Integrated list of problems

Problem 1. Give an example of an involutive Banach algebra that is not a C^*-algebra.

Problem 2. Show that a left unit element is also a right one, which has $1^* = 1$; that the identity element is unique and that $\|1\| = 1$. It is called the *unit of algebra*.

Problem 3. Verify that the algebra $C(X)$ formed by all continuous complex-valued functions on a compact space X and the algebra $C_0(X)$ of all continuous complex-valued functions on a locally compact space X tending to 0 at infinity (that is, $f : X \to \mathbb{C}$ such that for any $\varepsilon > 0$ there exists a compact $K \subseteq X$ such that $\sup\{|f(x)| \mid x \in K\} < \varepsilon$) are commutative C^*-algebras if the *supremum-norm*: $\|f\| = \sup_{x\in X} |f(x)|$, is taken as the norm and the pointwise multiplication is taken as the multiplication. Moreover, the algebra $C(X)$ is unital.

Problem 4. Verify that the algebra $\mathbb{B}(H)$ of all bounded operators acting on a Hilbert space H is a C^*-algebra with identity. Here as a norm we take the *operator norm* $\|a\| = \sup_{h\in H,\, \|h\|\leq 1} \|a(h)\|$, and the multiplication is the composition of operators.

Problem 5. Show that "sum" norm turns A^+ into an involutive Banach algebra, but not into a C^*-algebra.

Problem 6. Prove that A is an ideal in A^+.

Problem 7. Prove that if $a_0 \in A$ is invertible and $\|a - a_0\| < \frac{1}{\|a_0^{-1}\|}$, then a is also invertible, and $a^{-1} = \sum_{n=0}^{\infty}[a_0^{-1}(a_0 - a)]^n a_0^{-1}$.

Problem 8. Show that the quasi-spectrum always contains zero.

Problem 9. Let a and b be commuting elements of a Banach algebra. Then the product ab is invertible if and only if each of the elements a and b are invertible.

Problem 10. Show that $r(a) \leq \|a\|$ for any $a \in A$.

Problem 11. Prove that M_A is a locally compact Hausdorff space, and M_{A^+} is its one-point compactification.

Problem 12. Check that if $\varphi : A \to \mathbb{C}$ is a (non-zero) multiplicative functional on A, then the formula $\widetilde{\varphi}((a, \lambda)) = \varphi(a) + \lambda$ defines a unique extension of φ to a multiplicative functional on A^+.

Problem 13. Any ideal I of a commutative unital Banach algebra is contained in some maximum ideal.

Problem 14. Prove the Gelfand theorem for a non-unital algebra.

Problem 15. If a is an invertible element, then the algebra $C^*(a)$ is unital. In this case $C^*(a) = C^*(1, a)$.

Problem 16. Prove that $f(\mathrm{Sp}(a)) = \mathrm{Sp}(f(a))$ approximating f by polynomials, and correctly stating what it means that the image is continuous under a uniform approximation, and using the isometricity of the Gelfand transform.

Problem 17. Show that if $a \geqslant 0$ and $0 \geqslant a$, then $a = 0$; and also that $-\|a\|1 \leqslant a \leqslant \|a\|1$ for every self-adjoint a.

Problem 18. If $0 \leqslant a \leqslant b$, then $\|a\| \leqslant \|b\|$.

Problem 19. Prove that an algebra with countable approximate unit does not have to be separable.

Problem 20. Prove that a positive element of an arbitrary C^*-subalgebra is a positive element of the entire algebra.

Problem 21. Let $\varphi : A \to B$ be a $*$-homomorphism of non-unital algebras. Prove that there is a unique unital $*$-homomorphism $\varphi^+ : A^+ \to B^+$, extending φ.

Problem 22. Let $\varphi : A \to B$ be a $*$-homomorphism of algebras, with A non-unital, and B unital. Prove that there is a unique unital $*$-homomorphism $\varphi^{(+)} : A^+ \to B$, extending φ.

