ORTHOGONAL COMPLEMENTS AND ENDOMORPHISMS OF HILBERT MODULES AND C^*-ELLIPTIC COMPLEXES[1]

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1. Introduction

In the present paper we discuss some properties of endomorphisms of C^*-Hilbert modules and C^*-elliptic complexes. The main results of this paper can be considered as an attempt to answer the question: what kinds of good properties can one expect for an operator on a Hilbert module, which represents an element of a compact group? These results are new, but we have to recall some first steps made by us before to make the present paper self-contained.

In §2 we define the Lefschetz numbers "of the first type" of C^*-elliptic complexes, taking values in $K_0(A) \otimes \mathbb{C}$, A being a complex C^*-algebra with unity, and prove some properties of them.

The averaging theorem 3.2 was discussed in brief in [15] and was used there for constructing an index theory for C^*-elliptic operators. In this theorem we do not restrict the operators to admit a conjugate, but after averaging they even become unitary. This raises the following question: is the condition on an operator on a Hilbert module to represent an element of a compact group so strong that it automatically has to admit a conjugate?

The example in section 4 gives a negative answer to this question. Also we get an example of closed submodule in Hilbert module which has a complement but has no orthogonal complement.

In §5 we define the Lefschetz numbers of the second type with values in $HC_0(A)$. We prove that these numbers are connected via the Chern character in algebraic K-theory. These results were discussed in [18] and we only recall them.

In §6 we get similar results for $HC_{2l}(A)$. We have to use in a crucial way the properties of representations.

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2. Preliminaries

We consider the Hilbert C^*-module $l_2(P)$, where P is a projective module over C^*-algebra A with unity (see [10, 4, 13, 15]).

2.1. Lemma. *Let $P^+(A)$ be the positive cone of the C^*-algebra A. For every bounded A-homomorphism $F : l_2(P) \to l_2(P)$ and every $u \in l_2(P)$ we have*

$$\langle Fu, Fu\rangle \leqslant \|F\|^2 \langle u, u\rangle$$

in $P^+(A)$.

Proof. For $c \in P^+(A)$ we have $c \leqslant \|c\| \, 1_A$. So if $\langle u, u\rangle = 1_A$, then

$$\langle Fu, Fu\rangle \leqslant \|Fu\|^2 \, 1_A \leqslant \|F\|^2 \langle u, u\rangle .$$

Let now $\langle u, u\rangle$ be equal to $\alpha \in P^+(A)$, where α is an invertible element of A. We put $v = (\sqrt{\alpha})^{-1} u$. Then $u = \sqrt{\alpha}\, v$ and $\langle v, v\rangle = 1_A$. So

$$\langle Fv, Fv\rangle \leqslant \|F\|^2 \langle v, v\rangle ,$$

$$\langle Fu, Fu\rangle = \sqrt{\alpha} \, \langle Fv, Fv\rangle \left(\sqrt{\alpha}\right)^* \leqslant \sqrt{\alpha} \, \|F\|^2 \langle v, v\rangle \left(\sqrt{\alpha}\right)^* = \|F\|^2 \langle u, u\rangle .$$

Elements u with invertible $\langle u, u\rangle$ are dense in $l_2(A)$ (this is a consequence of Lemma 2 of [4]), so the continuity of the A-product gives the statement for $l_2(A)$. For $l_2(P)$ we have to use the stabilization theorem [10]. ∎

Let us recall the basic ideas of [16, 17].

2.2. Definition. Let $p : F \to X$ be a G-$\mathbb{C}$-bundle over a locally compact Hausdorff G-space X. Let $\Lambda(p^* F, s_F)$ be the well known complex of G-$\mathbb{C}$-bundles (see [5]) with, in general, non-compact support. Let a complex (E, α) represent an element $a \in K_G(X; A)$ (see [15, sect. 1.3]); then $(p^* E, p^* \alpha) \otimes \Lambda(p^* F, s_F)$ has compact support and defines an element of $K_G(F; A)$. We get the *Thom isomorphism* of $R(G)$-modules

$$\varphi = \varphi_A^F : K_G(X; A) \to K_G(F; A).$$

If we pass to K_G^1 by the Bott periodicity [15, 1.2.4], we can define

$$\varphi : K_G^*(X; A) \to K_G^*(F; A).$$

2.3. Theorem. *If X is separable and metrizable, then φ is an isomorphism.*

With the help of this theorem we can define the *Gysin homomorphism* $i_! : K_G(TX;A) \to K_G(TY;A)$ and the *topological index*

$$\text{t-ind}_G^X = \text{t-ind}_{G,A}^X : K_G(TX;A) \to K^G(A)$$

in a way similar to the case $A = \mathbb{C}$ [5]. Here $i : X \to Y$ is a G-inclusion of smooth manifolds and TX, TY are (co)tangent bundles.

We need the following property of the Gysin homomorphism.

2.4. Lemma. *Let $i : Z \to X$ be a G-inclusion of smooth manifolds, N its normal bundle. Then the homomorphism*

$$(di)^* i_! : K_G(TZ;A) \to K_G(TZ;A)$$

is the multiplication by

$$[\lambda_{-1}(N \otimes_{\mathbb{R}} \mathbb{C})] = \sum (-1)^i [\Lambda^i (N \otimes_{\mathbb{R}} \mathbb{C})] \in K_G(Z),$$

where Λ^i are the exterior powers, and we consider $K_G(TZ;A)$ as a $K_G(Z)$-module in the usual way.

2.5. Theorem. *Let* a-ind $D \in K^G(A)$ *be the analytic index of a pseudo differential equivariant C^*-elliptic operator* [15] , $\sigma(D) \in K_G(TX;A)$ *its symbol's class. Then*

$$\text{t-ind}_{G,A}^X \, \sigma(D) = \text{a-ind}\, D.$$

Now for the completeness of this text we recall a generalization of the result of [1]. Let, as above, G be a compact Lie group, X a G-space, X^g the set of fixed points of $g : X \to X$, $i : X^g \to X$ the inclusion.

2.6. Definition. Let E be a G-invariant A-complex on X, $\sigma(E)$ its sequence of symbols (see [15]) , $u = [\sigma(E)] \in K_G(TX;A)$, $\text{ind}_{G,A}^X(u) \in K_0(A) \otimes R(G)$. The *Lefschetz number of the first type* is

$$L_1(g,E) = \text{ind}_{G,A}^X(u)(g) \in K_0(A) \otimes \mathbb{C}.$$

2.7. Theorem. *Using the notation as above we have*

$$L_1(g,E) = (\text{ind}_{1,A}^{X^g} \otimes 1) \left(\frac{i_* u(g)}{\lambda_{-1}(N^g \otimes_{\mathbb{R}} \mathbb{C})(g)} \right).$$

Also we need the following theorem from [12].

