Functionals on $l_2(A)$, Kuiper and Dixmier-Douady type theorems for C*-Hilbert modules

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Abstract

Let us denote by $\mathrm{End}_A l_2(A)$ the Banach algebra of all bounded A-homomorphisms of Hilbert A-module $l_2(A)$, and by $\mathrm{End}^*_A l_2(A)$ the C^*-algebra of operators, admitting adjoint. Let $\mathrm{GL}\,(A)$ and $\mathrm{GL}^*(A)$ denote the correspondent groups of invertible operators. In the present paper we give a simple proof of the theorem of Cuntz and Higson on the contractibility of $\mathrm{GL}^*(A)$ for A with strictly positive element. We prove the contractibility $\mathrm{GL}\,(A)$ in some special cases, in particular, for A, being a subalgebra of algebra of compact operators in separable Hilbert space, and for $A = C_0(M)$, where M is a finite-dimensional manifold. We prove some generalizations of the theorem of Dixmier and Douady to the cases of $\mathrm{GL}\,(A)$ and $\mathrm{GL}^*(A)$ for σ–unital A.

1 Introduction

Let us denote by $\mathrm{End}_A l_2(A)$ the Banach algebra of all bounded A-homomorphisms of Hilbert A-module $l_2(A)$, and by $\mathrm{End}^*_A l_2(A)$ the C^*-algebra of operators, admitting adjoint. Let $\mathrm{GL}\,(A)$ and $\mathrm{GL}^*(A)$ denote the correspondent groups of invertible operators. The question about the contractibility of general linear groups is very important for K–theory to construct classifying spaces in terms of Fredholm operators. To this problem a series of papers is devoted: [13, 7, 21, 14]. The author used these results to construct the classifying spaces of K-theory $K^{p,q}(X; A)$ [20] which arises in analytical approach to the Novikov Conjecture on higher signatures. In paper [2] J. Cuntz and N. Higson proved the contractibility of $\mathrm{GL}^*(A)$ for A with strictly positive element (or, equivalent, with countable approximate unit = σ-unital).

In the present paper we give a simple proof of the theorem of Cuntz and Higson, distinguished from original, and based on generalization of a construction of homotopy from [17]. We also show, that the similar reasonings are aplicable to prove the contractibility

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GL (A) in some special cases, in particular, for A, being a subalgebra of algebra of compact operators in separable Hilbert space, and for $A = C_0(M)$, where M is a finite-dimensional manifold.

In the classical paper of Dixmier and Douady [3] it is proved the contractibility of the group of unitary operators in Hilbert space with the respect to strong topology. We prove some generalizations of this theorem to the cases of GL (A) and GL$^*(A)$ for σ–unital A. Instead of strong topology we use here the strict topology.

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2 Preliminary notes

It is known, that the set of invertible operators in a Banach space is open with the respect to the topology of a norm, while the set of bounded A-homomorphisms is closed in the set of all endomorphisms. Thus, GL is an open set in a Banach space. The similar argument is valid for GL*. According to the Milnor theorem [12] such sets have the homotopy type of CW-compexes, and, therefore, by the theorem of Whitehead, strong and weak homotopy triviality are equivalent for them. We have proved the following statement.

Lemma 2.1 *To prove the contractibility* GL (*resp.,* GL*) *it is sufficient to verify the following. Let $f : S \to$ GL be a continuous map of a sphere of arbitrary dimension. Then f is homotopic to the map to the single point* Id $\in$ GL . *The similar statement holds for* GL*. □

Let us produce one more reduction. To consider simultaneously case GL and case GL*, we shall enter a common notation: $\mathcal{G} :=$ GL (resp., GL*), $\mathcal{E}(\mathcal{M}) := \mathrm{End}_A(\mathcal{M})$ (resp., $\mathrm{End}^*_A(\mathcal{M})$).

