DISCRETE GROUPS ACTIONS AND CORRESPONDING MODULES

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ABSTRACT. We address the problem of interrelations between the properties of an action of a discrete group Γ on a compact Hausdorff space X and the algebraic and analytical properties of the module of all continuous functions $C(X)$ over the algebra of invariant continuous functions $C_\Gamma(X)$? The present paper is a continuation of our joint paper with M. Frank and V. Manuilov. Here we prove some statements inverse to the ones obtained in that paper: we deduce properties of actions from properties of modules. In particular, it is proved that if for a uniformly continuous action the module $C(X)$ is finitely generated projective over $C_\Gamma(X)$, then the cardinality of orbits of the action is finite and fixed. Sufficient conditions for existence of natural conditional expectations $C(X) \to C_\Gamma(X)$ are obtained.

1. INTRODUCTION

Given a discrete countable group Γ acting on a compact Hausdorff space X, one can talk about the following notions:

(1) The orbits (their cardinality and other dynamical properties).

(2) In some cases using invariant means on functions on Γ it is possible to define so-called conditional expectations $E_\Gamma : C(X) \to C_\Gamma(X)$ and study their properties.

(3) It is possible to consider properties of the module of all continuous functions $C(X)$ over the algebra of invariant continuous functions $C_\Gamma(X)$. In some cases with the help of E_Γ it can be equipped with a $C_\Gamma(X)$-valued C*-inner product.

The study of the relations between these properties was started systematically in [5] (for preliminary and related research see also [1, 2, 4, 7, 8, 10, 14, 15, 18] and the overview in [5]).

In the present paper we prove some statements inverse to the ones obtained in the aforementioned paper: from some properties of modules (3) we deduce some properties of actions (1). In particular, it is proved that if for a uniformly continuous action the module $C(X)$ is finitely generated projective over $C_\Gamma(X)$, then the cardinality of orbits is finite and fixed.

Some sufficient conditions of the Lyapunov type for existence of natural conditional expectations $C(X) \to C_\Gamma(X)$ are obtained.

In the last section we give a description of the class of module-infinite algebras introduced in the previous sections.

2. Preliminaries and reminding

The necessary information about Hilbert C*-modules can be found in [11] and [12].

Suppose, Γ is a discrete countable group and X is a locally compact Hausdorff Γ-space. Let us denote by $M_\Gamma(f)$ or simply by $M(f)$ the invariant mean of a bounded function $f : \Gamma \to \mathbb{C}$ if it exists (if Γ is amenable or f is almost periodic [6]). Let $C_\Gamma(X)$ be the algebra of continuous invariant functions. The algebra of all continuous functions $C(X)$ is a module over $C_\Gamma(X)$. Let us denote by L_g the left translation on functions on Γ: $(L_g\varphi)(h) := \varphi(gh)$.

Let us reformulate some definitions and statements from [5]. Actions will always be assumed to satisfy the following condition.

Definition 1. [5, Def. 2.1] An action of a topological group G on a locally compact Hausdorff space X is *uniformly continuous* if for every point $x \in X$ and every neighborhood U_x of x there exists a neighborhood V_x of x such that $g(V_x) \subseteq U_x$ for every $g \in G_x$, where G_x is the isotropy subgroup at the point x.

Definition 2. (cf. [5, Def. 2.3]) Let us define a conditional expectation $E_\Gamma : C(X) \to C_\Gamma(X)$ by the formula

$$(E_\Gamma(\varphi))(x) := M_\Gamma(\varphi_x), \qquad \text{where} \qquad \varphi_x(g) := \varphi(gx).$$

Remark 3. Of course, this expectation E_Γ is not always defined. The obstructions are 1) existence of an averaging at each point, 2) the resulting function $E_\Gamma(\varphi)$ can be discontinuous. In their absence one gets a linear mapping $E_\Gamma : C(X) \to C_\Gamma(X)$. It is evident that since $M_\Gamma(1) = 1$, this mapping is identical on invariant functions, hence a projection. The inequality ([6]) $\inf\{f(g)\} \le M_\Gamma(f) \le \sup\{f(g)\}$ shows that $\|E_\Gamma\| = 1$ and so we really have a conditional expectation.

