ORTHOGONAL COMPLEMENTS AND ENDOMORPHISMS OF HILBERT MODULES AND $C^*$-ELLIPTIC COMPLEXES$^1$

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1. Introduction

In the present paper we discuss some properties of endomorphisms of $C^*$-Hilbert modules and $C^*$-elliptic complexes. The main results of this paper can be considered as an attempt to answer the question: what kinds of good properties can one expect for an operator on a Hilbert module, which represents an element of a compact group? These results are new, but we have to recall some first steps made by us before to make the present paper self-contained.

In §2 we define the Lefschetz numbers “of the first type” of $C^*$-elliptic complexes, taking values in $K_0(A) \otimes \mathbb{C}$, $A$ being a complex $C^*$-algebra with unity, and prove some properties of them.

The averaging theorem 3.2 was discussed in brief in [15] and was used there for constructing an index theory for $C^*$-elliptic operators. In this theorem we do not restrict the operators to admit a conjugate, but after averaging they even become unitary. This raises the following question: is the condition on an operator on a Hilbert module to represent an element of a compact group so strong that it automatically has to admit a conjugate?

The example in section 4 gives a negative answer to this question. Also we get an example of closed submodule in Hilbert module which has a complement but has no orthogonal complement.

In §5 we define the Lefschetz numbers of the second type with values in $HC_0(A)$. We prove that these numbers are connected via the Chern character in algebraic $K$-theory. These results were discussed in [18] and we only recall them.

In §6 we get similar results for $HC_2(A)$. We have to use in a crucial way the properties of representations.

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2. Preliminaries

We consider the Hilbert $C^*$-module $l_2(P)$, where $P$ is a projective module over $C^*$-algebra $A$ with unity (see [10, 4, 13, 15]).

2.1. Lemma. Let $P^+(A)$ be the positive cone of the $C^*$-algebra $A$. For every bounded $A$-homomorphism $F: l_2(P) \to l_2(P)$ and every $u \in l_2(P)$ we have

$$\langle Fu, Fu \rangle \leq \|F\|^2 \langle u, u \rangle$$

in $P^+(A)$.

Proof. For $c \in P^+(A)$ we have $c \leq \|c\| 1_A$. So if $\langle u, u \rangle = 1_A$, then

$$\langle Fu, Fu \rangle \leq \|Fu\|^2 1_A \leq \|F\|^2 \langle u, u \rangle.$$ 

Let now $\langle u, u \rangle$ be equal to $\alpha \in P^+(A)$, where $\alpha$ is an invertible element of $A$. We put $v = (\sqrt{\alpha})^{-1} u$. Then $u = \sqrt{\alpha} v$ and $\langle v, v \rangle = 1_A$. So

$$\langle Fv, Fv \rangle \leq \|F\|^2 \langle v, v \rangle,$$

$$\langle Fu, Fu \rangle = \sqrt{\alpha} \langle Fv, Fv \rangle (\sqrt{\alpha})^* \leq \sqrt{\alpha} \|F\|^2 \langle v, v \rangle (\sqrt{\alpha})^* = \|F\|^2 \langle u, u \rangle.$$ 

Elements $u$ with invertible $\langle u, u \rangle$ are dense in $l_2(A)$ (this is a consequence of Lemma 2 of [4]), so the continuity of the $A$-product gives the statement for $l_2(A)$. For $l_2(P)$ we have to use the stabilization theorem [10].

Let us recall the basic ideas of [16, 17].

2.2. Definition. Let $p : F \to X$ be a $G$-$\mathbb{C}$-bundle over a locally compact Hausdorff $G$-space $X$. Let $\Lambda(p^* F, s_F)$ be the well known complex of $G$-$\mathbb{C}$-bundles (see [5]) with, in general, non-compact support. Let a complex $(E, \alpha)$ represent an element $a \in K_G(X; A)$ (see [15, sect. 1.3]); then $(p^* E, p^* \alpha) \otimes \Lambda(p^* F, s_F)$ has compact support and defines an element of $K_G(F; A)$. We get the Thom isomorphism of $R(G)$-modules

$$\varphi = \varphi^F_A : K_G(X; A) \to K_G(F; A).$$

If we pass to $K^1_G$ by the Bott periodicity [15, 1.2.4], we can define

$$\varphi : K^*_G(X; A) \to K^*_G(F; A).$$
2.3. Theorem. If $X$ is separable and metrizable, then $\varphi$ is an isomorphism.

With the help of this theorem we can define the Gysin homomorphism $i^! : K_G(TX; A) \to K_G(TY; A)$ and the topological index

$$t\text{-}\text{ind}_G^X = t\text{-}\text{ind}_{G,A}^X : K_G(TX; A) \to K_G^0(A)$$

in a way similar to the case $A = \mathbb{C}$ [5]. Here $i : X \to Y$ is a $G$-inclusion of smooth manifolds and $TX, TY$ are (co)tangent bundles.

We need the following property of the Gysin homomorphism.

2.4. Lemma. Let $i : Z \to X$ be a $G$-inclusion of smooth manifolds, $N$ its normal bundle. Then the homomorphism

$$(di)^* i^! : K_G(TZ; A) \to K_G(TZ; A)$$

is the multiplication by

$$[\lambda_i^{-1}(N \otimes \mathbb{C})] = \sum (-1)^i [\Lambda^i(N \otimes \mathbb{C})] \in K_G(Z),$$

where $\Lambda^i$ are the exterior powers, and we consider $K_G(TZ; A)$ as a $K_G(Z)$-module in the usual way.

2.5. Theorem. Let $a\text{-}\text{ind} D \in K_G^0(A)$ be the analytic index of a pseudo differential equivariant $C^*$-elliptic operator [15], $\sigma(D) \in K_G(TX; A)$ its symbol’s class. Then

$$t\text{-}\text{ind}_{G,A}^X \sigma(D) = a\text{-}\text{ind} D.$$

Now for the completeness of this text we recall a generalization of the result of [1]. Let, as above, $G$ be a compact Lie group, $X$ a $G$-space, $X^g$ the set of fixed points of $g : X \to X$, $i : X^g \to X$ the inclusion.

2.6. Definition. Let $E$ be a $G$-invariant $A$-complex on $X$, $\sigma(E)$ its sequence of symbols (see [15]), $u = [\sigma(E)] \in K_G(TX; A)$, $\text{ind}_{G,A}^X(u) \in K_0(A) \otimes R(G)$. The Lefschetz number of the first type is

$$L_1(g, E) = \text{ind}_{G,A}^X(u)(g) \in K_0(A) \otimes \mathbb{C}.$$

2.7. Theorem. Using the notation as above we have

$$L_1(g, E) = (\text{ind}_{G,A}^{X,g}) (\frac{i^* u(g)}{\lambda_i^{-1}(N^g \otimes \mathbb{C})(g)}).$$

Also we need the following theorem from [12].

