Program

1. Metric spaces and topological spaces: basic concepts.
2. Continuous maps. Connectedness and arc connectedness.
3. Hausdorff and normal spaces. The refinement theorem. Uryson's lemma.
4. Partition of unity on normal spaces. Normality of Hausdorff compacts. Continuous bijection onto a Hausdorff space.
5. Definitions of a manifold and of a smooth map. Examples and non-examples.
6. Existence of an open balls atlas and of smooth partition of unity.
7. Definitions of tangent vectors and their equivalence.
8. Definitions of tangent map and their equivalence. Regular values.
9. Definition of a submanifold. Submanifolds and embeddings.
10. Pre-image of a regular value as a submanifold.
11. Whitney theorem. Tangent bundle.
12. Manifolds with boundary.
13. Orientation, examples. Orientation of boundary.
14. Riemannian metric, existence, bilinear forms on tangent spaces.
15. Lie groups: main theorems. Classical matrix groups as Lie groups.
16. Tensor fields and basic tensor operations. Symmetric and alternating tensors.
17. Exterior product. Bases. Differential forms of maximal degree.
18. Fiber bundles, morphisms, cocycle approach.
19. Vector bundles. Example: tensor bundle. Principal bundles.
20. Operations on vector bundles. Tensor fields as sections of vector bundles.
21. Covariant differentiation on $\mathbb{R}^n$.
22. Properties of a connection. Definition of ∇ by its properties.
23. Levi-Civita connection, existence and uniqueness theorem, properties.
24. The parallel transport w.r.t an affine connection.
25. Definition of a geodesic, existence and uniqueness theorem.
26. Theorem about short geodesics.
27. Exterior derivative and its properties. Closed and exact forms. De Rham cohomology.

28. Exterior derivatives and pull-back. Pull-back in cohomology.

29. Differential forms on $M \times I$. Cohomology and homotopies.

30. Definition of integral of a form. Its properties. The general Stokes formula.

31. Riemann curvature tensor (two approaches). Symmetries of the Riemann curvature tensor (without proof).

32. Flatness of an affine connection. Transport along infinitely small coordinate square and along two homotopic paths.

33. Lie algebra of a Lie group.

34. Maurer-Cartan forms. Trivializations of TG of a Lie group G.

35. Vertical tangent bundle. Ehresmann connection. Existence.

36. Pull-back of an Ehresmann connection. Parallel transport for Ehresmann connections.

37. Connector and its properties.

38. Koszul connection(s). Relation between Ehresmann connection and Koszul connection.

39. Fiberwise inner products. Orthogonal complement of a subbundle. Stable trivializations.

40. Homotopies and isomorphisms of vector bundles.

41. Equivalences of bundles. Definitions of reduced K-groups. Grothendieck group of a semigroup. Cancellation property.

42. Definitions of K-groups. Their connection with reduced K-groups. Homotopy properties.

List of problems

1. Suppose that $f : M \to N$ is smooth and $Q_0 \in N$ is a regular value of f. Then $M_{Q_0} := f^{-1}(Q_0)$ is a smooth submanifold $\dim M_{Q_0} = \dim M - \dim N$. As a local coordinates in some neighborhood on M_{Q_0} one can take some $(m-n)$ coordinates of M.

2. Give an example of immersion, which is bijective on its image, but is not an embedding.

3. Find an example of smooth homeomorphism, which is not a diffeomorphism.

4. Find an example of a manifold and two non-compatible smooth structures on it, i.e., two smooth atlases (U_i, φ_i) and (V_j, ψ_j) such that $\{(U_i, \varphi_i), (V_j, \psi_j)\}$ is not a smooth atlas.

5. Let $f : X \to X$ be a continuous self-map of a Hausdorff space. Prove that the set of fixed points $F_f := \{x \in X \mid f(x) = x\}$ is closed.

6. Find an example of connected space, which is not arc-connected.

7. Prove that ny interval $[a, b] \subset \mathbb{R}$ is connected and arc connected.

8. Prove that (a, b), $[a, b)$ and $[a, b]$ (subsets of real line) are pair-wise non-homeomorphic.

9. Give an example of a continuous bijection, which is not a homeomorphism.

10. Suppose, $X = F_1 \cup F_2$, where F_1 and F_2 are closed subsets, and $f : X \to Y$ is a map. Then f is continuous iff $f|_{F_1} : F_1 \to Y$ and $f|_{F_2} : F_2 \to Y$ are continuous.