Remark 4. As it was shown in [5, p. 840], in the case of finite orbits the averaging over orbits gives the same map E_Γ. Indeed, let $T := \{g_1, \ldots, g_N\}$ be representatives of left cosets of Γ_x in Γ. Then

$$
\begin{aligned}
M_\Gamma(\varphi_x) &= M_\Gamma\left(\sum_{g_k \in T} \varphi_x(g_k)\chi_{g_k H}\right) = \sum_{g_k \in T} \varphi(g_k x)M_\Gamma(\chi_{g_k H}) \\
&= \sum_{g_k \in T} \varphi(g_k x)M_\Gamma(\chi_H) = \text{const} \cdot \sum_{g_k \in T} \varphi(g_k x),
\end{aligned}
$$

where χ are the characteristic functions. Taking $f = 1$ we obtain const $= 1/N$. This is a proof for amenable groups. For an almost periodic function it is evident

that the averaging over the finite orbit gives a constant function, which is a convex combination of N translations of the original function.

Definition 5. If E_Γ is well-defined one can equip $C(X)$ with the structure of a pre-Hilbert C*-module over $C_\Gamma(X)$ by the formula

$$\langle \varphi, \psi \rangle_E := E_\Gamma(\varphi^* \psi).$$

Remark 6. Let us remind that a conditional expectation $E : A \to B \subset A$ is said to be *faithful* if $E(x^*x) = 0$ implies $x = 0$ for $x \in A$. This is always true for E_Γ.

Proposition 7. [5, Prop 1.1] *Let A be a C*-algebra and $E : A \to B \subseteq A$ be a conditional expectation with fixed point set B. There then exists a finite real number $K \geq 1$ such that the mapping $(K \cdot E - \mathrm{id}_A)$ is positive if and only if E is faithful and the (right) pre-Hilbert B-module $\{A, E(\langle \cdot, \cdot \rangle_A)\}$ is complete with respect to the norm $\|E(\langle \cdot, \cdot \rangle_A)\|_B^{1/2}$ (where $\langle a, b \rangle_A = a^*b$ for $a, b \in A$).*

Theorem 8. (combination of [5, Lemmas 2.9 and 2.11 and Theorem 2.12]) *Suppose a discrete group Γ acts on a locally compact Hausdorff space X in such a way that $k := \max\{\#(\Gamma x) : x \in X\} < +\infty$. Then E_Γ is well-defined.*

If X is a normal space, then $K(E_\Gamma) = k$. Hence, by Prop. 7, $C(X)$ is a Hilbert C*-module over $C_\Gamma(X)$.

Theorem 9. [5, Theorem 3.6] *Suppose, X is a compact Hausdorff space, Γ is a discrete group acting in a uniformly continuous manner on X, and all orbits consist of the same finite number of points. Then the Hilbert $C_\Gamma(X)$-module $\{C(X), E_\Gamma(\langle \cdot, \cdot \rangle)\}$ is finitely generated and projective.*

A Hilbert B-module M is called *self-dual* if there is a natural isomorphism of M onto the module M' of all anti-B-linear bounded functionals on M. If B is unital, then any finitely generated projective module is self-dual. If $M'' = M$, then the module M is called (B-)*reflexive*.

Theorem 10. [5, Theorem 3.9] *Let X be a compact Hausdorff space and Γ a discrete group acting on X uniformly continuously. If there exists an integer n such that the cardinalities of all orbits do not exceed n and the number of orbits whose cardinality is strictly less than n is finite, then the Hilbert $C_\Gamma(X)$-module $\{C(X), E_\Gamma(\langle ., . \rangle)\}$ is $C_\Gamma(X)$-reflexive.*

The following final form of the previous theorem was obtained by V. Seregin.

Theorem 11. [17] *Let X be a compact Hausdorff space and Γ a discrete group acting in a uniformly continuous manner on X. If Γ acts in such a way that the cardinalities of all orbits are uniformly bounded, then the Hilbert $C_\Gamma(X)$-module $\{C(X), E_\Gamma(\langle ., . \rangle)\}$ is $C_\Gamma(X)$-reflexive.*

From here on X is **compact** and **separable**, hence metrizable and normal.

3. INVERSE THEOREMS

In this section we will obtain some theorems, inverse to the main statements of the previous section.