Lemma 2.2 (a variant of the Atiyah theorem about small balls) *Let $f : S \to \mathcal{G}$ be a continuous map of a sphere of arbitrary finite dimension. Then f is homotopic to a map f' such that $f'(S)$ is a finite polyhedron in $\mathcal{E}(l_2(A))$, laying in $\mathcal{G}$ together with the homotopy.*

Proof: Let $\varepsilon > 0$ be such that ε-neighborhood of the compact set $f(S)$ lays in $\mathcal{G}$. Let us choose a fine simplicial subdivision of the sphere S, such that $diam(f(\sigma)) < \varepsilon/2$ for any simplex σ of this subdivision. It is possible to do this, since S is compact. Let f' be a piecewise linear map, being the extension of the restriction f to the 0-dimensional sceleton. Thus $diam(f'(\sigma)) \le diam(f(\sigma)) < \varepsilon/2$ for any s. For any point $s \in S$ there exists a vertex $s_i \in S$, such that $\|f(s) - f'(s_i)\| = \|f(s) - f(s_i)\| < \varepsilon/2$ and $\|f'(s) - f'(s_i)\| < \varepsilon/2$, hence the segment $[f(s), f'(s)] \subset \mathcal{G}$ for any point $s \in S$. Therefore, the linear homotopy $f_t(s) = tf'(s) + (1-t)f(s)$ is in $\mathcal{G}$. Passing to a subdivision of $f'(S)$, we obtain a structure of simplicial complex. □

Remark 2.3 Let us remark, that this argument is not valid for other topologies, which we shall consider. For example, with the respect to the strong topology on operators in a Hilbert space, the sequence Id_n converges to Id, where Id_n has the matrix $diag(1, \dots, 1, 0, 0, \dots)$ (unit up to n-th place). So that with the respect to this topology the general linear group is not an open set.

One more step from the original work of Kuiper [9] is universal. Let us denote orthogonal (A-Hilbert) sum by $\oplus$ and Banach one by $\widetilde{\oplus}$.

Lemma 2.4 *Subset $V \subset \mathcal{G}$, defined as*

$$V = \{g \in \mathcal{G} \mid g|_{H'} = \mathrm{Id}_{H'},\ g(H_1) = H_1\},$$

where

$$l_2(A) = H' \oplus H_1, \qquad H' \cong H_1 \cong l_2(A),$$

is contractible in $\mathcal{G}$ to $1 \in \mathcal{G}$.

Proof: Let us represent H' as

$$H' = H_2 \oplus H_3 \oplus \dots, \qquad H_i \cong l_2(A),$$

so that $l_2(A) = H_1 \oplus H_2 \oplus H_3 \oplus \dots$. The matrix of g with the respect to this decomposition has the form

$$m(1,1) = u = g|_{H_1}, \quad m(i,i) = 1 \in \mathcal{E}(H_i),\ i > 1, \quad m(i,j) = 0,\ i \neq j,$$

$$g = diag(u, 1, 1, 1, \dots) = diag(u, u^{-1}u, 1, u^{-1}u, 1, \dots).$$

We want so to define a homotopy $g_t \in \mathcal{G}$, $t \in [0, \pi]$, in such a way that

$$g_0 = g, \qquad g_{\pi/2} = diag(u, u^{-1}, u, u^{-1}, u, \dots), \qquad g_\pi = diag(1, 1, 1, \dots) = \mathrm{Id} \in \mathcal{G}.$$

For this purpose let us put for $t \in [0, \pi/2]$

$$m_t(1,1) = u,$$

for $i \geq 1$

$$\begin{pmatrix} m_t(2i,2i) & m_t(2i,2i+1) \\ m_t(2i+1,2i) & m_t(2i+1,2i+1) \end{pmatrix} =$$

$$= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix},$$

$$m_t(r,s) = 0 \qquad \text{for remaining } r,\ s.$$

Let us put for $t \in [\pi/2, \pi]$

$$\begin{pmatrix} m_t(2i-1,2i-1) & m_t(2i-1,2i) \\ m_t(2i,2i-1) & m_t(2i,2i) \end{pmatrix} =$$

$$= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix},$$

$$m_t(r,s) = 0 \qquad \text{for remaining } r,\ s. \qquad \square$$

Lemma 2.5 *Subset $W \subset \mathcal{G}$, defined as*

$$W = \{g \in \mathcal{G} \mid g|_{H'} = \mathrm{Id}_{H'}\},$$

where

$$l_2(A) = H' \oplus H_1, \qquad H' \cong H_1 \cong l_2(A),$$

is contractible inside $\mathcal{G}$ to

$$V = \{g \in \mathcal{G} \mid g|_{H'} = \mathrm{Id}_{H'},\, g(H_1) = H_1\}.$$

Proof: With the respect to the decomposition $l_2(A) = H' \oplus H_1$ we define a homotopy by the formula