2.8. Theorem. Let $M$ be a countably generated Hilbert $A$-module. Then we have a $G$-$A$-isomorphism

$$M \cong \bigoplus \pi \text{Hom}_G(V_\pi, M) \otimes \mathbb{C} V_\pi,$$

where $\{V_\pi\}$ is a complete family of irreducible unitary complex finite dimensional representations of $G$, non-isomorphic to each other. In $\text{Hom}_G(V_\pi, M) \otimes \mathbb{C} V_\pi$ the algebra $A$ acts on the first factor and $G$ on the second.
3. An averaging theorem

Let us recall some facts about the integration of operator-valued functions (see [9, §3]). Let \( X \) be a compact space, \( A \) be a \( C^* \)-algebra, \( \varphi : C(X) \to A \) be an involutive homomorphism of algebras with unity, and \( F : X \to A \) be a continuous map, such that for every \( x \in X \) the element \( F(x) \) commutes with the image of \( \varphi \). In this case the integral

\[
\int_X F(x) \, d\varphi \in A
\]

can be defined in the following way. Let \( X = \bigcup_{i=1}^n U_i \) be an open covering and

\[
\sum_{i=1}^n a_i(x) = 1
\]

be a corresponding partition of unity. Let us choose the points \( \xi_i \in U_i \) and compose the integral sum

\[
\sum (F, \{U_i\}, \{a_i\}, \{\xi_i\}) = \sum_{i=1}^n F(\xi_i)\varphi(a_i).
\]

If there is a limit of such integral sums then it is called the corresponding integral.

If \( X = G \) then it is natural to take \( \varphi \) equal to the Haar measure

\[
\varphi : C(X) \to \mathbb{C}, \quad \varphi(\alpha) = \int_G \alpha(g) \, dg
\]

(though this is only a positive linear map, not a *-homomorphism) and to define for a norm-continuous \( Q : G \to L(H) \)

\[
\int_G Q(g) \, dg = \lim_{i} \sum_{i} Q(\xi_i) \int_G a_i(g) \, dg.
\]

If we have \( Q : G \to P^+(A) \subset L(H) \), then, since

\[
\int_G a_i(g) \, dg \geq 0,
\]

we get

\[
\sum_{i} Q(\xi_i) \cdot \int_G a_i(g) \, dg \in P^+(A)
\]

and

\[
\int_G Q(g) \, dg \in P^+(A)
\]

(the cone \( P^+(A) \) is convex and closed). So we have proved the following lemma.
3.1. Lemma. Let $Q : G \to P^+(A)$ be a continuous function. Then for the integral in the sense of [9] we have
\[
\int_G Q(g) \, dg \geq 0.
\]

3.2. Theorem. Let $GL$ be the group of all bounded $A$-linear automorphisms of $l_2(A)$ (see [14]). Let $g \mapsto T_g$ ($g \in G, T_g \in GL$) be a representation of $G$ such that the map
\[
G \times l_2(A) \to l_2(A), \quad (g, u) \mapsto T_g u
\]
is continuous. Then on $l_2(A)$ there is an $A$-product equivalent to the original one and such that $g \mapsto T_g$ is unitary with respect to it.

Proof. Let $\langle \, , \, \rangle'$ be the original product. We have a continuous map
\[
G \to A, \quad x \mapsto \langle T_x u, T_x v \rangle'
\]
for every $u$ and $v$ from $l_2(A)$. We define the new product by
\[
\langle u, v \rangle = \int_G \langle T_x u, T_x v \rangle' \, dx,
\]
where the integral can be defined in the sense of either of the two definitions from [9, p. 810] because the map is continuous with the respect to the norm of the $C^\ast$-algebra. It is easy to see that this new product is a $A$-sesquilinear map $l_2(A) \times l_2(A) \to A$. Lemma 3.1 shows that $\langle u, u \rangle \geq 0$. Let us show that this map is continuous. Let us fix $u \in l_2(A)$. Then
\[
x \mapsto T_x (u), \quad G \to l_2(A)
\]
is a continuous map defined on a compact space and so the set $\{T_x(u) | x \in G\}$ is bounded. Hence by the principle of uniform boundness [2, v. 2, p. 309]
\[
(1) \quad \lim_{v \to 0} T_x (v) = 0
\]
uniformly with respect to $x \in G$. If $u$ is fixed then
\[
\|T_x(u)\| \leq M_u = \text{const}
\]
and by (1)
\[
\| \langle u, v \rangle \| = \| \int_G \langle T_x(u), T_x(v) \rangle' \, dx \| \leq M_u \cdot \text{vol} G \cdot \sup_{x \in G} \| T_x(v) \| \to 0 \quad (v \to 0).
\]
This gives the continuity at 0 and hence everywhere. For $T_x u = (a_1(x), a_2(x), \ldots) \in l_2(A)$ the equation $\langle u, u \rangle = 0$ takes the form
\[
\int_G \sum_{i=1}^{\infty} a_i(x) a_i^\ast(x) \, dx = 0.
\]
Let \( A \) be realized as a subalgebra of the algebra of all bounded operators in the Hilbert space \( L \) with inner product \((\cdot,\cdot)_L\). For every \( p \in L \) we have

\[
0 = \left( \left( \int_G \sum_{i=1}^{\infty} a_i(x) a_i^*(x) \, dx \right) p, p \right)_L = \int_G \left( \sum_{i=1}^{\infty} a_i(x) a_i^*(x) p, p \right) \, dx = \int_G \left( \sum_{i=1}^{\infty} (a_i(x)p, a_i^*(x)p)_L \right) \, dx
\]

(cf. [9]). Hence \( a_i(x) = 0 \) almost everywhere, and thus \( a_i(x) = 0 \) for every \( x \) because of the continuity, and \( T_x u = 0 \). In particular, \( u = 0 \).