Let us remind the following statement.

Theorem 12. *Let Z be an infinite compact Hausdorff space. There is no (complete) Hilbert space structure on $C(Z)$ with the norm satisfying $\|f\|_h \leq \|f\|_{C(Z)}$.*

Proof. By [3, Theorem II.2.5] the two norms have to be equivalent. But $C(Z)$ is reflexive if and only if Z is finite [3, IV.13.15]. $\square$

Theorem 13. *Let Γ be a discrete group acting uniformly continuously on a compact separable Hausdorff topological space X. Then the following properties are equivalent*

    (i) *cardinalities of all orbits are uniformly bounded;*

    (ii) *E_Γ is well-defined and there exists a finite real number $K \geq 1$ such that the mapping $(K \cdot E - \mathrm{id}_A)$ is positive;*

    (iii) *E_Γ is well-defined and the corresponding pre-Hilbert module is complete;*

    (iv) *E_Γ is well-defined and the corresponding pre-Hilbert module is reflexive.*

Proof. (i)$\Rightarrow$(iv) is Theorem 11.

(iv)$\Rightarrow$(iii) is evident.

(ii)$\Leftrightarrow$(iii) is Proposition 7.

In order to prove (ii)$\Rightarrow$(i), let us suppose the opposite.

Notice that if $Y \subset X$ is a closed invariant subspace, then the pre-Hilbert $C_\Gamma(Y)$-module $C(Y)$ is also complete (see Lemma 19).

Assume first that there is an infinite orbit Γx. Taking $Y = \overline{\Gamma x}$ and applying the remark above we obtain that $C(Y)$ could be equipped with the Hilbert space structure, because $C_\Gamma(Y) \cong \mathbb{C}$. Contradiction with Theorem 12.

The other case is the case of finite orbits of arbitrary cardinality. Here we can argue as in [5, Theorem 2.12]. By Proposition 7 there exists a real number $K \geq 1$ such that $K \cdot E_\Gamma(\varphi) \geq \varphi$ for any positive function $\varphi : X \to \mathbb{C}$. Choose a point $x \in X$ with $k_x := \#(\Gamma x) > K$. Then we can obtain a neighborhood U_x of x such that $g_i U_x \cap g_j U_x = \emptyset$ for $i \neq j$ and $\{g_1 = 1, g_2, \ldots, g_m\} = T_x := \Gamma/\Gamma_x$, where Γ_x is the isotropy subgroup. For a continuous non-negative function f with support inside U_x (cf. [16, Th. V.17.4] for the existence) we have the inequality

$$K \cdot E_\Gamma(f)(x) = K \frac{1}{k_x} \cdot \sum_{g_a \in T_x} f(g_a x) < f(x).$$

This contradicts the definition of K. $\square$

Theorem 14. *Let Γ be a discrete group acting uniformly continuously on a compact separable Hausdorff connected topological space X. Suppose that the averaging over orbits defines a structure of a self-dual Hilbert module on $C(X)$ over the algebra of invariant continuous functions $C_\Gamma(X)$. If all orbits are finite, then they have the same cardinality.*

Proof. Let $W \subset X$ be the open subset consisting of all points from orbits of the maximal length k. Suppose $W \neq X$ and take $x \in \overline{W} \cap (X \setminus W)$ (since X is connected, this is possible).

Then for any neighborhood U of x there exist $y \in U \cap W$ and $g \in \Gamma_x$ such that $gy \neq y$, where Γ_x is the stabilizer of x. Indeed, suppose the opposite: there exists a neighborhood U such that any element $y \in U \cap W$ is Γ_x invariant, i.e. $\Gamma_x \subset \Gamma_y$. Then $\#(\Gamma x) \geq \#(\Gamma y)$, we have a contradiction.

