'. Let $x_1, \dots, x_m$ be the points with orbits shorter than n. It is possible to choose open neighbourhoods $U_1, \dots, U_m$ of these points in such a way that each neighbourhood U_i would be invariant with respect to the action of the subgroup G_{x_i} and if for some $h \in G$ one has $hx_i = x_j$ then $hU_i = U_j$. Denote by Y the G-invariant compact set $X \setminus (U_1 \cup \ldots \cup U_m)$. Let $F \in A'$ be a $C^G(X)$-valued functional on the module A. Consider its restriction on the Hilbert $C^G(Y)$-module $\{C(Y), E_G(\langle\cdot,\cdot\rangle)\}$. For a function $g \in C(Y)$ we take its extension $\widetilde{g} \in C(X)$ and define $F|_Y$ by the equality $F|_Y(g) = F(\widetilde{g})|_Y$. This definition does not depend on the choice of an extension $\widetilde{g}$. If $Y' \supset Y$ is also a compact G-invariant subspace not containing the points $x_1, \dots, x_m$ then $(F|_{Y'})|_Y = F|_Y$. Since the orbit of each point of the set Y has constant length, by the theorem 4.4.6 the $C^G(Y)$-module $\{C(Y), E_G(\langle\cdot,\cdot\rangle)\}$ is finitely generated and projective, therefore auto-dual. Denote by $C(X \setminus \{x_1, \dots, x_m\})$ the set of continuous functions on the *noncompact* space $X \setminus \{x_1, \dots, x_m\}$. The restriction on this space defines a map

$$A' \longrightarrow C(X \setminus \{x_1, \dots, x_m\}). \tag{15}$$

It is easy to verify that the map (15) is an monomorphism.

Let us study local properties of functionals from A' close to the points $x_1, \dots, x_m$. Let x_0 be one of these points. It has such neighbourhood U_{x_0} that if $gx_0 = x_0$ then $gU_{x_0} = U_{x_0}$. The group G_{x_0} contains a normal subgroup G_0 of the elements, which do not move points from the neighbourhood U_{x_0}. Choose a representative g_i in each coset G/G_{x_0}. Then outside the point x_0 the action of the functional $F \in A'$ can be written as

$$F(f)(x) = \frac{1}{n} \sum_{i=1}^{n} F^*(g_i x) \cdot f(g_i x), \tag{16}$$

and this action can be continuously extended to the point x_0. Let $x_0 = x^0, x^1, \dots, x^{k-1}$ be the orbit of the point x_0. Then it is possible to write the sum (16) in the form

$$F(f)(x) = \frac{1}{n} \sum_{j=0}^{k-1} \left(\sum_{i:g_i x_0 = x^j} F^*(g_i x) \cdot f(g_i x) \right) .$$

Passing to the limit (which exists by supposition), we obtain

$$F(f)(x_0) = \lim_{x \to x_0} \frac{1}{n} \sum_{j=0}^{k-1} \left(\sum_{i:g_i x_0 = x^j} F^*(g_i x) \cdot f(x^j) \right) ,$$

hence there exists (for $f \equiv 1 \in C(X)$) the limit

$$\frac{1}{n} \lim_{x \to x_0} \sum_{i:g_i x_0 = x^j} F(g_i x)$$

for any $x \in X \setminus \{x_1, ..., x_m\}$. Remind that the function $F(x)$ is defined only outside of the point x_0. If we would like the action F on the Hilbert $C^G(Y)$-module $\{C(Y), E_G(\langle \cdot, \cdot \rangle)\}$ to be of the form (16) on the whole X, it is necessary to define the function $F(x)$ at the point x_0 by the equality

$$F(x_0) = \frac{1}{n} \lim_{x \to x_0} \sum_{i:g_i x_0 = x^j} F(g_i x). \tag{17}$$

To complete the proof we need the following lemma.

Lemma 4.4.10 *The module A' is isomorphic to the module of all bounded functions $F(x)$ on X which are continuous on $X \setminus \{x_1, \dots, x_m\}$, and satisfy the condition* (17).

Proof: We need to show that the image of the monomorphism (15) consists of bounded functions. Suppose the inverse. Then there exists such point $\bar{x}$ that $|F(\bar{x})| > n \cdot \|F\|$, where $\|F\|$ is the norm of F in the dual Banach $C^G(X)$-module A'. Moreover, it is possible to choose a neighbourhood $U_{\bar{x}}$ of the point $\bar{x}$ so that $U_{\bar{x}} \cap g_i U_{\bar{x}} = \emptyset$ for those elements of the group G, for which $g_i \bar{x} \neq \bar{x}$. Consider such function $f \in C(X)$, that $f(\bar{x}) = 1$ and $\operatorname{supp} f \subset U_{\bar{x}}$. Then it follows from the equality (16) that

$$F(f)(\bar{x}) = \frac{1}{n} F^*(\bar{x}) \cdot f(\bar{x}),$$

and the inequality

$$|F(f)(\bar{x})| = \frac{1}{n} |F^*(\bar{x})| \cdot |f(\bar{x})| = \frac{1}{n} |F(\bar{x})| > \|F\|,$$

gives a contradiction. $\square$

Now, having the above description of the dual module A', it is possible to describe the bidual module A''. Since there exists the canonical inclusion $A'' \subset A'$, and the inner product on A can be in a natural way extended to an inner product on A'' (see Theorem 4.1.4), making it a Hilbert module, it is sufficient to verify, on which functions from A' it is possible to extend the $C^G(X)$-valued inner product. Consider a function F from $A' \cap A''$. Adding to it (if necessary) a continuous function from A we can suppose that $F(x_0) = 0$. Then the $C^G(X)$-valued inner product of F by itself is an element of $C^G(X)$ having the form

$$\langle F, F \rangle(x) = E(|F(x)|^2) = \frac{1}{n} \sum_{i:g_i x_0 = x^j} |F(g_i x)|^2 \tag{18}$$

for all $x \in X$. But, as $\langle F, F\rangle(x_0) = 0$, it follows from the supposition $F \in A''$ that

$$\lim_{x \to x_0} F(g_i x) = 0$$

for each summand of the equality (18). Therefore, we obtain from (17) that the function F is continuous at the point x_0, hence the module A is reflexive. $\square$

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