$$f_t(s) = \begin{pmatrix} 1 & \beta(s)(1-t) \\ 0 & \gamma(s) \end{pmatrix}.$$

$$F_t(s) = \begin{pmatrix} 1 & \beta(1-t) \\ 0 & \gamma \end{pmatrix}.$$

Let the operator $\begin{pmatrix} \varphi & \psi \\ \chi & \xi \end{pmatrix}$ be the inverse to $\begin{pmatrix} 1 & \beta \\ 0 & \gamma \end{pmatrix}$. Then

$$\begin{pmatrix} \varphi & \psi \\ \chi & \xi \end{pmatrix} \begin{pmatrix} 1 & \beta \\ 0 & \gamma \end{pmatrix} = \begin{pmatrix} \varphi & \varphi\beta + \psi\gamma \\ \chi & \chi\beta + \xi\gamma \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & \beta \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} \varphi & \psi \\ \chi & \xi \end{pmatrix} = \begin{pmatrix} \varphi + \beta\chi & \psi + \beta\xi \\ \gamma\chi & \gamma\xi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

whence

$$\varphi = 1, \qquad \chi = 0, \qquad \gamma\xi = \xi\gamma = 1,$$
$$\beta + \psi\gamma = 0, \qquad \psi + \beta\xi = 0,$$

and

$$\begin{pmatrix} 1 & \psi(1-t) \\ 0 & \xi \end{pmatrix} \begin{pmatrix} 1 & \beta(1-t) \\ 0 & \gamma \end{pmatrix} = \begin{pmatrix} 1 & \beta(1-t) + (1-t)\psi\gamma \\ 0 & \xi\gamma \end{pmatrix} = \begin{pmatrix} 1 & (1-t)\cdot 0 \\ 0 & 1 \end{pmatrix},$$

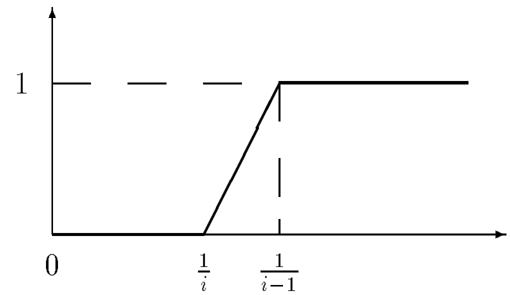
$$\begin{pmatrix} 1 & \beta(1-t) \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} 1 & \psi(1-t) \\ 0 & \xi \end{pmatrix} = \begin{pmatrix} 1 & \psi(1-t) + \beta\xi(1-t) \\ 0 & \gamma\xi \end{pmatrix} = \begin{pmatrix} 1 & (1-t)\cdot 0 \\ 0 & 1 \end{pmatrix}.$$

Hence, the homotopy lies in $\mathcal{G}$. □

3 Technical lemmas

Let by $\mathcal{K}_A$ be denoted the C^*-algebra of A-compact operators on $l_2(A)$, by $\mathbf{LM}(\mathcal{K}_A) \cong \mathrm{End}_A l_2(A)$ the algebra of the left multipliers, by $\mathbf{M}(\mathcal{K}_A) \cong \mathrm{End}^*_A l_2(A)$ the C^*-algebra of multipliers and by $\mathbf{QM}(\mathcal{K}_A) \cong \mathrm{End}_A(l_2(A), l_2(A)')$ the space of quasi-multipliers (see [1, 11, 6, 8, 16, 18]).

Let α be a strictly positive element in σ-unital algebra A, $\alpha_i := \varphi_i(\alpha)$ be a countable approximate unit, where φ_i has the graph



$\omega_i := (\alpha_i - \alpha_{i-1})^{1/2}$ для $i \geq 3$ и $\omega_2 = \alpha_2^{1/2}$, так что

$$\omega_j \alpha_i = \alpha_i \omega_j = 0, \quad j = i+2, i+3, \ldots, \qquad \omega_j \alpha_i = \alpha_i \omega_j = \omega_j, \quad j = 1, \ldots, i-1. \quad (1)$$

Since there is no unit in A, the notion of "standard base" $\{e_i\}$ of module $l_2(A)$ makes no sense. Nevertheless, it is possible to define properly elements $e_i\gamma$ for any $\gamma \in A$, namely,

$$e_i\gamma := (0, \ldots, 0, \gamma, 0, \ldots), \qquad \gamma \text{ at } i\text{-th place.}$$

Let us denote the correspondent orthoprojections on these one-dimensional submodules E_i by Q_i.