So, we can choose a sequence of elements $y_n \in W$ of different orbits (due to finiteness of orbits) and a sequence $g_n \in \Gamma_x$ such that $y_n \to x$ $(n \to \infty)$ and $y'_n := g_n y_n \neq y_n$. Since X is normal, we can choose (passing to a subsequence, if necessary) two sequences of open neighborhoods $W \supset V_n \ni y_n$ and $V'_n = g_n V_n \ni y'_n$ non-intersecting with each other and with x and satisfying the following condition: for any neighborhood V of x the sets V_n and V'_n are inside V for all large n. Take

continuous functions

$$\varphi_n : X \to [0,1], \qquad \varphi_n(y_n) = 1, \qquad \operatorname{supp} \varphi_n \subset V_n,$$

$$\varphi_n' : X \to [-1,0], \qquad \varphi_n'(y_n') = -1, \qquad \operatorname{supp} \varphi_n' \subset V_n',$$

$$\varphi_n'(g_n z) = -\varphi_n(z).$$

Define the function $\varphi : X \to [-1,1]$ to be equal to φ_n on V_n, to φ_n' on V_n', and vanish in other points. It is continuous in all points except of x. Then $F(f) := E_\Gamma(\varphi f)$ is a $C_\Gamma(X)$ functional on $C(X)$. It can not be presented as $E_\Gamma(g f)$ with $g \in C(X)$. Indeed, the desired g has to coincide with φ on the open invariant set W. Contradiction with self-duality. $\qquad \square$

Lemma 15. *Suppose X is a compact normal space and $C(X)$ is a finitely generated module over $C_\Gamma(X)$. Then the cardinality of each orbit is finite.*

Proof. Suppose there exists an infinite orbit. Then the dimension (algebraic) of the space of continuous functions on it (or its closure) is infinite. Hence it can not be finitely generated over constants (invariant functions). By the Tietze theorem all of these functions are the restrictions of some continuous functions to the closure of the orbit. Hence the same statement is true for all functions. $\qquad \square$

Definition 16. A C*-algebra A is called *module-infinite* (MI) if each countably generated Hilbert A-module is projective finitely generated if and only if it is self-dual. Let us remark that a projective finitely generated module over a unital algebra is always self-dual. We will give a description of MI algebras in Section 6.

Theorem 17. *Let Γ be a discrete group acting uniformly continuously on a compact separable Hausdorff connected topological space X. Suppose the averaging over orbits defines a structure of a self-dual Hilbert module on $C(X)$ over the algebra $C_\Gamma(X)$. If $C_\Gamma(X)$ is MI, then all orbits have the same cardinality.*

Proof. By Lemma 15 the orbits are finite. Suppose that there are orbits of cardinalities k and smaller. Let $W \subset X$ be the open subset formed by all points from the orbits of cardinality $\geq k$. Then $W \neq X$ and we can take $x \in \overline{W} \cap (X \setminus W)$ (since X is connected, it is possible). So $\#(\Gamma x) = m < k$. The proof could be completed as for Theorem 14. $\qquad \square$

4. LOCALIZATION AND GLOBALIZATION

It turns out that we can deduce facts about cardinality of orbits from a purely algebraic data.

Let us start from the following evident lemma, which formulates an idea from the previous section in a more general way.

Lemma 18. *Let $C(X)$ be m-generated over $C_\Gamma(X)$. If Y is a closed Γ-invariant subspace of X, then $C(Y)$ is m-generated over $C_\Gamma(Y)$.*

Lemma 19. *Suppose, X is a compact separable Hausdorff space, E_Γ is well-defined and the corresponding pre-Hilbert module is complete. If $Y \subset X$ is a closed invariant subspace, then the pre-Hilbert $C_\Gamma(Y)$-module $C(Y)$ is also complete.*

Proof. By Proposition 7 there exists a real number $K \geq 1$ such that $K \cdot E_\Gamma(\varphi) \geq \varphi$ for any positive function $\varphi : X \to \mathbb{C}$. Let $\widehat{\varphi} : Y \to \mathbb{C}$ be a positive function. By the

Tietze theorem, there is a positive continuous extension of $\widehat{\varphi}$ up to $\varphi : X \to \mathbb{C}$. We have

$$K \cdot E_\Gamma(\varphi) \geq \varphi, \qquad K \cdot E_\Gamma(\varphi)|_Y \geq \varphi|_Y, \qquad K \cdot E_\Gamma(\widehat{\varphi}) \geq \widehat{\varphi}.$$

Applying Proposition 7 in the opposite direction we complete the proof. $\square$

Lemma 20. *If an orbit $Y = \Gamma x$ consists exactly of k points, then $C(Y)$ is the free k-generated module over $C_\Gamma(Y)$.*