11. A connected orientable manifold can be oriented exactly in two ways.

12. A path *changing the orientation* is a closed path ($\gamma(0) = \gamma(1)$) such that there exists a collection of charts $U_1, \ldots, U_k$, which cover this path, each chart intersects only with its two neighboring charts, the intersections are connected, and all Jacobians of transition maps are positive, except for one. Prove that a manifold is not orientable iff there exists a changing the orientation path for it.

13. A *local orientation* is a choice of orientation (i.e., a basis) in each tangent space. A local orientation is *locally constant,* if, for each connected chart U the standard basis ∂_i defines a local orientation (over this chart), which is either the same as the local orientation in all points, or is the opposite to it in all points. Prove that a (connected) manifold is orientable iff it has a locally constant local orientation.

14. Prove that spheres S^n, for any n , and the torus T^2 are orientable.

15. Prove that any complex analytical manifold is orientable (as a real manifold).

16. Prove that a Möbius strip and the projective plane $\mathbb{R}P^2$ are non-orientable manifolds.

17. Prove that $T_e Sp(2n, \mathbb{K})$ is defined by $JA^T J = A$.

18. Prove that $T_e U(n)$ is defined by $\overline{A}^T = -A$.

19. Prove that $T_e SO(n)$ is defined by $A^T = -A$.
20. Prove that $T_e O(n)$ is defined by $A^T = -A$.
21. Prove that the matrix group $SO(n)$ is a Lie group and closed Lie subgroup of $GL(n, \mathbb{R})$.
22. Prove that the matrix group $SL(n, \mathbb{K})$ is a Lie group and closed Lie subgroup of $GL(n, \mathbb{K})$.
23. Prove that the matrix group $U(n)$ is a Lie group and closed Lie subgroup of $GL(n, \mathbb{C})$.
24. Prove that the matrix group $O(n)$ is a Lie group and closed Lie subgroup of $GL(n, \mathbb{R})$.
25. Prove that $A \in Sp(2n, \mathbb{K})$ iff $A^T J A = J$, where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$.
26. Let a Lie group H be an open subgroup of a Lie group G. Prove that H is closed.
27. Prove that any tensor of type $(1, 1)$, which is invariant under orthogonal coordinate changes, is a scaling of δ^i_j (i.e. is equal to $\lambda \delta^i_j$).
28. Prove that any tensor with $p + q = 3$ invariant w.r.t. any coordinate changes is equal to 0.
29. Prove that C^i_i, $C^i_j C^j_i$, $C^i_j C^j_k C^k_i$, can be expressed in terms of coefficients of the polynomial $\det(C - \lambda E)$.
30. Show by example that a transposition of an upper and a lower indexes is not a tensor operation. Consider the case of a tensor of type $(1, 1)$ (linear operator). Conclude in particular that the property of a matrix of an operator to be symmetric $C^i_j = C^j_i$ depends on coordinate system.
31. Suppose that a tensor field X is of type $(1, 0)$ and W is of type $(0, 1)$. Find the rank of $X \otimes W$.
32. Represent the trace of a matrix as a result of tensor operations.
33. Represent the determinant of a matrix as a result of tensor operations.
34. Find the type of tensors formed by coefficients of

    (a) vector product,

    (b) mixed (triple) product

    of vectors in $\mathbb{R}^3$. Prove that these tensors are obtained from each other by index raising and lowering.
35. Consider the Möbius band E_M as the following quotient space of $\mathbb{R} \times (-1, 1)$:

$$E_M = (\mathbb{R} \times (-1, 1))/\sim, \text{ where } (x, t) \sim (x + 2\pi n, (-1)^n t), \quad n \in \mathbb{Z}.$$

    For

$$S^1 = \mathbb{R}/\approx, \text{ where } x \approx x + 2\pi n, \quad n \in \mathbb{Z},$$

    define $\pi : E_M \to S^1$ by $\pi([x, t]) = [x]$. Prove that this is a fiber bundle. Find an appropriate cocycle with $G = \mathbb{Z}_2$, $F = (-1, 1)$.

36. Prove that $\pi : \mathbb{R} \to S^1$, $S^1 \subset \mathbb{C}$, $\pi(t) = e^{2\pi i t}$, is a covering with $F = \mathbb{Z}$.