2.8. Theorem. *Let M be a countably generated Hilbert A-module. Then we have a G-A-isomorphism*

$$M \cong \oplus_\pi \text{Hom}_G(V_\pi, M) \otimes_{\mathbb{C}} V_\pi,$$

where $\{V_\pi\}$ is a complete family of irreducible unitary complex finite dimensional representations of G, non-isomorphic to each other. In $\text{Hom}_G(V_\pi, M) \otimes_{\mathbb{C}} V_\pi$ the algebra A acts on the first factor and G on the second.

3. An averaging theorem

Let us recall some facts about the integration of operator-valued functions (see [9, §3]). Let X be a compact space, A be a C^*-algebra, $\varphi : C(X) \to A$ be an involutive homomorphism of algebras with unity, and $F : X \to A$ be a continuous map, such that for every $x \in X$ the element $F(x)$ commutes with the image of φ. In this case the integral

$$\int_X F(x)\, d\varphi \quad \in \quad A$$

can be defined in the following way. Let $X = \cup_{i=1}^n U_i$ be an open covering and

$$\sum_{i=1}^n \alpha_i(x) = 1$$

be a corresponding partition of unity. Let us choose the points $\xi_i \in U_i$ and compose the integral sum

$$\sum (F, \{U_i\}, \{\alpha_i\}, \{\xi_i\}) = \sum_{i=1}^n F(\xi_i)\varphi(\alpha_i).$$

If there is a limit of such integral sums then it is called the corresponding integral.

If $X = G$ then it is natural to take φ equal to the Haar measure

$$\varphi : C(X) \to \mathbb{C}, \quad \varphi(\alpha) = \int_G \alpha(g)\, dg$$

(though this is only a positive linear map, not a $*$-homomorphism) and to define for a norm-continuous $Q : G \to L(H)$

$$\int_G Q(g)\, dg = \lim \sum_i Q(\xi_i) \int_G \alpha_i(g)\, dg.$$

If we have $Q : G \to P^+(A) \subset L(H)$, then, since

$$\int_G \alpha_i(g)\, dg \geqslant 0,$$

we get

$$\sum_i Q(\xi_i) \cdot \int_G \alpha_i(g)\, dg \quad \in \quad P^+(A)$$

and

$$\int_G Q(g)\, dg \quad \in \quad P^+(A)$$

(the cone $P^+(A)$ is convex and closed). So we have proved the following lemma.

3.1. Lemma. *Let $Q: G \to P^+(A)$ be a continuous function. Then for the integral in the sense of* [9] *we have*

$$\int_G Q(g)\,dg \geqslant 0. \qquad \blacksquare$$

3.2. Theorem. *Let* GL *be the group of all bounded A-linear automorphisms of $l_2(A)$ (see* [14]*). Let $g \mapsto T_g \quad (g \in G, T_g \in \mathrm{GL})$ be a representation of G such that the map*

$$G \times l_2(A) \to l_2(A), \quad (g,u) \mapsto T_g u$$

is continuous. Then on $l_2(A)$ there is an A-product equivalent to the original one and such that $g \mapsto T_g$ is unitary with respect to it.

Proof. Let $\langle\ ,\ \rangle'$ be the original product. We have a continuous map

$$G \to A, \quad x \mapsto \langle T_x u, T_x v\rangle'$$

for every u and v from $l_2(A)$. We define the new product by

$$\langle u,v\rangle = \int_G \langle T_x u, T_x v\rangle'\,dx,$$

where the integral can be defined in the sense of either of the two definitions from [9, p. 810] because the map is continuous with the respect to the norm of the C^*-algebra. It is easy to see that this new product is a A-sesquilinear map $l_2(A) \times l_2(A) \to A$. Lemma 3.1 shows that $\langle u,u\rangle \geqslant 0$. Let us show that this map is continuous. Let us fix $u \in l_2(A)$. Then

$$x \mapsto T_x(u), \qquad G \to l_2(A)$$

is a continuous map defined on a compact space and so the set $\{T_x(u) | x \in G\}$ is bounded. Hence by the principle of uniform boundness [2, v. 2, p. 309]

$$\lim_{v\to 0} T_x(v) = 0 \tag{1}$$

uniformly with respect to $x \in G$. If u is fixed then

$$\|T_x(u)\| \leq M_u = const$$

and by (1)

$$\|\langle u,v\rangle\| = \|\int_G \langle T_x(u), T_x(v)\rangle'\,dx\| \leq M_u \cdot vol\,G \cdot \sup_{x\in G}\|T_x(v)\| \to 0 \quad (v \to 0).$$

This gives the continuity at 0 and hence everywhere. For $T_x u = (a_1(x), a_2(x), \dots) \in l_2(A)$ the equation $\langle u,u\rangle = 0$ takes the form

$$\int_G \sum_{i=1}^{\infty} a_i(x) a_i^*(x)\,dx = 0.$$

Let A be realized as a subalgebra of the algebra of all bounded operators in the Hilbert space L with inner product $(\ ,\ )_L$. For every $p \in L$ we have

$$\begin{aligned} 0 &= \left(\left(\int_G \sum_{i=1}^{\infty} a_i(x)a_i^*(x)\,dx\right) p,\, p\right)_L \\ &= \int_G \left(\sum_{i=1}^{\infty} a_i(x)a_i^*(x)p,\, p\right)_L dx = \int_G \left(\sum_{i=1}^{\infty}(a_i(x)p,\, a_i^*(x)p)_L\right) dx \end{aligned}$$

(cf. [9]). Hence $a_i(x) = 0$ almost everywhere, and thus $a_i(x) = 0$ for every x because of the continuity, and $T_x u = 0$. In particular, $u = 0$.