Since every \( T_y \) is an automorphism, we have (cf. [9])

\[
\langle T_y u, T_y v \rangle = \int_G \langle T_{xy} u, T_{xy} v \rangle' \, dx = \int_G \langle T_z u, T_z v \rangle' \, dz = \langle u, v \rangle.
\]

Now we will show the equivalence of the two norms and, in particular, the continuity of the representation. There is a number \( N > 0 \) such that \( \|T_x\|' \leq N \) for every \( x \in G \). So by [9]

\[
\|u\|^2 = \|\langle u, u \rangle\|_A = \|\int_G \langle T_x u, T_x u \rangle' \, dx\|_A \leq \left( \sup_{x \in G} \|T_x u\|' \right)^2 \leq N^2(\|u\|').
\]

On the other hand, applying 2.1 and 3.1 we have

\[
\langle u, u \rangle' = \int_G \langle T_{x^{-1}} T_{y} u, T_{x^{-1}} T_{y} u \rangle' \, dg \leq \int_G \|T_{x^{-1}}\|^2 \langle T_{y} u, T_{y} u \rangle' \, dg \leq \int_G N^2 \langle T_{y} u, T_{y} u \rangle' \, dg = N^2 \int_G \langle T_{y} u, T_{y} u \rangle' \, dg = N^2 \langle u, u \rangle.
\]

Then

\[
(\|u\|')^2 = \|\langle u, u \rangle'\|_A \leq N^2 \|\langle u, u \rangle\|_A = N^2 \|u\|^2.
\]

**3.3. Remark.** \( l_2(P) \) is a direct summand in \( l_2(A) \), so 3.2 holds for \( l_2(P) \).

4. **Complements and orthogonal complements**

Let us recall some preliminary statements.

4.1. **Lemma.** 1. An \( A \)-linear operator \( F : M \rightarrow H_A \) always admits a conjugate if \( M \in \mathcal{P}(A) \) — the category of finitely generated projective modules.

2. Let \( 0_A \leq \alpha < 1_A \). Then \( \|\alpha\| < 1 \).

3. Let \( \alpha \geq 0, \alpha = \beta \beta^*, 1 - \alpha > 0 \). Then \( 1 - \beta \) is an isomorphism.

Here the strong inequality means that the spectrum of the operator is bounded away from zero.
4.2. Example. Let \( A = C[0, 1], \{\varepsilon_i\} \) be the standard basis of \( H_A \). Let

\[
\varphi_i(x) = \begin{cases} 
0 & \text{on } [0, \frac{1}{2}] \text{ and } [\frac{1}{2}, 1], \\
1 & \text{at } x_i = \frac{1}{2} \left( \frac{1}{2} + \frac{1}{2} \right), \\
\text{linear on } [\frac{1}{2}, x_i] \text{ and } [x_i, 1]\end{cases}
\]

\( i = 2, 3, \ldots \). Let

\[
h_i = \frac{\varepsilon_i + \varphi_i \varepsilon_1}{(1 + \varphi_i^2)^{1/2}} \quad (i = 2, 3, \ldots)
\]

be an orthonormal system of vectors which generates \( H_1 \subset H_A, H_1 \cong H_A \). Then \( H_1 \oplus \text{span}_A(\varepsilon_1) = H_A \). Indeed, all \( \varepsilon_i \in H_1 + \text{span}_A(\varepsilon_1) \), and if

\[
x = (\alpha_1, \alpha_2, \ldots) \in H_1 \cap \text{span}_A(\varepsilon_1),
\]

\[
x = (\alpha_1, 0, \ldots) = \sum_{i=2}^{\infty} \beta_i h_i,
\]

then all \( \beta_i = 0, x = 0 \). However the module \( H_1 \) does not have an orthogonal complement. More precisely we have the following situation. Let \( y = \sum_{j=1}^{\infty} \alpha_j e_j \) be in \( H_1^\perp \). Then

\[
\langle \sum_{j=1}^{\infty} \alpha_j e_j, h_i \rangle = 0 \quad \text{for } i = 2, 3, \ldots, \quad \text{so } \alpha_i + \alpha_1 \varphi_i = 0 \quad (i = 2, 3, \ldots), \quad \text{and } \alpha_i = - \alpha_1 \varphi_i,
\]

hence

\[
y = (\alpha_1, -\alpha_1 \varphi_2, -\alpha_1 \varphi_3, \ldots).
\]

This is possible if and only if the function \( \alpha_1 \) vanishes at 0: \( \alpha_1(0) = 0 \). If \( H_1 \oplus H_1^\perp = H_A \), then for some \( \alpha_1 \) we have \( e_1 = y + \sum_{i=2}^{\infty} \beta_i h_i \). In particular the series \( \sum_{i=2}^{\infty} \beta_i h_i \) converges and

\[
1 = \alpha_1 + \sum_{i=2}^{\infty} \frac{\beta_i \varphi_i}{(1 + \varphi_i^2)^{1/2}}.
\]

But \( \|\beta_i\|_A \to 0 \), so for

\[
\gamma = \sum_{i=2}^{\infty} \frac{\beta_i \varphi_i}{(1 + \varphi_i^2)^{1/2}}
\]

we get \( \gamma(0) = 0 \), as well as for \( \alpha_1 \). We come to a contradiction.

Let us investigate the involution \( J \) which determines a representation of \( \mathbb{Z}_2 \):

\[
J(x) = \begin{cases} 
x & \text{if } x \in H_1, \\
-x & \text{if } x \in \text{span}_A(\varepsilon_1),
\end{cases}
\]

This operator does not admit a conjugate. Indeed, let \( J^* \) exist. Then \( (J^*)^2 = J^2 = \text{Id} \), so \( J^* \) is also an involution.

\[
J^* x = x \quad \Leftrightarrow \quad (J^* x, y) = (x, y) \quad \forall y \quad \Leftrightarrow \quad (x, J y) = (x, y) \quad \forall y \quad \Leftrightarrow \quad (x, (J - 1) y) = 0 \quad \forall y \quad \Leftrightarrow \quad x \perp \text{Im}(J - 1) \quad \Leftrightarrow \quad x \perp \text{span}_A(\varepsilon_1),
\]
and $J^*x = -x \iff x \perp H_1$. But $H_1$ has no orthogonal complement and so the involution $J^*$ can not be defined. Nevertheless for the $A$-product averaged by the action of $\mathbb{Z}_2$

$$\langle x, y \rangle_2 = 1/2(\langle x, y \rangle + \langle Jx, Jy \rangle)$$

we get if $x \in H_1$, $y \in \text{span}_A(e_1)$: $\langle x, y \rangle_2 = 1/2(\langle x, y \rangle + \langle x, -y \rangle) = 0$, so the $+$ and $-$ subspaces of the involution are orthogonal to each other, and $J_{(2)}^* = J$.

Let us recall the definition of $A$-Fredholm operator [11, 13]. The theorem which will be proved is the crucial one for the possibility of construction of Sobolev chains in the $C^*$-case.