Problem 23. Verify that

- If S is self-adjoint, then so is S'.
- The commutant of any set is a unital algebra.
- The commutant of any set is weakly closed.
- Thus, S' is the von Neumann algebra for any self-adjoint set S.
- If $S_1 \subset S_2$, then $S_1' \supset S_2'$.
- Always $S \subset S''$.
- Therefore $S' = S'''$, $S'' = S''''$, etc.

Problem 24. Prove that if a self-adjoint element a in a unital C^*-algebra has $\mathrm{Sp}(a) = \{0, 1\}$, then a is a nonscalar idempotent.

Problem 25. Let u_λ, $\lambda \in \Lambda$, be some approximate unit in a unital algebra. Prove that $1 = \lim_{\lambda\in\Lambda} u_\lambda$.

Problem 26. There is a natural bijection between self-adjoint linear functionals on A and (real) linear functionals on A_{sa}.

Problem 27. If A is unital, then a representation π is non-degenerate if and only if $\pi(1) = 1$.

Problem 28. Prove that the matrix algebra M_n is simple for any n

Problem 29. Prove that the image of the matrix algebra M_n under a $*$-homomorphism is either a zero algebra or an algebra isomorphic to M_n.

Problem 30. Draw Bratteli diagram (for some defining sequence) of the following AF-algebra: the algebra of compact operators $\mathbb{K}(H)$.

Problem 31. Draw Bratteli diagram (for some defining sequence) of the following AF-algebra: the unitization $\mathbb{K}(H)^+$ of the algebra of compact operators $\mathbb{K}(H)$.

Problem 32. Draw Bratteli diagram (for some defining sequence) of the following AF-algebra: the closure of the union of $A_p = M_{2^p}$, with embeddings $A_p \subset A_{p+1}$ of multiplicity 2 according to the formula $a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ (CAR algebra).

Problem 33. Draw Bratteli diagram (for some defining sequence) of the following AF-algebra: $C(K)$, where K is the Cantor set obtained from $[0,1]$ by successive removing the middle third of the corresponding intervals. If K_p is a set, obtained at the pth step of this process, then A_p is an algebra of continuous functions constant on intervals of K_p.

Problem 34. Draw Bratteli diagram (for some defining sequence) of the following AF-algebra: $C(X)$, where $X := \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$, and A_k consists of all functions constant on $[0, 1/2^k]$.

Problem 35. Let $\pi : A \to \mathbb{B}(H)$ be a degenerate representation. Let us denote by H_0 the invariant subspace $H_0 := \{\xi \in H \colon \pi(a)(\xi)0 \text{ for any } a \in A\}$. Prove that π induces a representation $\pi' : A \to \mathbb{B}(H/H_0)$, and if π was a faithful representation (an injective homomorphism), then so is π'.

Problem 36. Let $\pi : A \to \mathbb{B}(H)$ be a degenerate representation. Let us denote by H_0 the invariant subspace $H_0 := \{\xi \in H \colon \pi(a)(\xi)0 \text{ for any } a \in A\}$. Prove that π induces a representation $\pi' : A \to \mathbb{B}(H/H_0)$, and if π was a faithful representation (an injective homomorphism), then so is π'.

Problem 37. Prove that a representation $\pi : A \to \mathbb{B}(H)$ is non-degenerate if and only if for some approximate unit u_λ of the algebra A the following condition is satisfied: for any vector $\xi \in H$ there is a λ such that that $u_\lambda(\xi) \neq 0$.

Problem 38. Let A be a C^*-algebra, $a \in A$, $p, q \in A$ — orthogonal projections (i.e. self-adjoint idempotents with $pq = 0$). Show that if a is positive and $pap = 0$, then $paq = 0$.

Problem 39. Let A be a C^*-algebra, $a \in A$. Let us denote by aAa the set of all elements of the form aba, where $b \in A$, and by $\overline{aAa}$ the closure of this set. A C^*-subalgebra $B \subset A$ *is hereditary* if the conditions $0 \leq a \leq b$ and $b \in B$ imply that $a \in B$.