Since every T_y is an automorphism, we have (cf. [9])

$$\langle T_y u, T_y v\rangle = \int_G \langle T_{xy}u, T_{xy}v\rangle'\,dx = \int_G \langle T_z u, T_z v\rangle'\,dz = \langle u, v\rangle .$$

Now we will show the equivalence of the two norms and, in particular, the continuity of the representation. There is a number $N > 0$ such that $\|T_x\|' \le N$ for every $x \in G$. So by [9]

$$\|u\|^2 = \|\langle u,u\rangle\|_A = \|\int_G \langle T_x u, T_x u\rangle'\,dx\|_A \le \left(\sup_{x\in G}\|T_x u\|'\right)^2 \le N^2(\|u\|')^2.$$

On the other hand, applying 2.1 and 3.1 we have

$$\begin{aligned} \langle u,u\rangle' &= \int_G \langle T_{g^{-1}}T_g u, T_{g^{-1}}T_g u\rangle'\,dg \le \int_G \|T_{g^{-1}}\|^2 \langle T_g u, T_g u\rangle'\,dg \\ &\le \int_G N^2 \langle T_g u, T_g u\rangle'\,dg = N^2 \int_G \langle T_g u, T_g u\rangle'\,dg = N^2 \langle u,u\rangle . \end{aligned}$$

Then

$$(\|u\|')^2 = \|\langle u,u\rangle'\|_A \le N^2\|\langle u,u\rangle\|_A = N^2\|u\|^2. \qquad \blacksquare$$

3.3. Remark. $l_2(P)$ is a direct summand in $l_2(A)$, so 3.2 holds for $l_2(P)$.

4. Complements and orthogonal complements

Let us recall some preliminary statements.

4.1. Lemma. *1. An A-linear operator $F : M \to H_A$ always admits a conjugate if $M \in \mathcal{P}(A)$ — the category of finitely generated projective modules.*

2. Let $0_A \le \alpha < 1_A$. Then $\|\alpha\| < 1$.

3. Let $\alpha \ge 0$, $\alpha = \beta\beta^*$, $1 - \alpha > 0$. Then $1 - \beta$ is an isomorphism.

Here the strong inequality means that the spectrum of the operator is bounded away from zero.

4.2. Example. Let $A = C[0,1]$, $\{e_i\}$ be the standard basis of H_A. Let

$$\varphi_i(x) = \begin{cases} 0 & \text{on } [0, \frac{1}{i}] \text{ and } [\frac{1}{i-1}, 1], \\ 1 & \text{at } x_i = \frac{1}{2}\left(\frac{1}{i} + \frac{1}{i-1}\right), \\ \text{linear on } [\frac{1}{i}, x_i] \text{ and } [x_i, \frac{1}{i-1}], \end{cases}$$

$i = 2, 3, \dots$. Let

$$h_i = \frac{e_i + \varphi_i e_1}{(1 + {\varphi_i}^2)^{1/2}} \qquad (i = 2, 3, \dots)$$

be an orthonormal system of vectors which generates $H_1 \subset H_A$, $H_1 \cong H_A$. Then $H_1 \oplus \operatorname{span}_A(e_1) = H_A$. Indeed, all $e_i \in H_1 + \operatorname{span}_A(e_1)$, and if

$$x = (\alpha_1, \alpha_2, \dots) \in H_1 \cap \operatorname{span}_A(e_1),$$

$$x = (\alpha_1, 0, \dots) = \sum_{i=2}^{\infty} \beta_i h_i,$$

then all $\beta_i = 0$, $x = 0$. However the module H_1 does not have an orthogonal complement. More precisely we have the following situation. Let $y = \sum_{j=1}^{\infty} \alpha_j e_j$ be in $H_1^{\perp}$. Then $\left\langle \sum_{j=1}^{\infty} \alpha_j e_j, h_i \right\rangle = 0$ for $i = 2, 3, \dots$, so $\alpha_i + \alpha_1 \varphi_i = 0 \quad (i = 2, 3, \dots)$, and $\alpha_i = -\alpha_1 \varphi_i$, hence

$$y = (\alpha_1, -\alpha_1\varphi_2, -\alpha_1\varphi_3, \dots).$$

This is possible if and only if the function α_1 vanishes at 0: $\alpha_1(0) = 0$. If $H_1 \oplus H_1^{\perp} = H_A$, then for some α_1 we have $e_1 = y + \sum_{i=2}^{\infty} \beta_i h_i$. In particular the series $\sum_{i=2}^{\infty} \beta_i \bar{\beta}_i$ converges and

$$1 = \alpha_1 + \sum_{i=2}^{\infty} \frac{\beta_i \varphi_i}{(1 + {\varphi_i}^2)^{1/2}}.$$

But $\|\beta_i\|_A \to 0$, so for

$$\gamma = \sum_{i=2}^{\infty} \frac{\beta_i \varphi_i}{(1 + {\varphi_i}^2)^{1/2}}$$

we get $\gamma(0) = 0$, as well as for α_1. We come to a contradiction.

Let us investigate the involution J which determines a representation of $\mathbb{Z}_2$:

$$J(x) = \begin{cases} x & \text{if } x \in H_1, \\ -x & \text{if } x \in \operatorname{span}_A e_1, \end{cases}$$

This operator does not admit a conjugate. Indeed, let J^* exist. Then $(J^*)^2 = {J^2}^* = \mathrm{Id}$, so J^* is also an involution.

$$\begin{aligned} J^* x = x \quad &\Leftrightarrow \quad (J^* x, y) = (x, y) \quad \forall y \quad \Leftrightarrow \quad (x, Jy) = (x, y) \quad \forall y \quad \Leftrightarrow \\ &\Leftrightarrow \quad (x, (J-1)y) = 0 \quad \forall y \quad \Leftrightarrow x \perp \operatorname{Im}(J-1) \quad \Leftrightarrow \\ &\Leftrightarrow \quad x \perp \operatorname{span}_A(e_1), \end{aligned}$$

and $J^*x = -x \quad \Leftrightarrow \quad x \perp H_1$. But H_1 has no orthogonal complement and so the involution J^* can not be defined. Nevertheless for the A-product averaged by the action of $\mathbb{Z}_2$

$$\langle x, y\rangle_2 = 1/2(\langle x, y\rangle + \langle Jx, Jy\rangle)$$

we get if $x \in H_1$, $y \in \operatorname{span}_A(e_1)$: $\quad \langle x, y\rangle_2 = 1/2(\langle x, y\rangle + \langle x, -y\rangle) = 0$, so the + and − subspaces of the involution are orthogonal to each other, and $J^*_{(2)} = J$.

Let us recall the definition of A-Fredholm operator [11, 13]. The theorem which will be proved is the crucial one for the possibility of construction of Sobolev chains in the C^*-case.

4.3. Definition. A bounded A-operator $F : H_A \to H_A$ admitting a conjugate is called Fredholm, if there exist decompositions of the domain of definition $H_A = M_1 \oplus N_1$ and the range $H_A = M_2 \oplus N_2$ where M_1, M_2, N_1, N_2 are closed A-modules, N_1, N_2 have a finite number of generators, and such that the operator F has in these decompositions the following form

$$F = \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix},$$

where $F_1 : M_1 \to M_2$ is an isomorphism.