4.3. Definition. A bounded $A$-operator $F : H_A \to H_A$ admitting a conjugate is called Fredholm, if there exist decompositions of the domain of definition $H_A = M_1 \oplus N_1$ and the range $H_A = M_2 \oplus N_2$ where $M_1, M_2, N_1, N_2$ are closed $A$-modules, $N_1, N_2$ have a finite number of generators, and such that the operator $F$ has in these decompositions the following form

$$F = \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix},$$

where $F_1 : M_1 \to M_2$ is an isomorphism.

4.4. Lemma. Let $J : H_A \to H_A$ be a self adjoint injection. Then $J$ is an isomorphism. Here injection means an injective $A$-homomorphism with closed range.

Proof. Let us consider $J_1 = J : H_A \to J(H_A)$. It is an isomorphism of Hilbert modules admitting a conjugate $J_1^* = J^*|_{J(H_A)} = J|_{J(H_A)}$. Let $J_2 = J(J_1^*J_1)^{-1/2}$; then $\langle J_2x, J_2y \rangle = \langle x, y \rangle$ for every $x, y \in H_A$. We have $J_2(H_A) = J(H_A)$ and $J_2^*J_2 = 1$. Let $z \in H_A$ be an arbitrary element. Then

$$z = J_2J_2^*z + (z - J_2J_2^*z), \quad J_2J_2^*z \in J_2(H_A)$$

and

$$J_2^*(z - J_2J_2^*z) = J_2^*z - (J_2^*J_2)J_2^*z = J_2^*z - J_2^*z = 0,$$

so $(z - J_2J_2^*z) \in \text{Ker} J_2^*$, but

$$x \in \text{Ker} J_2^* \iff \forall y : \quad \langle J_2^*x, y \rangle = 0 \iff \forall y : \quad \langle x, J_2y \rangle = 0 \iff x \in J_2(H_A)^\perp.$$

Hence $J_2J_2^*z \in J_2(H_A)$, $(z - J_2J_2^*) \in J_2(H_A)^\perp$, and

$$H_A = J_2(H_A) \bigoplus J_2(H_A)^\perp = J(H_A) \bigoplus J(H_A)^\perp.$$

So, if $J(H_A)^\perp = 0$, then $J$ is an isomorphism. Let $x \in J(H_A)^\perp$, then $x \in J^*(H_A)^\perp$, so $\forall y : \quad \langle x, J^*y \rangle = 0 \quad \text{or} \quad \forall y : \quad \langle Jx, y \rangle = 0$, and $x \in \text{Ker} J$. But $J$ is an injection, and so, $x = 0$. ■
4.5. Lemma. Let $F : M \to H_A$ be an injection admitting a conjugate. Then

$$F(M) \bigoplus F(M)^\perp = H_A.$$ 

Proof. We can assume by the stabilization theorem that $M = H_A^1 \cong H_A$. Then $F^* F : H_A^1 \to H_A^1$ is a self adjoint operator. Let $\|x\| = 1$, then

$$\|Fx\|^2 = \|\langle Fx, Fx \rangle \| \geq c^2$$

by injectivity and

$$\|F^*Fx\| = \|F^*Fx\| \|x\| \geq \|\langle F^*Fx, x \rangle \| = \|\langle Fx, Fx \rangle \| \geq c^2.$$ 

So $F^* F : H_A^1 \to H_A^1$ is a self adjoint injection and it is an isomorphism by the previous lemma. Moreover, $F^* F \geq 0$, and so, $(F^* F)^{-1/2}$ can be defined. Hence $U = F(F^* F)^{-1/2} : M \to H_A$ (which is an injection with $U(M) = F(M)$) is well defined. We have $U^* U = \text{Id}_M$. Let $z \in H_A$ be an arbitrary element. Then

$$z = UU^* z + (z - UU^* z), \quad U^*(z - UU^* z) = U^* z - (U^* U)U^* z = U^* z - U^* z = 0.$$ 

Since $y \in \text{Ker} U^* \Leftrightarrow \langle U^* y, x \rangle = 0 \\forall x \Leftrightarrow \langle y, Ux \rangle = 0 \\forall x \Leftrightarrow y \perp \text{Im} U$ we get

$$U^* U z \in \text{Im} U = \text{Im} F, \quad (z - UU^* z) \in (\text{Im} U)^\perp.$$ 

The proof is finished because $z$ is an arbitrary element. \hfill \Box

4.6. Lemma. Let $H_A = M \oplus N$, $p : H_A \to M$ be a projection, $N$ be a finitely generated projective module. Then $M \bigoplus M^\perp = H_A$ if and only if $p$ admits a conjugate.

Proof. If there exists $p^*$, then there exists $(1 - p)^* = 1 - p^*$, so by [11] $\text{Ker}(1 - p) = M$ is the kernel of a self adjoint projection.

To prove the converse statement let us start from the case where $N$ is a free module and let us prove first that $H_A = N^\perp + M^\perp$. By the Kasparov stabilization theorem we can assume that

$$N = \text{span}_A \langle e_1, \ldots, e_n \rangle, \quad N^\perp = \text{span}_A \langle e_{n+1}, e_{n+2}, \ldots \rangle.$$ 

Let $g_i$ be the image of $e_i$ by the projection of $N$ on $M^\perp$:

$$e_1 = f_1 + g_1, \ldots, e_n = f_n + g_n, \quad f_i \in M, g_i \in M^\perp.$$ 

This projection is an isomorphism of $A$-modules $N \cong M^\perp$, so the elements $g_1, \ldots, g_n$ are free generators and $\langle g_k, g_k \rangle > 0_A$. Hence, if

$$f_k = \sum_{k=1}^{\infty} f^i_k e_i, \quad \text{then} \quad e_k - f^i_k e_k = \sum_{i \neq k} f^i_k e_i + g_k,$$
On the other hand
\[ 1 = \langle e_k, e_k \rangle = \langle f_k, f_k \rangle + \langle g_k, g_k \rangle, \quad 1 - (f_k^*) (f_k^*)^* \geq \langle g_k, g_k \rangle > 0. \]

Then by 2.1 the element \( 1 - f_k^* \) is invertible in \( A \),
\[ e_k = \frac{1}{1 - f_k^*} \left( \sum_{i \neq k} f_i^* e_i + g_k \right) \in N^\perp + M^\perp \quad (k = 1, \ldots, n), \]
so, \( N^\perp + M^\perp = H_A \). Let \( x \in N^\perp \cap M^\perp \). Every \( y \in H_A = M \oplus N \) has the form \( y = m + n \), so \( \langle x, y \rangle = \langle x, m \rangle + \langle x, n \rangle = 0 \), in particular, \( \langle x, x \rangle = 0 \) and \( x = 0 \). Hence, \( H_A = N^\perp \oplus M^\perp \).