Lemma 3.1 *The injection $i : A \to l_2(A)$, defined by the formula*

$$x \mapsto \sum_i e_{k(i)} \omega_i x, \qquad k(1) < k(2) < k(3) < \ldots,$$

remain the inner product and admits adjoint. In particular, the image $\operatorname{Im} i$ *is defined by a selfadjoint projection of the form*

$$p = ii^*. \quad (2)$$

Proof: First of all,

$$\begin{aligned} \langle ix, iy \rangle &= \langle \textstyle\sum_i e_{k(i)} \omega_i x, \sum_i e_{k(i)} \omega_i y \rangle = \sum_i \langle e_{k(i)} \omega_i x, e_{k(i)} \omega_i y \rangle = \\ &= \textstyle\sum_i x^* \omega_i \omega_i y = x^* y = \langle x, y \rangle. \end{aligned}$$

Let us consider operator $t : l_2(A) \to A$ of the form

$$t(z) := \sum_i \langle e_{k(i)} \omega_i, z \rangle = \sum_i \omega_i z_{k(i)}.$$

This series satisfies to the Cauchy criterion: if number m is so great, that

$$\sum_{i=m+1}^{\infty} z_i^* z_i < \delta,$$

then

$$\| \sum_{i=s}^{r} \omega_i z_{k(i)} \| \leq \| \sum_{i=s}^{r} \omega_i^2 \|^{1/2} \cdot \| \sum_{i=s}^{r} z_{k(i)}^* z_{k(i)} \|^{1/2} \leq 1 \cdot \delta.$$

The same reasoning for $s = 1$ implies the relation $\|t(z)\| \leq \|z\|$. Also, $\langle ix, z \rangle = \langle x, tz \rangle$, i. e., $t = i^*$.

Let us consider arbitrary elements $x, y \in A$. Then

$$(i^*ix)^*y = \langle i^*ix, y\rangle = \langle ix, iy\rangle = \langle x, y\rangle = x^*y.$$

Since y is an arbitrary element, we conclude, that $i^*ix = x$ and $i^*i = \mathrm{Id}$. Hence,

$$ii^*ii^* = ii^*,$$

i. e., p is a projection. Since $i^*i = \mathrm{Id}$, i^* is an epimorphism and $\mathrm{Im}\, i = \mathrm{Im}\, p$ (see also [10, Sect. 3]). □

We need some more strong variant of this lemma.

Lemma 3.2 *The injection $J : l_2(A) \to l_2(A)$ under the formula*

$$(a_1, a_2, \ldots) \mapsto \sum_j \sum_i v_{ij} a_j, \qquad \langle v_{ij}, v_{ij}\rangle = \omega_i^2, \qquad v_{ij} \in M_{k(i,j)},$$

$$l_2(A) = M_1 \oplus M_2 \oplus \ldots, \qquad M_r = \{\, (0, \ldots, 0, a_{s(r)}, \ldots, a_{s(r+1)-1}, 0, \ldots)\,\},$$
$$\{k(1,1); k(1,2), k(2,1); k(1,3), k(2,2), k(3,1); \ldots\} = \{1, 2, \ldots\},$$

remains the inner product and admits an adjoint. In particular, the image is defined by a selfadjoint projection of the form JJ^*.

Proof: Let $x = (a_1, a_2, \ldots) \in l_2(A)$, $y = (b_1, b_2, \ldots) \in l_2(A)$. Then

$$\begin{aligned}\langle Jx, Jy\rangle &= \langle \textstyle\sum_j \sum_i v_{ij} a_j, \sum_j \sum_i v_{ij} b_j\rangle = \sum_j \sum_i a_j^* \omega_i^2 b_j = \sum_j a_j^* \left(\sum_i \omega_i^2\right) b_j = \\ &= \textstyle\sum_j a_j^* b_j = \langle x, y\rangle.\end{aligned}$$