Proof. $C_\Gamma(Y) \cong \mathbb{C}$ and $C(Y) \cong \mathbb{C}^k$ as a $\mathbb{C}$-module. $\square$

Lemma 21. *If an orbit $Y = \Gamma x$ is infinite, then $C(\overline{Y})$ is not a finitely generated module over $C_\Gamma(\overline{Y})$.*

Proof. $C_\Gamma(\overline{Y}) \cong \mathbb{C}$, and the algebra of continuous functions on an infinite compact Hausdorff space is $\mathbb{C}$-infinite. $\square$

We need the following statement.

Proposition 22. *Let $C(X)$ be m-generated over $C_\Gamma(X)$, where X is a compact separable Hausdorff space. Then the cardinality of each orbit does not exceed m.*

Proof. The statement is a combination of Lemmas 18, 20, and 21. $\square$

Theorem 23. *Suppose, an action of discrete group Γ on a compact separable Hausdorff space X is uniformly continuous. Let $C(X)$ be a finitely generated projective module over $C_\Gamma(X)$. Then all orbits have the same finite cardinality.*

Proof. By Proposition 22 the cardinality of orbits is bounded by m. Hence, by Theorem 8 E_Γ is well-defined and the pre-Hilbert module $C(X)$ over $C_\Gamma(X)$ is complete. Since it is finitely generated projective, it ia self-dual. So, we have both properties and we do not need MI to complete the proof as in Theorem 17. $\square$

5. Existence theorem

Starting from here we will concentrate on some specific properties of an action of an (infinite) discrete group G on a compact Hausdorff separable space X. Since such spaces are metrizable, we may fix a metric ρ on X. Of course, the considerations below can be formulated using topology and neighborhoods instead of the distance and ε-neighborhoods.

In the present section we will obtain some sufficient conditions for E_Γ to be well-defined.

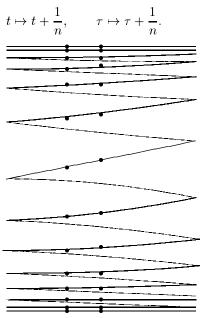Let us start with a couple of examples.

Example 24. Let $X \subset \mathbb{R}^3$ consist of 2 circles

$$S_\pm : \qquad \begin{cases} x &= \cos 2\pi t \\ y &= \sin 2\pi t \qquad t \in [0,1], \\ z &= \pm 1, \end{cases}$$

and non-uniform spiral

$$\Sigma : \qquad \begin{cases} x &= \cos 2\pi \tau \\ y &= \sin 2\pi \tau \qquad \tau \in [0,1]. \\ z &= \frac{2}{\pi} \cdot \arctan \tau, \end{cases}$$

Let the generator g of $\Gamma = \mathbb{Z}$ act on all three components by

$$t \mapsto t + \frac{1}{n}, \qquad \tau \mapsto \tau + \frac{1}{n}.$$

Since $\mathbb{Z}$ is amenable, the pointwise value of E_Γ is defined. If $\varphi : X \to \mathbb{R}_+ \subset \mathbb{C}$ is a non-negative continuous function with

$$\varphi|_{S_+} = 0, \qquad \varphi|_\Sigma = 1 \quad \text{for} \quad t \leq 0, \qquad \varphi|_{S_-} = 1,$$

then $E_\Gamma(\varphi)|_{S_+} = 0$ and $E_\Gamma(\varphi)|_\Sigma > 0$, so that the function $E_\Gamma(\varphi)$ is discontinuous.

In this example the action is not uniformly continuous and we always emphasized that these are the bad ones. Unfortunately, the next example shows that the condition on action to be uniformly continuous does not help much either.

Example 25. Let X and Γ be as in the previous example, but the action be defined by

$$t \mapsto t + \alpha, \qquad \tau \mapsto \tau + \alpha,$$

where α is a positive irrational number. Then the isotropy group of each point of X is trivial. Hence, the condition of uniform continuity holds automatically. Nevertheless, a discontinuous $E_\Gamma(\varphi)$ corresponds to the same function φ.