Let us consider
\[ q = \begin{cases} 1 & \text{on } N^\perp, \\ 0 & \text{on } M^\perp. \end{cases} \]

It is a bounded projection because \( H_A = N^\perp \oplus M^\perp \). Let \( x + y \in M \oplus N \), \( x + y \in N^\perp \oplus M^\perp \).

Then
\[ \langle p(x + y), x_1 + y_1 \rangle = \langle x, x_1 + y_1 \rangle = \langle x, x_1 \rangle, \]
\[ \langle x + y, q(x_1 + y_1) \rangle = \langle x + y, x_1 \rangle = \langle x, x_1 \rangle. \]

Hence, there exists \( p^* = q \).

To prove the general case let \( \hat{H}_A = H_A \hat{\oplus} \hat{N} \) with \( N \hat{\oplus} \hat{N} \) a free module. Then, by the previous case,
\[ M \hat{\oplus} \hat{M} = \hat{H}_A, \]
\[ M \hat{\oplus} (M^\perp \hat{\oplus} \hat{N}) = H_A \hat{\oplus} \hat{N}, \]
\[ M \hat{\oplus} M^\perp = H_A. \]

4.7. **Theorem.** In the decomposition in the definition of A-Fredholm operator we can always assume \( M_0 \) and \( M_1 \) admitting an orthogonal complement. More precisely, there exists a decomposition for \( F \)
\[ \left( \begin{array}{cc} F_3 & 0 \\ 0 & F_4 \end{array} \right) : H_A = V_0 \oplus W_0 \rightarrow V_1 \oplus W_1 = H_A, \]
such that \( V_0^\perp \hat{\oplus} V_0 = H_A, \) \( V_1^\perp \hat{\oplus} V_1 = H_A \), or (by the previous lemma it is just the same) such that the projections
\[ p_0 : V_0 \oplus W_0 \rightarrow V_1, \quad p_1 : V_1 \oplus W_1 \rightarrow V_1 \]
admit conjugates.

**Proof.** Let \( W_0 = N_0, V_0 = W_0^\perp \). This orthogonal complement exists by [4], and \( F|_{W_0^\perp} \) is an isomorphism. Indeed, if \( x_n \in W_0^\perp \), then let \( x_n = x_n^1 + x_n^2, \) \( x_n^1 \in M_0, x_n^2 \in W_0, \| x_n \| = 1. \)
Let us assume that \( \| F x_n \| \to 0 \). Then \( \| F x_1^n + F x_2^n \| \to 0 \), and, since \( F x_1^n \in V_1, F x_2^n \in W_1, V_1 \oplus W_1 = H_A \), then this means that \( \| F x_1^n \| \to 0 \) and \( \| F x_2^n \| \to 0 \), and, since \( F_1 \) is an isomorphism, then \( \| x_1^n \| \to 0 \). If \( a_1, \ldots, a_s \) are the generators of \( W_0 = N_0 \), then

\[
0 = \langle x_n, a_j \rangle = \langle x_1^n, a_j \rangle + \langle x_2^n, a_j \rangle,
\]

\[
\| x_1^n, a_j \| = \| x_1^n, a_j \| \leq \| x_1^n \| \| a_j \| \to 0 \quad (n \to \infty)
\]

for any \( j = 1, \ldots, s \). Hence, since \( x_2^n \in N \), we have \( x_2^n \to 0 \quad (n \to \infty) \) and \( x_n = x_1^n + x_2^n \to 0 \), but this contradicts the equality \( \| x_n \| = 1 \). This contradiction shows that \( F|_{W_0^\perp} \) is an isomorphism.

Let \( V_1 = F(V_0) \). Since \( W_0 = N_0 \), we can assume that \( W_1 = N_1 \). Indeed, any \( y \in H_A \) has the form \( y = m_1 + n_1 = F(m_0) + n_1 \), where \( m_1 \in M_1, n_1 \in N_1, m_0 \in M_0 \). On the other hand, \( m_0 = v_0 + n_0 \), where \( v_0 \in V_0, n_0 \in W_0 = N_0 \), and

\[
y = F(v_0 + n_0) + n_1 = F(v_0) + (F(n_0) + n_1) \in V_1 + N_1.
\]

Hence, \( H_A = V_1 + W_1 \).

Let \( y \in V_1 \cap W_1 = V_1 \cap N_1 \), so that \( n_1 = y = F(v_0), n_1 \in N_1, v_0 \in V_0 \). Let us decompose \( v_0 + n_0 \), where \( m_0 \in M_0, n_0 \in N_0 \). Then

\[
n_1 = F(m_0) + F(n_0),
\]

\[
F(m_0) = n_1 - F(n_0), \quad F(m_0) \in M_1, \quad n_1 - F(n_0) \in N_1.
\]

Hence \( F(m_0) = 0, \quad n_1 - F(n_0) = 0 \), and since \( F : M_0 \cong M_1 \), then \( m_0 = 0 \). We have \( v_0 \in V_0 = N_0^\perp \) and hence,

\[
0 = \langle v_0, n_0 \rangle = \langle m_0 + n_0, n_0 \rangle = \langle n_0, n_0 \rangle, \quad n_0 = 0.
\]

So, \( v_0 = m_0 + n_0 = 0, y = F(v_0) = 0 \). Hence \( V_1 \cap W_1 = 0 \) and \( H_A = V_1 \oplus W_1 \).

By 4.5 \( V_1 \) has an orthogonal complement \( V_1^\perp, V_1 \oplus V_1^\perp = H_A \), and this completes the proof. □

4.8. Remark. If we do not restrict the operator \( F \) to admit a conjugate, we can assert that there exists a decomposition

\[
F : N_0^\perp \oplus N_0 \to M_1 \oplus L_n,
\]

where \( L_n = \text{span}_A(e_1, \ldots, e_n) \), but \( M_1 \) may have no orthogonal complement. This result was proved in [6].