37. Prove that $\pi : S^1 \to S^1$, $S^1 \subset \mathbb{C}$, $\pi(z) = z^2$, is a covering with $F = \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$.

38. Find the appropriate cocycle for the covering $\pi : \mathbb{R} \to S^1$, $S^1 \subset \mathbb{C}$, $\pi(t) = e^{2\pi i t}$.

39. Find the appropriate cocycle for the covering $\pi : S^1 \to S^1$, $S^1 \subset \mathbb{C}$, $\pi(z) = z^2$.

40. Prove that there is no sections for the covering $\pi : \mathbb{R} \to S^1$, $S^1 \subset \mathbb{C}$, $\pi(t) = e^{2\pi i t}$.

41. Prove that there is no sections for the covering $\pi : S^1 \to S^1$, $S^1 \subset \mathbb{C}$, $\pi(z) = z^2$.

42. Consider S^{2n-1} as the subset of $\mathbb{C}^n$ given by $S^{2n-1} = \{z \in \mathbb{C}^n \colon \|z\| = 1\}$, where $z = (z^1, \dots, z^n)$ and $\|z\| = \sum \overline{z^i} z^i$. Let $S^1 = U(1)$ act on S^{2n-1} by $(a, z) \mapsto az = (az^1, \dots, az^n)$. The quotient (the space of orbits) is $\mathbb{C}P^{n-1}$. We obtain the *Hopf map* $\pi_n : S^{2n-1} \to \mathbb{C}P^{n-1}$. Prove that this is a principal $U(1)$-bundle (*Hopf bundle*).

43. Let $\pi_1 : E_1 \to M$ and $\pi_2 : E_2 \to M$ be vector bundles and let $\Delta : M \to M \times M$ be diagonal map $P \mapsto (P, P)$. Then one can define $\pi_{E_1 \times E_2} : E_1 \times E_2 \to M \times M$. Verify that this is a vector bundle. Prove that the Whitney sum $E_1 \oplus E_2$ is naturally isomorphic to the pull-back $\Delta^* \pi_{E_1 \times E_2}$.

44. Prove that the velocity field of a geodesic of a Levi-Civita connection has constant length (i.e. its parametrization is a scaling of the arc length one).

45. Prove that if two geodesics are tangent to each other in some point (with the same velocity), then they coincide.

46. Prove that a parallel transport of a vector v along a geodesic conserves the angle between v and the curve (i.e., the velocity vector).

47. Describe geometrically the parallel transport for the Levi-Civita connection on a surface in $\mathbb{R}^3$ (projection).

48. Prove that a curve on a surface in $\mathbb{R}^3$ is a geodesics iff its normal (the second derivative for the natural parametrization = parametrization by the arc length) is orthogonal to tangent plane at each point.

49. Find geodesics on the standard sphere S^2 (without direct calculation).

50. Find geodesics on the standard sphere S^2 (direct calculation).

51. Find geodesics on the pseudosphere = the upper half-plane with coordinates (x, y) and the metric $ds^2 = \frac{dx^2 - dy^2}{x^2}$.

52. Prove that if two surfaces in $\mathbb{R}^3$ are tangent to each other (tangent planes coincide) along a curve then two respective parallel transports along this curve coincide.

53. Find the rotation angle for the parallel transport of a vector along the circle being the base of the standard round cone.

54. Find the rotation angle for the parallel transport of a vector along the circle being a parallel of the standard sphere.

55. Find the de Rham cohomology of an interval (a, b) and Eucledean space $\mathbb{R}^n$.
56. Find the de Rham cohomology of the circle S^1.
57. Find the de Rham cohomology of sphere S^2.
58. Find the de Rham cohomology of the plane $\mathbb{R}^2$ without one point.
59. Find the de Rham cohomology of the plane $\mathbb{R}^2$ without two points.
60. Prove the *Poincare lemma*: any closed form on any manifold is locally exact.
61. Prove that the general Stokes formula implies Green's formula from vector calculus.
62. Prove that the general Stokes formula implies divergence (Gauss–Ostrogradsky) theorem from vector calculus,
63. Prove that the general Stokes formula implies the classical Stokes formula from vector calculus:
64. Find the explicit form of the Maurer-Cartan form of $G = SO(2)$.
65. Find $G(\mathbb{N})$, $\mathbb{N} = \{0, 1, 2, \dots\}$.
66. Prove that generally $\mathrm{Vect}_{\mathbb{K}}(X)$ has no cancellation property.