Remark 26. Let in any of these examples $x \in \Sigma$. Then for the aforementioned φ the function $\varphi_x : \mathbb{Z} \to \mathbb{C}$ is not almost periodic. Of course, for amenable $\Gamma = \mathbb{Z}$ this is not an obstruction, but a similar phenomenon can occur for non-amenable groups too.

Now we introduce a condition which is sufficient to overcome these difficulties.

Definition 27. An action of a group Γ on a metric space X is called (Lyapunov) *stable* if for any $\varepsilon > 0$ and any $x \in X$ there exist $\delta > 0$ such that

$$\rho(gx, gy) < \varepsilon \quad \text{for any} \quad g \in \Gamma \quad \text{if} \quad \rho(x, y) < \delta.$$

The following statement is evident.

Proposition 28. *If an action is stable, then it is uniformly continuous.*

Proposition 29. *Let a discrete group Γ act on a compact separable Hausdorff space X in a stable way. If $\varphi : X \to \mathbb{C}$ is a continuous function, then for any $x \in X$ the function $\varphi_x : \Gamma \to \mathbb{C}$, $\varphi_x(g) := \varphi(gx)$, is almost periodic, and therefore its mean $M(\varphi_x)$ is well-defined.*

Proof. Let $\varepsilon > 0$ be arbitrary. Since X is compact and φ is continuous, it is uniformly continuous, in particular, there exists $\widetilde{\varepsilon}$ such that

$$(1) \qquad\qquad |\varphi(y) - \varphi(z)| < \varepsilon \qquad \text{if} \quad \rho(z, y) < \widetilde{\varepsilon}.$$

For each point gx of the orbit let us find an ε_g-neighborhood U_g with the following property:

$$(2) \qquad\qquad \rho(hgx, hy) < \widetilde{\varepsilon} \qquad \text{for any } h \in \Gamma \text{ if } y \in U_g.$$

Let $U_{g_1}, \ldots, U_{g_m}$ be a finite subcovering of the (closure of) the orbit Γx. Then the functions $L_{g_s}\varphi_x$, $s = 1, \ldots, m$, form an ε-net for the set $L_g\varphi_x$, $g \in \Gamma$, with respect to the uniform norm. Indeed, for any $g \in \Gamma$ we can choose $s_0 \in \{1, \ldots, m\}$ such that $y = gx \in U_{g_{s_0}}$. Then by (1) and (2) we have

$$\sup_{h \in \Gamma} |(L_g\varphi_x)(h) - (L_{g_{s_0}}\varphi_x)(h)| = \sup_{h \in \Gamma} |\varphi(hgx) - \varphi(hg_{s_0}x)| < \varepsilon.$$

$\square$

Theorem 30. *Let a discrete group Γ act on a compact separable Hausdorff space X. If the action is stable, then E_Γ is well-defined.*

Proof. By Proposition 29 we only need to verify continuity of the mean $M(\varphi_x)$ in $x \in X$. Let $x \in X$ and $\varepsilon > 0$ be arbitrary. Let us remind (cf. [6, pp. 250–251]) that we can choose $h_1, \ldots, h_p \in \Gamma$ in such a way that the uniform distance on $\Gamma \times \Gamma$ between the function

$$\frac{1}{p} \sum_{j=1}^{p} D_{h_j}\varphi_x : \Gamma \times \Gamma \to \mathbb{C} \qquad (\text{where} \quad (D_h\psi)(g_1, g_2) := \psi(g_1 h g_2))$$

and some constant is less then ε. In this case the uniform distance satisfies the inequality

$$\left\| M(\varphi_x) - \frac{1}{p} \sum_{j=1}^{p} D_{h_j}\varphi_x \right\|_u < 2\varepsilon.$$

Let us choose a δ-neighborhood U of x such that

$$|\varphi(gy) - \varphi(gx)| < \varepsilon, \qquad \text{for any } g \in \Gamma,\ y \in U.$$

This neighborhood can be found as in the proof of Proposition 29: $\varepsilon \to \widetilde{\varepsilon} \to \delta$. More precisely, first we can find $\widetilde{\varepsilon} > 0$ such that $|\varphi(y) - \varphi(z)| < \varepsilon$ whenever $\rho(y, z) < \widetilde{\varepsilon}$ (using compactness of X). Then, by stability of the action, we can find for our fixed x a number $\delta > 0$ such that $\rho(gy, gx) < \widetilde{\varepsilon}$ for any $g \in \Gamma$ whenever $\rho(y, x) < \delta$.