4.4. Lemma. *Let $J : H_A \to H_A$ be a self adjoint injection. Then J is an isomorphism. Here injection means an injective A-homomorphism with closed range.*

Proof. Let us consider $J_1 = J : H_A \to J(H_A)$. It is an isomorphism of Hilbert modules admitting a conjugate $J_1^* = J^*|_{J(H_A)} = J|_{J(H_A)}$. Let $J_2 = J(J_1^* J_1)^{-1/2}$; then $\langle J_2 x, J_2 y\rangle = \langle x, y\rangle$ for every $x, y \in H_A$. We have $J_2(H_A) = J(H_A)$ and $J_2^* J_2 = 1$. Let $z \in H_A$ be an arbitrary element. Then

$$z = J_2 J_2^* z + (z - J_2 J_2^* z), \quad J_2 J_2^* z \in J_2(H_A)$$

and

$$J_2^*(z - J_2 J_2^* z) = J_2^* z - (J_2^* J_2) J_2^* z = J_2^* z - J_2^* z = 0,$$

so $(z - J_2 J_2^* z) \in \operatorname{Ker} J_2^*$, but

$$\begin{aligned} x \in \operatorname{Ker} J_2^* \quad &\Leftrightarrow \quad \forall y : \quad \langle J_2^* x, y\rangle = 0 \quad \Leftrightarrow \\ &\Leftrightarrow \quad \forall y : \quad \langle x, J_2 y\rangle = 0 \quad \Leftrightarrow \quad x \in J_2(H_A)^\perp. \end{aligned}$$

Hence $J_2 J_2^* z \in J_2(H_A)$, $(z - J_2 J_2^*) \in J_2(H_A)^\perp$, and

$$H_A = J_2(H_A) \widehat{\bigoplus} J_2(H_A)^\perp = J(H_A) \widehat{\bigoplus} J(H_A)^\perp.$$

So, if $J(H_A)^\perp = 0$, then J is an isomorphism. Let $x \in J(H_A)^\perp$, then $x \in J^*(H_A)^\perp$, so $\forall y : \quad \langle x, J^* y\rangle = 0$ or $\forall y : \quad \langle Jx, y\rangle = 0$, and $x \in \operatorname{Ker} J$. But J is an injection, and so, $x = 0$. ■

4.5. Lemma. *Let $F : M \to H_A$ be an injection admitting a conjugate. Then*

$$F(M) \widehat{\bigoplus} F(M)^\perp = H_A.$$

Proof. We can assume by the stabilization theorem that $M = H_A^1 \cong H_A$. Then $F^*F : H_A^1 \to H_A^1$ is a self adjoint operator. Let $\|x\| = 1$, then

$$\|Fx\|^2 = \| \langle Fx, Fx \rangle \| \geqslant c^2$$

by injectivity and

$$\|F^*Fx\| = \|F^*Fx\| \, \|x\| \geqslant \| \langle F^*Fx, x \rangle \| = \| \langle Fx, Fx \rangle \| \geqslant c^2.$$

So $F^*F : H_A^1 \to H_A^1$ is a self adjoint injection and it is an isomorphism by the previous lemma. Moreover, $F^*F \geqslant 0$, and so, $(F^*F)^{-1/2}$ can be defined. Hence $U = F(F^*F)^{-1/2} : M \to H_A$ (which is an injection with $U(M) = F(M)$) is well defined. We have $U^*U = \mathrm{Id}_M$. Let $z \in H_A$ be an arbitrary element. Then

$$z = UU^*z + (z - UU^*z), \quad U^*(z - UU^*z) = U^*z - (U^*U)U^*z = U^*z - U^*z = 0.$$

Since $y \in \operatorname{Ker} U^* \quad \Leftrightarrow \quad \langle U^*y, x \rangle = 0 \, \forall x \quad \Leftrightarrow \langle y, Ux \rangle = 0 \, \forall x \quad \Leftrightarrow \quad y \perp \operatorname{Im} U$ we get

$$U^*Uz \in \operatorname{Im} U = \operatorname{Im} F, \quad (z - UU^*z) \in (\operatorname{Im} U)^\perp.$$

The proof is finished because z is an arbitrary element. ■

4.6. Lemma. *Let $H_A = M \oplus N$, $p : H_A \to M$ be a projection, N be a finitely generated projective module. Then $M \widehat{\bigoplus} M^\perp = H_A$ if and only if p admits a conjugate.*

Proof. If there exists p^*, then there exists $(1-p)^* = 1 - p^*$, so by [11] $\operatorname{Ker}(1-p) = M$ is the kernel of a self adjoint projection.

To prove the converse statement let us start from the case where N is a free module and let us prove first that $H_A = N^\perp + M^\perp$. By the Kasparov stabilization theorem we can assume that

$$N = \operatorname{span}_A \langle e_1, \dots, e_n \rangle, \quad N^\perp = \operatorname{span}_A \langle e_{n+1}, e_{n+2}, \dots \rangle.$$

Let g_i be the image of e_i by the projection of N on $M^\perp$:

$$e_1 = f_1 + g_1, \dots, e_n = f_n + g_n, \quad f_i \in M, \, g_i \in M^\perp.$$

This projection is an isomorphism of A-modules $N \cong M^\perp$, so the elements $g_1, \dots, g_n$ are free generators and $\langle g_k, g_k \rangle > 0_A$. Hence, if

$$f_k = \sum_{k=1}^{\infty} f_k^i e_i, \quad \text{then} \quad e_k - f_k^k e_k = \sum_{i \neq k} f_k^i e_i + g_k.$$

On the other hand

$$1 = \langle e_k, e_k \rangle = \langle f_k, f_k \rangle + \langle g_k, g_k \rangle, \quad 1 - (f_k^k)(f_k^k)^* \geqslant \langle g_k, g_k \rangle > 0.$$

Then by 2.1 the element $1 - f_k^k$ is invertible in A,

$$e_k = \frac{1}{1 - f_k^k}\left(\sum_{i \neq k} f_k^i e_i + g_k\right) \in N^\perp + M^\perp \quad (k = 1, \dots, n),$$

so, $N^\perp + M^\perp = H_A$. Let $x \in N^\perp \cap M^\perp$. Every $y \in H_A = M \oplus N$ has the form $y = m + n$, so $\langle x, y \rangle = \langle x, m \rangle + \langle x, n \rangle = 0$, in particular, $\langle x, x \rangle = 0$ and $x = 0$. Hence, $H_A = N^\perp \oplus M^\perp$. Let us consider

$$q = \begin{cases} 1 & \text{on } N^\perp, \\ 0 & \text{on } M^\perp. \end{cases}$$

It is a bounded projection because $H_A = N^\perp \oplus M^\perp$. Let $x + y \in M \oplus N$, $x_1 + y_1 \in N^\perp \oplus M^\perp$. Then

$$\langle p(x+y), x_1 + y_1 \rangle = \langle x, x_1 + y_1 \rangle = \langle x, x_1 \rangle,$$
$$\langle x + y, q(x_1 + y_1) \rangle = \langle x + y, x_1 \rangle = \langle x, x_1 \rangle.$$

Hence, there exists $p^* = q$.