5. LEFSCHETZ NUMBERS WITH VALUES IN \( HC_0(A) \)

5.1. Definition. Let \( \{ e_1, e_2, \ldots \} \) be an \( A \)-orthobasis of \( H_A = l_2(A) \) (the Hilbert module over \( A \)) with \( A \)-inner product \( (\cdot, \cdot) \). Let \( S \in \text{End}_A^* H_A \) (the \( A \)-linear endomorphisms of \( H_A \) admitting an adjoint) and \( S(e_i) = 0 \) (\( i > k \)). We define the trace of \( S \) by

\[
t(S, \{ e_i \}, k) = \sum_{i=1}^\infty f((S e_i, e_i)) = \sum_{i=1}^k f(S_i^j),
\]

where \( f : A \to A/[A, A] = HC_0(A) \), \( S_j^i \) is the matrix of \( S \) with respect to \( \{ e_i \}, S_j^i \in A \).
5.2. Lemma. \( t(S, \{e_i\}, k) = t(S, \{e_i\}, l) := t(S, \{e_i\}) \) for \( l \geq k \).

The proofs of this lemma and the other statements of this Section can be found in [18].

5.3. Lemma. Let \( S, \{e_i\}, k \) be as in 5.1 and \( \{h_j\} \) a new \( A \)-basis of \( H_A \) (in general non-orthogonal). Then the series

\[
\sum_{r=1}^{\infty} f((S_h)_r^r)
\]

converges to \( t(S, \{e_i\}) \), where \( (S_h)_r^r \) are the matrix elements of \( S \) with respect to \( \{h_i\} \).

Let us note that a basis of \( H_A \) is a system of elements \( \{h_i\} \), such that \( h_i = Be_i \), where \( B \in GL^* \) (automorphisms admitting a conjugate). The matrix of \( S \) with respect to \( \{h_i\} \) is the matrix of \( B^{-1}SB \) with the respect to \( \{e_i\} \), i.e., \( (S_h)_j^i = (B^{-1}SB)_j^i = \langle B^{-1}SBe_i, e_j \rangle \).

So we can give instead of 5.1 the following correct definition.

5.4. Definition. Let \( S, N, \tilde{S} \in \text{End}_A H_A, M \oplus \tilde{N} \) Hilbert submodules of \( H_A, N \) finitely generated, \( H_A = M \oplus N, S|_M = 0 \). For an arbitrary basis \( \{e_i\} \) we define

\[
t(S) = \sum_{i=1}^{\infty} f(S_i^i).
\]

5.5. Lemma. Let \( M, \tilde{N}, S \) be as in 5.4, and \( \tilde{N} \) be a countably generated Hilbert \( A \)-module, \( H_A = H_A \bigoplus \tilde{N} \cong H_A, \)

\[
\tilde{S} = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} : H_A \bigoplus \tilde{N} \to H_A \bigoplus \tilde{N}.
\]

Then \( t(S) = t(\tilde{S}) \).

5.6. Lemma. Let \( M, N, S \) be as in 5.4, \( M \cong H_A, N = \tilde{N} \oplus \tilde{N}, S|_{\tilde{N}} = 0 \). Then

\[
t(S) = t(pSp),
\]

where \( p : M \oplus \tilde{N} \oplus \tilde{N} \to M \oplus \tilde{N} \) is a projection, and the sum on the right is in the space \( M \oplus \tilde{N} \cong H_A \). Let us notice, that if we denote by

\[
q : M \oplus N \to M, \quad p_1 : N \to \tilde{N}
\]

the projections, then they admit conjugates. Hence, the projection \( p = q + p_1(1 - q) \) admits one, too.

5.7. Corollary. If in 5.5 \( M \oplus \tilde{N} \) is orthogonal to \( \tilde{N} \), and \( \{h_i\} \) is an \( A \)-orthobasis of \( M \oplus \tilde{N} \), then

\[
\sum_{i=1}^{\infty} f(\langle Sh_i, h_i \rangle) \]


**Definition.** Let $F : H_A \to H_A$ be an $A$-Fredholm operator (admitting an adjoint),

$$
\begin{pmatrix}
F_1 & 0 \\
0 & F_2
\end{pmatrix} : H_A = M_0 \oplus N_0 \to M_1 \oplus N_1 = H_A \quad (D)
$$

a corresponding decomposition, restricted to satisfy the condition as in 4.7 (we always will assume this without specification). Let $S_0$ and $S_1$ be operators from $\text{End}_A H_A$, such that the diagram

$$
\begin{array}{ccc}
H_A & \xrightarrow{F} & H_A \\
\downarrow S_0 & & \downarrow S_1 \\
H_A & \xrightarrow{F} & H_A.
\end{array}
$$

commutes. Let

$$
S_0 = \begin{cases}
0 \text{ on } M_0, \\
S_0 \text{ on } N_0,
\end{cases} \quad \tilde{S}_1 = \begin{cases}
0 \text{ on } M_1, \\
S_1 \text{ on } N_1.
\end{cases}
$$

Let us define

$$
L(F, S, D) = t(\tilde{S}_0) - t(\tilde{S}_1).
$$

**5.9. Lemma.** Let

$$
\begin{align*}
H_A &= M_0 \oplus N_0 \to M_1 \oplus N_1 = H_A, \quad (D) \\
H_A &= M_0 \oplus N_0 \to \tilde{M}_1 \oplus \tilde{N}_1 = H_A \quad (\tilde{D})
\end{align*}
$$

then

$$
L(F, S, D) = L(F, S, \tilde{D}).
$$

**5.10. Lemma.** Let

$$
\begin{align*}
H_A &= (M_0 \oplus N_0) \oplus K_0 \to (M_1 \oplus N_1) \oplus K_1 = H_A, \quad (D_1) \\
H_A &= M_0 \oplus (N_0 \oplus K_0) \to M_1 \oplus (N_1 \oplus K_1) = H_A \quad (D_2)
\end{align*}
$$

be two decompositions for $F$. Then $L(F, S, D_1) = L(F, S, D_2)$.

**5.11. Lemma.** Let

$$
H_A = M_0 \oplus N_0 \to M_1 \oplus N_1 = H_A \quad (D)
$$

and

$$
H_A = \tilde{M}_0 \oplus \tilde{N}_0 \to \tilde{M}_1 \oplus \tilde{N}_1 = H_A \quad (\tilde{D})
$$

be two decompositions for $F$. Then $L(F, S, D) = L(F, S, \tilde{D})$. So $L$ does not depend on $D$ and we denote it by $L(F, S)$.

**5.12. Remark.** By the stabilization theorem and Lemma 5.5, we can define $L(F, S)$ for any countably generated Hilbert $A$-module instead of $H_A$. 