(1) Check that $\overline{aAa}$ is a C^*-subalgebra for any $a \in A$.
(2) Let $p \in A$ be a projection. Verify that pAp is closed.
(3) Show that pAp is hereditary for any projector p.
(4) Show that $\overline{aAa}$ is hereditary for any positive $a \in A$.

Problem 40. Let $X \subset \mathbb{R}$ be the set of points $1, 1/2, 1/3, \dots$ and 0. Let $C(X, M_2)$ be the set of all continuous functions on X with values in the matrix algebra M_2. Let $B_1 = \{f \in C(X, M_2) : f(0) \text{ is diagonal}\}$, $B_2 = \{f \in C(X, M_2) : f(0) \text{ has the form } \left(\begin{smallmatrix} * & 0 \\ 0 & 0 \end{smallmatrix}\right)\}$.

(1) Show that $C(X, M_2)$, B_1, B_2 are C^*-algebras.
(2) Find all (two-sided, closed) ideals in $C(X)$, $C(X, M_2)$, B_1, B_2.

Problem 41. Let A be a C^*-algebra, $J \subset A$ be an ideal, $a \in A$ is a self-adjoint element. Show that there exists an element $j \in J$ such that $\|[a]\| = \|a - j\|$, where $[a] \in A/J$ is the class $a + J$ of element a. Hint: decompose $a - \|[a]\| \cdot 1 = a_+ - a_-$ with positive a_+, a_- and show that $a_+ \in J$.

Problem 42. Let A be a C^*-algebra, $a \in A$ be a self-adjoint element. Show that if the spectrum $\sigma(a)$ is an infinite set, then A is infinite-dimensional.

Problem 43. Describe the GNS construction for the C^*-algebra $C[0,1]$ and for a positive linear functional φ

(1) $\varphi(f) = f(0)$,
(2) $\varphi(f) = \frac{1}{2}(f(0) + f(1))$,
(3) $\varphi(f) = \int_0^1 f(x)\,dx$,
where $f \in C[0,1]$.

Problem 44. Describe the GNS construction for the C^*-algebra M_n of complex $n\times n$-matrices and for a positive linear functional φ
(1) $\varphi(A) = a_{11}$,
(2) $\varphi(A) = \mathrm{tr}(A)$,
where $A = (a_{ij})_{i,j=1}^n \in M_n$.

Problem 45. Let π, σ be representations of a C^*-algebra A on the Hilbert spaces H_π and H_σ, and let a partial isometry $U : H_\pi \to H_\sigma$ satisfy the equality $\sigma(a)U = U\pi(a)$ for any $a \in A$. Show that the image (resp. orthogonal complement to the kernel) of U is an invariant subspace for $\sigma(A)$ (resp. for $\pi(A)$). (U is a partial isometry if U^*U and UU^* are projections)

Problem 46. (a) Let $M_n(A)$ be the set of all $n\times n$-matrices with coefficients from a C^*-algebra A. Show that on $M_n(A)$ there exists a C^*-norm.
(b) Let A be a C^*-algebra with norm $\|\cdot\|$, and let $\|\cdot\|'$ be another norm on A, equivalent to the first norm. Show that if $\|\cdot\|'$ is a C^*-norm, then these norms coincide. Deduce from this the uniqueness of C^*-norm on $M_n(A)$.

Problem 47. Let φ be a state on a C^*-algebra A. Suppose that for some self-adjoint element $a \in A$ one has the equality $\varphi(a^2) = \varphi(a)^2$. Show that it follows from this that $\varphi(ab) = \varphi(ba) = \varphi(a)\varphi(b)$ for any $b \in A$.