To prove the general case let $\tilde{H}_A = H_A \widehat{\bigoplus} \tilde{N}$ with $N \widehat{\bigoplus} \tilde{N}$ a free module. Then, by the previous case,

$$M \widehat{\bigoplus} \tilde{M} = \tilde{H}_A,$$

$$M \widehat{\bigoplus} (M^\perp \widehat{\bigoplus} \tilde{N}) = H_A \widehat{\bigoplus} \tilde{N},$$

$$M \widehat{\bigoplus} M^\perp = H_A. \blacksquare$$

4.7. Theorem. *In the decomposition in the definition of A-Fredholm operator we can always assume M_0 and M_1 admitting an orthogonal complement. More precisely, there exists a decomposition for F*

$$\begin{pmatrix} F_3 & 0 \\ 0 & F_4 \end{pmatrix} : H_A = V_0 \oplus W_0 \to V_1 \oplus W_1 = H_A,$$

such that $V_0^\perp \widehat{\bigoplus} V_0 = H_A$, $V_1^\perp \widehat{\bigoplus} V_1 = H_A$, or (by the previous lemma it is just the same) such that the projections

$$p_0 : V_0 \oplus W_0 \to V_1, \quad p_1 : V_1 \oplus W_1 \to V_1$$

admit conjugates.

Proof. Let $W_0 = N_0$, $V_0 = W_0^\perp$. This orthogonal complement exists by [4], and $F|_{W_0^\perp}$ is an isomorphism. Indeed, if $x_n \in W_0^\perp$, then let $x_n = x_1^n + x_2^n$, $x_1^n \in M_0$, $x_2^n \in W_0$, $\|x_n\| = 1$.

Let us assume that $\|Fx_n\| \to 0$. Then $\|Fx_1^n + Fx_2^n\| \to 0$, and, since $Fx_1^n \in V_1$, $Fx_2^n \in W_1$, $V_1 \oplus W_1 = H_A$, then this means that $\|Fx_1^n\| \to 0$ and $\|Fx_2^n\| \to 0$, and, since F_1 is an isomorphism, then $\|x_1^n\| \to 0$. If $a_1, \dots, a_s$ are the generators of $W_0 = N_0$, then

$$0 = \langle x_n, a_j \rangle = \langle x_1^n, a_j \rangle + \langle x_2^n, a_j \rangle ,$$

$$\| \langle x_2^n, a_j \rangle \| = \| \langle x_1^n, a_j \rangle \| \leqslant \|x_1^n\| \, \|a_j\| \to 0 \quad (n \to \infty)$$

for any $j = 1, \dots, s$. Hence, since $x_2^n \in N$, we have $x_2^n \to 0 \quad (n \to \infty)$ and $x_n = x_1^n + x_2^n \to 0$, but this contradicts the equality $\|x_n\| = 1$. This contradiction shows that $F|_{W_0^\perp}$ is an isomorphism.

Let $V_1 = F(V_0)$. Since $W_0 = N_0$, we can assume that $W_1 = N_1$. Indeed, any $y \in H_A$ has the form $y = m_1 + n_1 = F(m_0) + n_1$, where $m_1 \in M_1$, $n_1 \in N_1$, $m_0 \in M_0$. On the other hand, $m_0 = v_0 + n_0$, where $v_0 \in V_0$, $n_0 \in W_0 = N_0$, and

$$y = F(v_0 + n_0) + n_1 = F(v_0) + (F(n_0) + n_1) \in V_1 + N_1.$$

Hence, $H_A = V_1 + W_1$.

Let $y \in V_1 \cap W_1 = V_1 \cap N_1$, so that $n_1 = y = F(v_0)$, $n_1 \in N_1$, $v_0 \in V_0$. Let us decompose $v_0 + n_0$, where $m_0 \in M_0$, $n_0 \in N_0$. Then

$$\begin{aligned} &n_1 = F(m_0) + F(n_0), \\ &F(m_0) = n_1 - F(n_0), \quad F(m_0) \in M_1, \quad n_1 - F(n_0) \in N_1. \end{aligned}$$

Hence $F(m_0) = 0, \quad n_1 - F(n_0) = 0$, and since $F : M_0 \cong M_1$, then $m_0 = 0$. We have $v_0 \in V_0 = N_0^\perp$ and hence,

$$0 = \langle v_0, n_0 \rangle = \langle m_0 + n_0, n_0 \rangle = \langle n_0, n_0 \rangle , \quad n_0 = 0.$$

So, $v_0 = m_0 + n_0 = 0$, $y = F(v_0) = 0$. Hence $V_1 \cap W_1 = 0$ and $H_A = V_1 \oplus W_1$.

By 4.5 V_1 has an orthogonal complement $V_1^\perp$, $V_1 \widehat{\bigoplus} V_1^\perp = H_A$, and this completes the proof. ■

4.8. Remark. If we do not restrict the operator F to admit a conjugate, we can assert that there exists a decomposition

$$F : N_0^\perp \oplus N_0 \to M_1 \oplus L_n,$$

where $L_n = \operatorname{span}_A(e_1, \dots, e_n)$, but M_1 may have no orthogonal complement. This result was proved in [6].

5. Lefschetz numbers with values in $HC_0(A)$

5.1. Definition. Let $\{e_1, e_2, \dots\}$ be an A-orthobasis of $H_A = l_2(A)$ (the Hilbert module over A) with A-inner product $(\ , \ )$. Let $S \in \operatorname{End}_A^* H_A$ (the A-linear endomorphisms of H_A admitting an adjoint) and $S(e_i) = 0$ $(i > k)$. We define the trace of S by

$$t(S, \{e_i\}, k) = \sum_{i=1}^{\infty} f((Se_i, e_i)) = \sum_{i=1}^{k} f(S_i^i),$$

where $f : A \to A/[A, A] = HC_0(A)$, $\|S_j^i\|$ is the matrix of S with respect to $\{e_i\}$, $S_j^i \in A$.