5.13. **Definition.** Let $T = \{T_i\}$ be an endomorphism of an $A$-elliptic complex $E$:

\[
\begin{align*}
0 & \rightarrow \Gamma(E_0) \xrightarrow{d_0} \Gamma(E_1) \rightarrow \cdots \\
& \downarrow T_0 \quad \downarrow T_1 \\
0 & \rightarrow \Gamma(E_0) \xrightarrow{d_0} \Gamma(E_1) \rightarrow \cdots \\
\end{align*}
\]

$T_{i+1}d_i = d_iT_i$, $T_i \in \text{End}^*_A \Gamma(E_i)$.

Assume the following

5.14. **Condition.** Sobolev products in $\Gamma(E)$ can be chosen in such a way that

\[T_id_i^* = d_i^*T_{i+1}.\]

We take $E_{ev} = \oplus E_{2i}$, $E_{od} = \oplus E_{2i+1}$,

\[F = d + d^* : \Gamma(E_{ev}) \rightarrow \Gamma(E_{od}).\]

Then $F$ is an $A$-Fredholm operator and the diagram stated below commutes, where

\[
\begin{array}{ccc}
\Gamma(E_{ev}) & \xrightarrow{F} & \Gamma(E_{od}) \\
\downarrow s_0 & & \downarrow s_1 \\
\Gamma(E_{ev}) & \xrightarrow{F} & \Gamma(E_{od}).
\end{array}
\]

We define the Lefschetz number of the second type as

\[L_0(E, T, m) = L(F, S) \in HC_0(A),\]

where $m$ denotes the dependence on inner products (via $d^*$).

5.15. **Lemma.** Let $T = T_g$, $g \in G$ as in §2. Then the condition 5.14 is fulfilled.

5.16. **Theorem.** If $T = T_g$, $g \in G$, then

\[L_0(E, T_g, m_G) = \text{Ch}_0(L_1(g, E)),\]

where $\text{Ch}_0$ is the Chern character

\[\text{Ch}_0 : K_0(A) \rightarrow HC_0(A)\]

(see [3, 7, 8]), and

\[\text{Ch}_0(a \otimes z) = \text{Ch}_0(a)z, \quad z \in \mathbb{C}.\]
In particular, $L_0$ does not depend on $m_G$.

Proof. We have

$$L_1(g, E) = \text{ind}_{G, A}^X([\sigma(E)])(g) = \text{ind}_{G, A}^X(F)(g).$$

Let

$$M_0 \oplus N_0 \to M_1 \oplus N_1 \quad (D)$$

be a decomposition for $F$. Then by 2.8 and [15]

$$N_0 = \bigoplus_{k=1}^KV_k \otimes P_k, \quad N_1 = \bigoplus_{l=1}^LW_l \otimes Q_l,$$

where $V_k$ and $W_l$ are $\mathbb{C}$-vector spaces of irreducible representations of $G$, $P_k$ and $Q_l$ are $G$-trivial projective finitely generated $A$-modules. Then (representations are unitary)

$$\text{ind}_{G, A}^X(F) = \sum_{k=1}^K[P_k] \otimes \chi(V_k) - \sum_{l=1}^L[Q_l] \otimes \chi(W_l)$$

and

$$L_1(g, E) = \sum_{k=1}^K[P_k] \otimes \text{Trace}(g|V_k) - \sum_{l=1}^L[Q_l] \otimes \text{Trace}(g|W_l).$$

The end of the proof see in [18].

6. LEFSCHETZ NUMBERS WITH VALUES IN $HC_2l(A)$

Let $W^*A$ be the universal enveloping von Neumann algebra of the algebra $A$ with the norm topology. Let $U$ be a unitary operator in the Hilbert module $A^n$. Then

$$U = \int_{S^1} \epsilon^{i\varphi} dP(\varphi),$$

where $P(\varphi)$ is the projection valued measure valued in the space of matrices $M(n, W^*A)$, and the integral converges with respect to the norm. Let us associate with the integral sum

$$\sum_k \epsilon^{i\varphi_k} P(E_k)$$

the following class of the cyclic homology $HC_{2l}(M(n, W^*A))$:

$$\sum_k P(E_k) \otimes \ldots \otimes P(E_k) \cdot \epsilon^{i\varphi_k}.$$

Passing to the limit we get the following element

$$TU = \int_{S^1} \epsilon^{i\varphi} d(P \otimes \ldots \otimes P)(\varphi) \in HC_{2l}(M(n, W^*A)).$$

Then we define

$$T(U) = \text{Tr}_n^* TU \in HC_{2l}(W^*A).$$
6.1. Lemma. Let \( J : M = A^m \to N = A^n \) be an isomorphism, \( U_M : M \to M, U_N : N \to N \) be \( A \)-unitary operators and \( JU_M = U_N J \). Then
\[
T(U_M) = T(U_N).
\]

Proof. If
\[
U_M = \int_{S^1} e^{i \varphi} \, dP(\varphi),
\]
then
\[
U_N = JU_M J^{-1} = \int_{S^1} e^{i \varphi} \, dP J^{-1}(\varphi).
\]
To verify the equality \( T(U_M) = T(U_N) \) it is sufficient to verify that
\[
\text{Tr}_n^m \left[ \sum_k P(E_k) \otimes \ldots \otimes P(E_k) \cdot e^{i \varphi_k} \right] =
\]
\[
= \text{Tr}_n^m \left[ \sum_k JP(E_k) J^{-1} \otimes \ldots \otimes JP(E_k) J^{-1} \cdot e^{i \varphi_k} \right] \in HC_{2l}(W^* A),
\]
but this follows from well-definedness of the Chern character \( \text{Ch}^0_{2l} : K_0(B) \to HC_{2l}(B) \) (see [3, 8]).

Let now \( U \) be equal to \( U_g \), i.e. an operator representing \( g \in G \). Then (3) turns to be the sum associated with the decomposition from 2.8 and [15]
\[
A^n \cong \bigoplus_{k=1}^M Q_k \otimes V_k,
\]
where \( V_k \cong \mathbb{C}^{L_k} \), and \( Q_k \) are projective \( A \)-modules of finite type. Then
\[
U_g \left( \sum_{k=1}^M x_k \otimes v_k \right) = \sum_{k=1}^M x_k \otimes u^k_g v_k = \sum_{k=1}^M \sum_{l=1}^{L_k} x_k \otimes e^{i \varphi^k_l} v^l_k f_l,
\]
where \( f_1, \ldots, f_{L_k} \) is the diagonalizing basis for \( u^k_g \); \( v_k = \sum v^l_k f_l \). Then we can define
\[
(4) \quad \tau(U_g) = \sum_{k=1}^M \sum_{l=1}^{L_k} \text{Ch}^0_{2l}[P_k] \cdot \text{Tr}(u^k_g) \in HC_{2l}(A).
\]
We have \( T(U_g) = i_*(\tau(U_g)) \), where \( i : A \to W^* A \).