In particular, J is bounded. Let us consider operator $T : l_2(A) \to l_2(A)$ of the form

$$T(z) := (t_1, t_2, \ldots), \quad t_j := \sum_i \langle v_{ij}, z\rangle.$$

For this series the Cauchy criterion is carried out: let number $N = N(z)$ be so great, that $\|(1 - p_N)z\| < \delta$ and m be so great, that $s(k(m,j)) > N$ (j is fixed), (by [18])

$$\|\sum_{i=m}^{r} \langle v_{ij}, z\rangle\| = \|\left\langle \sum_{i=m}^{r} v_{ij}, (1 - p_N)z \right\rangle\| \le \left\| \left\langle \sum_{i=m}^{r} v_{ij}, \sum_{i=m}^{r} v_{ij} \right\rangle \right\|^{1/2} \cdot \|(1 - p_N)z\| \le 1 \cdot \delta.$$

For any r by [18] the following inequality holds

$$\sum_{i=1}^{r} \langle v_{ij}, z\rangle^* \sum_{i=1}^{r} \langle v_{ij}, z\rangle = \left\langle \sum_{i=1}^{r} v_{ij}, q_j z \right\rangle^* \left\langle \sum_{i=1}^{r} v_{ij}, q_j z \right\rangle \le \langle q_j z, q_j z\rangle,$$

where q_j is the orthoprojection on $\bigoplus_i M_{k(i,j)}$. Hence

$$t_j^* t_j \le \langle q_j z, q_j z\rangle, \qquad \langle T(z), T(z)\rangle \le \langle z, z\rangle.$$

So, T is bounded, and the fact, that it is the adjoint for J is obvious.

The proof of the second statement literally repeats the reasoning from the previous lemma. □

Let us consider an operator $F \in \mathrm{GL}$. Then, with the respect to the standard decomposition $l_2(A)$ into the direct sum of $E_i \cong A$, the operator F has a matrix F^i_j with the elements from $\mathbf{LM}(A)$. If $F \in \mathrm{GL}^*$, $F^i_j \in \mathbf{M}(A)$, since $(F^*)^i_j = (F^j_i)^*$. Let us note, that for any $b \in A$ and any $F \in \mathrm{GL}$ holds $\|F^i_{m_0}(b)\| \to 0$ as $i \to \infty$, because $\{F^i_{m_0}(b)\}_{i=1}^{\infty} = F(e_{m_0}b) \in l_2(A)$. For $F \in \mathrm{GL}^*$ holds $\|F^{m_0}_j(b)\| \to \infty$ as $j \to \infty$ as well, as it is proved in the following lemma.

Lemma 3.3 *For any $F \in \mathrm{GL}^*$, $\varepsilon > 0$ and $e_k\gamma$ there exists a number $m(k)$, such that for any $m \geq m(k)$ and $\varphi \in A$ with $\|\varphi\| \leq 1$ holds*

$$\|\langle e_k\gamma, Fe_m\varphi\rangle\| < \varepsilon.$$

Proof: Let us consider the bounded operator F^*. Since $F^*e_k\gamma \in l_2(A)$, there exists a number $m(k)$, such that

$$\|(1 - p_{m(k)})F^*e_k\gamma\| < \varepsilon, \qquad \|Q_mF^*e_k\gamma\| < \varepsilon, \quad (m > m(k)).$$

Hence,

$$\|\langle e_k\gamma, Fe_m\varphi\rangle\| = \|Q_mF^*e_k\gamma\| \cdot \|\varphi\| < \varepsilon, \quad (m > m(k)). \qquad \square$$

4 Proof of the Cuntz–Higson theorem

Lemma 4.1 *Let $F_r \in \mathrm{GL}^*$, $r = 1, \ldots, N$, be arbitrary operators, and $\varepsilon > 0$ be any number. Then we can choose such increasing non-intersecting sequences of natural numbers $i(k)$ and $j(k)$, that*

$$\|(1 - p_{j(s)})F_re_{i(k)}\alpha_k\| < \frac{1}{4} \cdot \frac{\varepsilon}{2^s \cdot 2^k}, \qquad s = k, k+1, \ldots, \quad r = 1, \ldots, N, \tag{3}$$

$$\|\langle F_re_{i(k)}\alpha_k, e_{j(s)}\alpha_s\rangle\| < \frac{1}{2} \cdot \frac{\varepsilon}{2^s \cdot 2^k}, \qquad s = 1, \ldots k-1, \quad r = 1, \ldots, N \tag{4}$$