5.2. Lemma. *$t(S,\{e_i\},k) = t(S,\{e_i\},l) := t(S,\{e_i\})$ for $l \geqslant k$.*

The proofs of this lemma and the other statements of this Section can be found in [18].

5.3. Lemma. *Let S, $\{e_i\}$, k be as in 5.1 and $\{h_j\}$ a new A-basis of H_A (in general non-orthogonal). Then the series*

$$\sum_{r=1}^{\infty} f((S_h)_r^r)$$

converges to $t(S,\{e_i\})$, where $(S_h)_r^p$ are the matrix elements of S with respect to $\{h_i\}$.

Let us note that a basis of H_A is a system of elements $\{h_i\}$, such that $h_i = Be_i$, where $B \in \mathrm{GL}^*$ (automorphisms admitting a conjugate). The matrix of S with respect to the $\{h_i\}$ is the matrix of $B^{-1}SB$ with the respect to $\{e_i\}$, i.e., $(S_h)_j^i = (B^{-1}SB)_j^i = \langle B^{-1}SBe_i, e_j\rangle$.

So we can give instead of 5.1 the following correct definition.

5.4. Definition. Let $S \in \mathrm{End}_A^* H_A$, M and N Hilbert submodules of H_A, N finitely generated, $H_A = M \oplus N$, $S|_M = 0$. For an arbitrary basis $\{e_i\}$ we define

$$t(S) = \sum_{i=1}^{\infty} f(S_i^i).$$

5.5. Lemma. *Let M, N, S be as in 5.4, and $\tilde N$ be a countably generated Hilbert A-module, $\tilde H_A = H_A \widehat{\bigoplus} \tilde N \cong H_A$,*

$$\tilde S = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} : H_A \widehat{\bigoplus} \tilde N \to H_A \widehat{\bigoplus} \tilde N.$$

Then $t(S) = t(\tilde S)$.

5.6. Lemma. *Let M, N, S be as in 5.4, $M \cong H_A$, $N = \bar N \oplus \bar{\bar N}$, $S|_{\bar{\bar N}} = 0$. Then*

$$t(S) = t(pSp),$$

where $p : M \oplus \bar N \oplus \bar{\bar N} \to M \oplus \bar N$ is a projection, and the sum on the right is in the space $M \oplus \bar N \cong H_A$. Let us notice, that if we denote by

$$q : M \oplus N \to M, \quad p_1 : N \to \bar N$$

the projections, then they admit conjugates. Hence, the projection $p = q + p_1(1-q)$ admits one, too.

5.7. Corollary. *If in 5.5 $M \oplus \bar N$ is orthogonal to $\bar{\bar N}$, and $\{h_i\}$ is an A-orthobasis of $M \oplus \bar N$, then*

$$t(S) = \sum_{i=1}^{\infty} f(\langle Sh_i, h_i\rangle). \blacksquare$$

Definition. Let $F: H_A \to H_A$ be an A-Fredholm operator (admitting an adjoint),

$$\begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix} : H_A = M_0 \oplus N_0 \to M_1 \oplus N_1 = H_A \qquad (D)$$

a corresponding decomposition, restricted to satisfy the condition as in 4.7 (we always will assume this without specification). Let S_0 and S_1 be operators from $\operatorname{End}^*_A H_A$, such that the diagram

$$\begin{array}{ccc} H_A & \xrightarrow{F} & H_A \\ {\scriptstyle S_0}\downarrow & & \downarrow{\scriptstyle S_1} \\ H_A & \xrightarrow{F} & H_A. \end{array}$$

commutes. Let

$$\tilde{S}_0 = \begin{cases} 0 & \text{on } M_0, \\ S_0 & \text{on } N_0, \end{cases} \qquad \tilde{S}_1 = \begin{cases} 0 & \text{on } M_1, \\ S_1 & \text{on } N_1. \end{cases}$$

Let us define

$$L(F, S, D) = t(\tilde{S}_0) - t(\tilde{S}_1).$$

5.9. Lemma. *Let*

$$H_A = M_0 \oplus N_0 \to M_1 \oplus N_1 = H_A, \qquad (D)$$
$$H_A = \tilde{M}_0 \oplus N_0 \to \tilde{M}_1 \oplus N_1 = H_A \qquad (\tilde{D})$$

then

$$L(F, S, D) = L(F, S, \tilde{D}).$$

5.10. Lemma. *Let*

$$H_A = (M_0 \oplus N_0) \oplus K_0 \to (M_1 \oplus N_1) \oplus K_1 = H_A, \quad (D_1)$$
$$H_A = M_0 \oplus (N_0 \oplus K_0) \to M_1 \oplus (N_1 \oplus K_1) = H_A \quad (D_2)$$

be two decompositions for F. Then $L(F, S, D_1) = L(F, S, D_2)$.

5.11. Lemma. *Let*

$$H_A = M_0 \oplus N_0 \to M_1 \oplus N_1 = H_A \qquad (D)$$

and

$$H_A = \bar{M}_0 \oplus \bar{N}_0 \to \bar{M}_1 \oplus \bar{N}_1 = H_A \qquad (\bar{D})$$

be two decompositions for F. Then $L(F, S, D) = L(F, S, \bar{D})$. So L does not depend on D and we denote it by $L(F, S)$.

5.12. Remark. By the stabilization theorem and Lemma 5.5, we can define $L(F, S)$ for any countably generated Hilbert A-module instead of H_A.

5.13. Definition. Let $T = \{T_i\}$ be an endomorphism of an A-elliptic complex E:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Gamma(E_0) & \xrightarrow{d_0} & \Gamma(E_1) & \longrightarrow & \dots \\ & & \downarrow T_0 & & \downarrow T_1 & & \\ 0 & \longrightarrow & \Gamma(E_0) & \xrightarrow{d_0} & \Gamma(E_1) & \longrightarrow & \dots \end{array},$$

$$T_{i+1} d_i = d_i T_i, \quad T_i \in \operatorname{End}^*_A \Gamma(E_i).$$

Assume the following

5.14. Condition. Sobolev products in $\Gamma(E)$ can be chosen in such a way that

$$T_i d_i^* = d_i^* T_{i+1}.$$

We take $E_{ev} = \oplus E_{2i}$, $E_{od} = \oplus E_{2i+1}$,

$$F = d + d^* : \Gamma(E_{ev}) \to \Gamma(E_{od}).$$

Then F is an A-Fredholm operator and the diagram stated below commutes, where

$$S_0 = \oplus T_{2i}, \qquad S_1 = \oplus T_{2i+1}.$$

$$\begin{array}{ccc} \Gamma(E_{ev}) & \xrightarrow{F} & \Gamma(E_{od}) \\ {\scriptstyle S_0}\downarrow & & \downarrow{\scriptstyle S_1} \\ \Gamma(E_{ev}) & \xrightarrow{F} & \Gamma(E_{od}). \end{array}$$

We define *the Lefschetz number of the second type* as

$$L_0(E, T, m) = L(F, S) \in HC_0(A),$$

where m denotes the dependence on inner products (via d^*).