A similar technique can be developed for a projective module \( N \) instead of \( A^n \). For this purpose we take \( N = q(A^n) \),
\[
U \oplus 1 : A^n \cong N \oplus (1 - q) A^n \to N \oplus (1 - q) A^n \cong A^n,
\]
\[
TU = \int_{S^1} e^{i \varphi} \, d(q P q \otimes \ldots \otimes q P q)(\varphi).
\]
The well-definedness is an immediate consequence of Lemma 6.1.

Let us consider a \( G \)-invariant \( A \)-elliptic complex \((E, d)\), and let the Sobolev \( A \)-products be chosen invariant, so that \( T_g = U_g \) are unitary operators (see §3).
6.2. Lemma. We can choose a decomposition for the $A$-Fredholm operator

$$ F = d + d^* : \Gamma(E_{ev}) \to \Gamma(E_{od}), $$

$$ F : M_0 \oplus N_0 \to M_1 \oplus N_1, \quad F : M_0 \cong M_1, $$
such that

$$ N_0 = \oplus_i N_{2i}, \quad N_{2i} \subset \Gamma(E_{2i}), $$

$$ N_1 = \oplus_i N_{2i+1}, \quad N_{2i+1} \subset \Gamma(E_{2i+1}), $$

where $N_m$ are projective invariant modules.

Proof. Let us assume that the complex consists of operators of the degree $m$, so $F = \partial + \partial^*$ is an $A$-Fredholm operator in the spaces $H^m(E_{ev}) \to H^0(E_{od})$. We can choose the basis in $H^m(E_{ev})$ (or the decomposition into modules $P_j$ in $l_2(P)$) in such a way that $e_{m+j} \in \Gamma(E_{2j})$, where $E_0, E_2, \ldots, E_{2j}, \ldots, E_{2m}$ are all non-zero terms of the complex, $s \in \mathbb{N}$, and $j = 0, \ldots, m$ (and in a similar way for $P_j$). As usual, without loss of generality we can assume that

$$ N_0 = \operatorname{span}_A(e_1, \ldots, e_{n_0}), \quad M_0 = \operatorname{span}_A(e_{n_0+1}, e_{n_0+2}, \ldots), $$

and $M_1 = F(M_0)$ has in $H^0(E_{od})$ the $A$-orthogonal complement $M_1^\perp$. Then for every $x \in M_1$, $y \in N_0$

$$ \langle x, Fy \rangle = \langleFx, y \rangle_0, $$

(5)

where the first brackets mean the pairing of a functional and an element. So, $F(N_0) \subset M_1^\perp$ and taking $N_1 = M_1^\perp$, we get a decomposition $F : M_0 \oplus N_0 \to M_1 \oplus N_1$.

Let

$$ y = y_1 + y_3 + \cdots + y_{2m+1} \in N_1 \subset H^0(E_{od}), \quad y_{2j+1} \in H^0(E_{2j+1}), $$

and

$$ x = x_0 + x_2 + \cdots + x_{2m} \in M_0 \subset H^m(E_{ev}), \quad x_{2j} \in H^m(E_{2j}). $$

Then $\langle Fx, y \rangle_0 = 0$, where

$$ Fx = d^*x_0 + \sum_{i=1}^m (dx_{2i-2} + d^*x_{2i}) + dx_{2m} \in 0 \oplus \oplus_{i=1}^m H^0(E_{2i+1}) \oplus 0. $$

Since $(E, d)$ is a complex, $d^2 = 0$ and

$$ \langle du, d^*v \rangle = \langle d^2u, v \rangle = 0, $$

so

$$ \langle y_{2j+1}, dx \rangle = 0, \quad \langle y_{2j+1}, d^*x_{2j+2} \rangle = 0 \quad (j = 0, 1, \ldots, m) $$

Hence $e_{2j+1} \in F(M_0) = M_1^\perp = N_1$, and

$$ N_1 = \oplus_i (N_1 \cap \Gamma(E_{2i+1})) = \oplus_i N_{2i+1}. $$

\]
6.3. Definition. The Lefschetz number $L_{2l}$ we define as

$$L_{2l}(E, U_g, m_G) = \sum_i (-1)^i \tau(U_g|N_i) \in HC_{2l}(A),$$

where $m_G$ denotes the dependence on inner products (via $d^*$).

Remark. For more general situations we hope to use $T$ instead of $\tau$.

6.4. Lemma. The definition of $L_{2l}$ is correct, i.e. this number does not depend on the choice of decompositions in Lemma 6.2.

Proof. For any two decompositions we can by use of projection (as in [13, 15]) replace $\tilde{N}_0$ by a module inside span$_A(e_1, \ldots, e_n)$ for a sufficiently great $n$ (we use the notation of Lemma 6.2). By 6.1 $\tau(U_g|N_i)$ does not change under this replacement. So we can assume that we have to compare the decomposition as in 6.2 and the decomposition

$$F: \tilde{M}_0 \oplus \tilde{N}_0 \to \tilde{M}_1 \oplus \tilde{N}_1,$$

$$\tilde{N}_0 = \oplus_i \tilde{N}_{2i}, \quad \tilde{N}_{2i} \subset N_{2i} \subset \Gamma(E_{2i}),$$

$$\tilde{N}_1 = \oplus_i \tilde{N}_{2i+1}, \quad \tilde{N}_{2i+1} \subset \Gamma(E_{2i}).$$

Hence by (5), $\tilde{N}_{2i+1} \subset N_{2i+1}$. Let $K_i = (\tilde{N}_i)^\bot_{\tilde{N}_i}$. Then $F: K_{2i} \cong K_{2i+1}$ and by Lemma 6.1 we get $\tau(U_g|K_{2i}) = \tau(U_g|K_{2i+1})$. Hence

$$\sum_i (-1)^i \tau(U_g|N_i) = \sum_i (-1)^i (\tau(U_g|\tilde{N}_i) + \tau(U_g|K_i)) =$$

$$= \sum_i (-1)^i (\tau(U_g|\tilde{N}_i)). \quad \blacksquare$$

6.5. Theorem. Let $\text{Ch}^0_{2l}(a \otimes z) = \text{Ch}^0_{2l}(a) \cdot z$, where $z \in \mathbb{C}$. Then

$$L_{2l}(E, U_g, m_G) = \text{Ch}^0_{2l}(L_1(g, E)),$$

in particular, $L_{2l}$ does not depend on $m_G$.

Proof. We get the statement immediately from (2) and (4). $\blacksquare$

References


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