Problem 48. Let $A = c$ be the C^*-algebra of all convergent sequences of complex numbers, $c = \{(a_n)_{n\in\mathbb{N}} : a_n \in \mathbb{C}; \lim_{n\to\infty} a_n \text{ exists}\}$. Let us consider it as a C^*-subalgebra of the algebra $\mathbb{B}(l_2)$ of bounded operators in the Hilbert space l_2 of square-integrable sequences. Find the first and second commutant, A' and A'', and (independently) the weak closure of A in $\mathbb{B}(l_2)$.

Problem 49. (a) Show that the weak topology is strictly weaker than the strong topology.
(b) Let $P \subset \mathbb{B}(H)$ be the set of all (self-adjoint) projections on a Hilbert space. Show that if $p_\lambda \to p$ weakly converges, where $p_\lambda \in P$ and $p \in P$, then $p_\lambda \to p$ strongly converges.
(c) Show that the strong limit of a sequence of (self-adjoint) projections is a projection.
(d) Find an example of a weakly convergent net $p_\lambda \to p$ with $p_\lambda \in P$ and $p \notin P$.

Problem 50. Let $H_n \subset H$ be the subspace of a Hilbert space H generated by the first n vectors of an orthonormal basis. In the set of all sequences $(m_1, m_2, \ldots)$, where $m_k \in \mathbb{B}(H_n) \subset \mathbb{B}(H)$, consider the subset A of all sequences such that

- $\sup_k \|m_k\| < \infty$;
- the sequences $(m_1, m_2, \ldots)$ and $(m_1^*, m_2^*, \ldots)$ are convergent in the strong topology.

Show that A is a C^*-algebra and that the mapping $(m_1, m_2, \ldots) \mapsto$ $s\text{-}\lim_{k\to\infty} m_k \in \mathbb{B}(H)$ is a surjective $*$-homomorphism of $A \to \mathbb{B}(H)$.

Problem 51. Let A be a commutative C^*-algebra and let π be its irreducible representation on a Hilbert space H. Show that $\dim H = 1$

Problem 52. Consider $C[0,1]$ as a C^*-subalgebra in $\mathbb{B}(H)$, where $H = L^2([0,1])$ (continuous functions act on H by multiplication).

(a) Check that $C[0,1] \cap \mathbb{K}(H) = 0$;

(b) Let φ be a linear functional on $C[0,1]$ defined by the equality $\varphi(f) = f(0)$, $f \in C[0,1]$. Find a sequence of $\{e_n\}_{n\in\mathbb{N}}$ vectors of unit length weakly converging to zero in H such that $\varphi(f) = \lim_{n\to\infty}\langle fe_n, e_n\rangle$ for any function $f \in C[0,1]$.

Problem 53. Operators a, b in a Hilbert space H are called *compalent* if there exists a unitary operator $u \in \mathbb{B}(H)$ such that $u^*au - b \in \mathbb{K}(H)$. Show that if self-adjoint operators a, b are compalent then their essential spectra coincide.

Problem 54. Show that any AF C^*-algebra without unity has an approximative unity consisting of an increasing sequence of projections.

Problem 55. (1) Show that $C[0,1]$ is not an AF-algebra.

(2) Construct an injective $*$-homomorphism $C[0,1]$ into the AF-algebra $C(K)$ of continuous functions on the Cantor set K. Hint: construct a function f on K that takes all rational values from $[0,1]$ and show that $C^*(f)$ is isometrically $*$-isomorphic $C(\mathrm{Sp}(f)) = C[0,1]$.

Problem 56. Let $A_n = M_{2^n}(\mathbb{C}) \oplus M_{2^n}(\mathbb{C})$, and let the embedding $\alpha_n : A_n \to A_{n+1}$ be given by the formula $\alpha_n : \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \mapsto \left(\begin{array}{cc|cc} a_1 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 \\ \hline 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & a_2 \end{array}\right)$, where $a_1, a_2 \in M_{2^n}(\mathbb{C})$.

(1) Find the Bratteli diagram for the AF algebra $A = \overline{\cup_{n=1}^{\infty} A_n}$;

(2) Find whether A is unital.