5.15. Lemma. *Let $T = T_g$, $g \in G$ as in §2. Then the condition 5.14 is fulfilled.*

5.16. Theorem. *If $T = T_g$, $g \in G$, then*

$$L_0(E, T_g, m_G) = \tilde{\operatorname{Ch}}^0_0(L_1(g, E)),$$

where Ch^0_0 is the Chern character

$$\operatorname{Ch}^0_0 : K_0(A) \to HC_0(A)$$

(see [3, 7, 8]), and

$$\tilde{\operatorname{Ch}}^0_0(a \otimes z) = \operatorname{Ch}^0_0(a) z, \qquad z \in \mathbb{C}.$$

In particular, L_0 does not depend on m_G.

Proof. We have

$$L_1(g,E) = \mathrm{ind}^X_{G,A}([\sigma(E)])(g) = \mathrm{ind}^X_{G,A}(F)(g).$$

Let

$$M_o \oplus N_0 \to M_1 \oplus N_1 \qquad (D)$$

be a decomposition for F. Then by 2.8 and [15]

$$N_0 = \bigoplus_{k=1}^{K} V_k \otimes P_k, \qquad N_1 = \bigoplus_{l=1}^{L} W_l \otimes Q_l,$$

where V_k and W_l are $\mathbb{C}$-vector spaces of irreducible representations of G, P_k and Q_l are G-trivial projective finitely generated A-modules. Then (representations are unitary)

$$\mathrm{ind}^X_{G,A}(F) = \sum_{k=1}^{K} [P_k] \otimes \chi(V_k) - \sum_{l=1}^{L} [Q_l] \otimes \chi(W_l)$$

and

(2) $$L_1(g,E) = \sum_{k=1}^{K} [P_k] \otimes \mathrm{Trace}(g|V_k) - \sum_{l=1}^{L} [Q_l] \otimes \mathrm{Trace}(g|W_l).$$

The end of the proof see in [18]. ■

6. Lefschetz numbers with values in $HC_{2l}(A)$

Let W^*A be the universal enveloping von Neumann algebra of the algebra A with the norm topology. Let U be a unitary operator in the Hilbert module A^n. Then

(3) $$U = \int_{S^1} e^{i\varphi}\, dP(\varphi),$$

where $P(\varphi)$ is the projection valued measure valued in the space of matrices $M(n, W^*A)$, and the integral converges with respect to the norm. Let us associate with the integral sum

$$\sum_k e^{i\varphi_k} P(E_k)$$

the following class of the cyclic homology $HC_{2l}(M(n, W^*A))$:

$$\sum_k P(E_k) \otimes \ldots \otimes P(E_k) \cdot e^{i\varphi_k}.$$

Passing to the limit we get the following element

$$\tilde{T}U = \int_{S^1} e^{i\varphi}\, d(P \otimes \ldots \otimes P)(\varphi) \in HC_{2l}(M(n, W^*A)).$$

Then we define

$$T(U) = \mathrm{Tr}^n_* \, \tilde{T}U \in HC_{2l}(W^*A).$$

6.1. Lemma. *Let $J: M = A^m \to N = A^n$ be an isomorphism, $U_M: M \to M$, $U_N: N \to N$ be A-unitary operators and $JU_M = U_N J$. Then*

$$T(U_M) = T(U_N).$$

Proof. If

$$U_M = \int_{S^1} e^{i\varphi}\, dP(\varphi),$$

then

$$U_N = JU_M J^{-1} = \int_{S^1} e^{i\varphi}\, dJPJ^{-1}(\varphi).$$

To verify the equality $T(U_M) = T(U_N)$ it is sufficient to verify that

$$\mathrm{Tr}_*^m \left[\sum_k P(E_k) \otimes \ldots \otimes P(E_k) \cdot e^{i\varphi_k} \right] =$$

$$= \mathrm{Tr}_*^n \left[\sum_k JP(E_k)J^{-1} \otimes \ldots \otimes JP(E_k)J^{-1} \cdot e^{i\varphi_k} \right] \in HC_{2l}(W^*A),$$

but this follows from well-definedness of the Chern character $\mathrm{Ch}_{2l}^0 : K_0(B) \to HC_{2l}(B)$ (see [3, 8]). ■

Let now U be equal to U_g, i.e. an operator representing $g \in G$. Then (3) turns to be the sum associated with the decomposition from 2.8 and [15]

$$A^n \cong \bigoplus_{k=1}^{M} Q_k \otimes V_k,$$

where $V_k \cong \mathbb{C}^{L_k}$, and Q_k are projective A-modules of finite type. Then

$$U_g \left(\sum_{k=1}^{M} x_k \otimes v_k \right) = \sum_{k=1}^{M} x_k \otimes u_g^k v_k = \sum_{k=1}^{M} \sum_{l=1}^{L_k} x_k \otimes e^{i\varphi_l^k} v_k^l f_l,$$

where $f_1, \ldots, f_{L_k}$ is the diagonalizing basis for u_g^k; $v_k = \sum v_k^l f_l$. Then we can define

$$\tau(U_g) = \sum_{k=1}^{M} \sum_{l=1}^{L_k} \mathrm{Ch}_{2l}^0[P_k] \cdot \mathrm{Trace}(u_g^k) \in HC_{2l}(A). \tag{4}$$

We have $T(U_g) = i_*(\tau(U_g))$, where $i: A \to W^*A$.

A similar technique can be developed for a projective module N instead of A^n. For this purpose we take $N = q(A^n)$,

$$U \oplus 1 : A^n \cong N \oplus (1-q)A^n \to N \oplus (1-q)A^n \cong A^n,$$

$$\tilde{T}U = \int_{S^1} e^{i\varphi}\, d(qPq \otimes \ldots \otimes qPq)(\varphi).$$

The well-definedness is an immediate consequence of Lemma 6.1.