Then for any $y \in U$ one has

$$\left(\frac{1}{p} \sum_{j=1}^{p} D_{h_j} \varphi_y \right) (g_1, g_2) =$$

$$= \left(\frac{1}{p} \sum_{j=1}^{p} D_{h_j} \varphi_x \right) (g_1, g_2) + \frac{1}{p} \sum_{j=1}^{p} \left(\varphi_y(g_1 h_j g_2) - \varphi_x(g_1 h_j g_2) \right)$$

$$= \left(\frac{1}{p} \sum_{j=1}^{p} D_{h_j} \varphi_x \right) (g_1, g_2) + \frac{1}{p} \sum_{j=1}^{p} \left(\varphi(g_1 h_j g_2 y) - \varphi(g_1 h_j g_2 x) \right).$$

Each term of the second summand is less then ε. Hence, the second summand is less then ε. Thus,

$$\left\| M(\varphi_x) - \frac{1}{p} \sum_{j=1}^{p} D_{h_j} \varphi_x \right\|_u < 3\varepsilon.$$

Therefore, considering $M(\varphi_x)$ as an arbitrary constant, we have

$$\left\| M(\varphi_y) - \frac{1}{p} \sum_{j=1}^{p} D_{h_j} \varphi_x \right\|_u < 6\varepsilon.$$

Finally,

$$|M(\varphi_y) - M(\varphi_x)| < 9\varepsilon.$$

$\square$

6. Commutative MI algebras

Definition 31. A commutative unital C^* algebra $A = C(Y)$ is said to be DI (*divisible infinite*) if for any infinite sequence u_i of its elements of norm $1 \geq \|u_i\| \geq C > 0$ there exist a subsequence $i(k)$ and elements $0 \leq b_k \in A$ of norm 1, such that
    (i) the supports of b_k in Y are pairwise disjoint, and
    (ii) for each k there exist points y_k, y_k' such that $b_k(y_k) = 1$, $y_k' \notin \cup_j \operatorname{supp} b_j$, $|u_{i(k)}(y_k')| \geq \delta$, $|u_{i(k)}(y_k)| \geq \delta$, and the sequences $\{y_k\}$ and $\{y_k'\}$ have a common accumulation point. In particular, $\sum_k b_k^s$ is a discontinuous function for any integer $s \geq 1$.

Theorem 32. *If a commutative unital C^*-algebra A is DI, then it is MI.*

Proof. We have to prove that any countable generated self-dual Hilbert A-module $\mathcal{M}$ is finitely generated projective. By the Kasparov stabilization theorem [9] (see also [11, 12]) one has $\mathcal{M} \oplus l_2(A) \cong l_2(A)$, where $l_2(A)$ is the standard Hilbert module (see, e.g. [12]). Denote by $p : l_2(A) \to \mathcal{M} \subset l_2(A)$ the corresponding orthoprojection. Let $p_j : l_2(A) \to E_j \cong A^j$ be the orthoprojection on the first j standard summands of $l_2(A)$. Two opportunities can arise: 1) $\|(1 - p_j)p\| \to 0$ as $j \to \infty$, and 2) the opposite case.

1) Let us show that in this case $\mathcal{M}$ is finitely generated projective. One can argue as in [13]: for a sufficiently large j the operator

$$J(x) = \begin{cases} p_j(x) & \text{if } x \in \mathcal{M}, \\ x & \text{if } x \in \mathcal{M}^{\perp} \cong l_2(A), \end{cases}$$

is close to identity, hence an isomorphism. It maps $\mathcal{M}$ isomorphically onto a direct summand of E_j.