Let us consider a G-invariant A-elliptic complex (E, d), and let the Sobolev A-products be chosen invariant, so that $T_g = U_g$ are unitary operators (see §3).

6.2. Lemma. *We can choose a decomposition for the A-Fredholm operator*

$$F = d + d^* : \Gamma(E_{ev}) \to \Gamma(E_{od}),$$

$$F : M_0 \oplus \tilde{N}_0 \to M_1 \oplus \tilde{N}_1, \quad F : M_0 \cong M_1,$$

such that

$$\tilde{N}_0 = \oplus_i N_{2i}, \qquad N_{2i} \subset \Gamma(E_{2i}),$$
$$\tilde{N}_1 = \oplus_i N_{2i+1}, \quad N_{2i+1} \subset \Gamma(E_{2i+1}),$$

where N_m are projective invariant modules.

Proof. Let us assume that the complex consists of operators of the degree m, so $F = d + d^*$ is an A-Fredholm operator in the spaces $H^m(E_{ev}) \to H^0(E_{od})$. We can choose the basis in $H^m(E_{ev})$ (or the decomposition into modules P_j in $l_2(P)$) in such a way that $e_{ms+j} \in \Gamma(E_{2j})$, where $E_0, E_2, \dots, E_{2j}, \dots, E_{2m}$ are all non-zero terms of the complex, $s \in \mathbb{N}$, $j = 0, \dots, m$ (and in a similar way for P_j). As usual, without loss of generality we can assume that

$$\tilde{N}_0 = \operatorname{span}_A(e_1, \dots, e_{n_0}), \quad M_0 = \operatorname{span}_A(e_{n_0+1}, e_{n_0+2}, \dots),$$

and $M_1 = F(M_0)$ has in $H^0(E_{od})$ the A-orthogonal complement $M_1^\perp$. Then for every $x \in M_1$, $y \in \tilde{N}_0$

$$\langle x, Fy \rangle = \langle Fx, y \rangle_0, \tag{5}$$

where the first brackets mean the pairing of a functional and an element. So, $F(\tilde{N}_0) \subset M_1^\perp$ and taking $\tilde{N}_1 = M_1^\perp$, we get a decomposition $F : M_0 \oplus \tilde{N}_0 \to M_1 \oplus \tilde{N}_1$.

Let

$$y = y_1 + y_3 + \dots + y_{2m+1} \in \tilde{N}_1 \subset H^0(E_{od}), \quad y_{2j+1} \in H^0(E_{2j+1}),$$

and

$$x = x_0 + x_2 + \dots + x_{2m} \in M_0 \subset H^m(E_{ev}), \quad x_{2j} \in H^m(E_{2j}).$$

Then $\langle Fx, y \rangle_0 = 0$, where

$$\begin{array}{ccccccccc} Fx & = & d^* x_0 & + & \sum_{i=1}^m (dx_{2i-2} + d^* x_{2i}) & + & dx_{2m} & \in \\ & \in & 0 & \oplus & \oplus_{i=1}^m H^0(E_{2i+1}) & \oplus & 0. & \end{array}$$

Since (E, d) is a complex, $d^2 = 0$ and

$$\langle du, d^* v \rangle = \langle d^2 u, v \rangle = 0,$$

so

$$\begin{array}{ll} \langle y_{2j+1}, dx_{2j} \rangle = 0, & \langle y_{2j+1}, d^* x2j + 2 \rangle = 0 \quad (j = 0, 1, \dots, m) \\ \langle y_{2j+1}, dx \rangle = 0, & \qquad\qquad \langle y_{2j+1}, d^* x \rangle = 0. \end{array}$$

Hence $e_{2j+1} \in F(M_0)^\perp = M_1^\perp = \tilde{N}_1$, and

$$\tilde{N}_1 = \oplus_i (\tilde{N}_1 \cap \Gamma(E_{2i+1})) = \oplus_i N_{2i+1}.$$

■

6.3. Definition. *The Lefschetz number L_{2l} we define as*

$$L_{2l}(E, U_g, m_G) = \sum_i (-1)^i \, \tau(U_g|N_i) \in HC_{2l}(A),$$

where m_G denotes the dependence on inner products (via d^*).

Remark. For more general situations we hope to use T instead of τ.

6.4. Lemma. *The definition of L_{2l} is correct, i.e. this number does not depend on the choice of decompositions in Lemma 6.2.*

Proof. For any two decompositions we can by use of projection (as in [13, 15]) replace $\tilde{N}_0$ by a module inside $\operatorname{span}_A(e_1, \dots, e_n)$ for a sufficiently great n (we use the notation of Lemma 6.2). By 6.1 $\tau(U_g|N_i)$ does not change under this replacement. So we can assume that we have to compare the decomposition as in 6.2 and the decomposition

$$\begin{aligned} F : \bar{M}_0 \oplus \tilde{\bar{N}}_0 &\to \bar{M}_1 \oplus \tilde{\bar{N}}_1, \\ \tilde{\bar{N}}_0 = \oplus_i \bar{N}_{2i}, &\quad \bar{N}_{2i} \subset N_{2i} \subset \Gamma(E_{2i}), \\ \tilde{\bar{N}}_1 = \oplus_i \bar{N}_{2i+1}, &\quad \bar{N}_{2i+1} \subset \Gamma(E_{2i}). \end{aligned}$$

Hence by (5), $\bar{N}_{2i+1} \subset N_{2i+1}$. Let $K_i = (\bar{N}_i)^{\perp}_{N_i}$. Then $F : K_{2i} \cong K_{2i+1}$ and by Lemma 6.1 we get $\tau(U_g|K_{2i}) = \tau(U_g|K_{2i+1})$. Hence

$$\begin{aligned} \sum_i (-1)^i \, \tau(U_g|N_i) &= \sum_i (-1)^i \left(\tau(U_g|\bar{N}_i) + \tau(U_g|K_i) \right) = \\ &= \sum_i (-1)^i \left(\tau(U_g|\bar{N}_i) \right). \qquad \blacksquare \end{aligned}$$

6.5. Theorem. *Let $\tilde{\mathrm{Ch}}^0_{2l}(a \otimes z) = \mathrm{Ch}^0_{2l}(a) \cdot z$, where $z \in \mathbb{C}$. Then*

$$L_{2l}(E, U_g, m_G) = \tilde{\mathrm{Ch}}^0_{2l}(L_1(g, E)),$$

in particular, L_{2l} does not depend on m_G.

Proof. We get the statement immediately from (2) and (4). ■

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