2) In this case (changing the standard decomposition, if necessary) we can find a sequence $j(k)$ such that for each k there exists an element $z_k \in \mathcal{M}$ of norm 1 such that its $j(k)$-th entry $z_k^{j(k)}$ with respect to the standard decomposition has the norm greater than some fixed $C > 0$. According to Def. 31 let us choose functions b_k (for the sake of notational brevity we assume that we do not need to pass to a subsequence the second time). Then the formula $\beta(x) = \sum_k b_k x^{j_k}$ defines a functional on $l_2(A)$. It is evident that it does not admit an adjoint on $l_2(A)$, because $b_k \not\to 0$ as $k \to \infty$. Let us show that there is no adjointable functional α on $l_2(A)$ such that $\alpha|_{\mathcal{M}} = \beta|_{\mathcal{M}}$, and hence, $\beta|_{\mathcal{M}}$ is a non-adjointable functional on $\mathcal{M}$ and $\mathcal{M}$ is not a self-dual module. Indeed, suppose, there exists an element $a = (a_1, a_2, \dots) \in \mathcal{M} \subset l_2(A)$ such that $\alpha(x) := \sum_i a_i x^i = \beta(x)$ for any $x \in \mathcal{M}$. Then

$$\langle a, a \rangle \leq \sum_k b_k^2 \tag{3}$$

(the last element is a bounded measurable, but discontinuous function, see Def. 31 (ii)), because for any continuous positive function f_k, which is equal to 1 on $\cup_{j \leq k} \operatorname{supp} b_j$ and 0 on $\cup_{j > k} \operatorname{supp} b_j$ (the existence follows from normality of Y) one has $\beta f_k = \alpha f_k + (\beta - \alpha) f_k$. But βf_k is an element of $l_2(A)$ (or, more precisely, an adjointable functional $\beta f_k(x) = \sum_{j \leq k} b_k x^{j_k}$). Let $b f_k$ be the corresponding element. Then $b f_k = a f_k + (b - a) f_k$ is the decomposition corresponding to $l_2(A) = \mathcal{M} \oplus l_2(A)$. Indeed, $a \in \mathcal{M}$, and for any $x \in M$, $\langle (b-a) f_k, x \rangle = 0$, because $\alpha(x) = \beta(x)$ for those x. The Pythagorean theorem for Hilbert modules shows that $\langle a, a \rangle f_k^2 \leq \sum_{j \leq k} b_j^2$. Taking all f_k's one obtains (3).

On the other hand,

$$\langle a, a \rangle (y_i) \geq \langle a, z_i \rangle \langle z_i, a \rangle (y_i) = \sum_{m,k} b_m z_i^{j(m)} \overline{z_i^{j(k)}} b_k (y_i) = b_i^2(y_i) z_i^{j(i)}(y_i) \geq \delta^2.$$

But from (3) one obtains $\langle a, a \rangle(y_i') = 0$. Hence, $\langle a, a \rangle$ does not belong to A. A contradiction.   □

Now we describe consequences of this fact for the case studied in the present paper.

Theorem 33. *A commutative separable unital C^*-algebra A is* MI *if and only if its Gelfand spectrum Y has no isolated points.*

Proof. If Y has an isolated point, a separable Hilbert space arises as one of self-dual modules, hence, MI does not hold.

Now, suppose, Y has no isolated points, in particular it is infinite. Since Y is a compact Hausdorff separable space, the topology is generated by some metric ρ. For any given sequence u_i of norm $\geq C$ we can find a sequence of different points $\widetilde{y}_i$ such that $|u_i(\widetilde{y}_i)| > 2C/3$.

Since Y is compact, one can pass to a convergent subsequence $y_k = \widetilde{y}_{i(k)}$. For a convergent sequence by induction we can choose ε_k such that the corresponding ε_k-neighborhoods U_k of y_k are pairwise disjoint. Then we can choose $y_k' \neq y_k$ inside these neighborhoods so close to y_k that $|u_{i(k)}| \geq C/2$. After that one can choose functions b_k such that $b_k(y_k) = 1$ and $\operatorname{supp} b_k \subset U_k \setminus y_k'$. It remains to take $\delta = C/2$.   □

Corollary 34. *Suppose, X is a compact connected separable Hausdorff Γ-space. Then $C_\Gamma(X)$ is MI if and only if X/Γ has at least two separated points.*

Proof. The Gelfand spectrum Y of $C_\Gamma(X)$ is a quotient space of X/Γ with the respect to the equivalence of non-separated points. Since X is connected, Y is connected too. So, by Theorem 33, $C_\Gamma(X)$ is MI if and only if Y is not reduced to one point. $\qquad\square$

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