Proof: Let us take $i(1) := 1$. Let us choose $j(1) > i(1)$ in such a way that

$$\|(1 - p_{j(1)})F_re_{i(1)}\alpha_1\| < \frac{1}{4} \cdot \frac{\varepsilon}{2^1 \cdot 2^1}, \quad r = 1, \ldots, N.$$

Let us discover $i(2) > j(1)$, such that (in the correspondence with Lemma 3.3)

$$\|\langle F_re_{i(2)}\alpha_2, e_{j(1)}\alpha_1\rangle\| < \frac{1}{2} \cdot \frac{\varepsilon}{2^1 \cdot 2^2}, \quad r = 1, \ldots, N.$$

Let us now choose $j(2) > i(2)$, such that

$$\|(1 - p_{j(2)})F_re_{i(k)}\alpha_k\| < \frac{1}{4} \cdot \frac{\varepsilon}{2^2 \cdot 2^k}, \qquad k = 1, 2, \quad r = 1, \ldots, N,$$

and such $i(3) > j(2)$, such that

$$\|\langle F_re_{i(3)}\alpha_3, e_{j(s)}\alpha_s\rangle\| < \frac{1}{2} \cdot \frac{\varepsilon}{2^s \cdot 2^k}, \qquad s = 1, 2, \quad r = 1, \ldots, N.$$

Let us continue the process by induction. Let $i(1),\dots,i(k-1)$ and $j(1),\dots,j(k-2)$ be already found in such a manner, that the conditions (3) and (4) hold for them. Let us find $j(k-1)>i(k-1)$, such that

$$\|(1-p_{j(k-1)})F_r e_{i(m)}\alpha_m\| < \frac{1}{4}\cdot\frac{\varepsilon}{2^{k-}\cdot 2^m},\qquad m=1,\dots k-1,\quad r=1,\dots,N,$$

and after that let us find $i(k)>j(k-1)$ in such a manner that

$$\|\langle F_r e_{i(k)}\alpha_k, e_{j(s)}\alpha_s\rangle\| < \frac{1}{2}\cdot\frac{\varepsilon}{2^s\cdot 2^k},\qquad s=1,\dots k-1,\quad r=1,\dots,N.$$

By induction we obtain the required statement. □

Let us define now embeddings J and J' similarly to the constructions in Lemma 3.2. For the definition of J we shall take some of $e_{i(g)}\alpha_g\omega_s$ as vectors v_{sj}, but so that $g=g(s,j)>s+j$, $g>s$, whence $e_{i(g)}\alpha_g\omega_s=e_{i(g)}\omega_s$ and $\langle v_{sj},v_{sj}\rangle=\omega_s^2$. Let us define similarly v'_{sm} for J', but taking $e_{j(k)}$ instead of $e_{i(k)}$. From the conditions (3) and (4) we obtain

$$\|\langle F_r v_{st}, v'_{nm}\rangle\| = \|\langle F_r e_{i(g(s,t))}\alpha_{g(s,t)}\omega_s, e_{j(h(n,m))}\alpha_{h(n,m)}\omega_n\rangle\| \le \|Q_{j(h(n,m))}F_r e_{i(g(s,t))}\alpha_{g(s,t)}\| \le$$

$$\le \|(1-p_{j(h(n,m)-1)})F_r e_{i(g(s,t))}\alpha_{g(s,t)}\| < \frac{1}{4}\cdot\frac{\varepsilon}{2^{h-1}\cdot 2^g},\ h\ge g,\ r=1,\dots,N. \qquad (5)$$

$$\|\langle F_r v_{st}, v'_{nm}\rangle\| = \|\langle F_r e_{i(g(s,t))}\alpha_{g(s,t)}\omega_s, e_{j(h(n,m))}\alpha_{h(n,m)}\omega_n\rangle\| < \frac{1}{2}\cdot\frac{\varepsilon}{2^h\cdot 2^g},$$
$$h<g,\qquad r=1,\dots,N. \qquad (6)$$

Let us denote by P and P' the correspondent orthoprojections. Then $PP'=P'P=0$. Let $x=(a_1,a_2,\dots)$ and $y=(b_1,b_2,\dots)$ be arbitrary vectors from $l_2(A)$ with the norm 1. Then for any $r=1,\dots,N$ by (5,6)

$$\|\langle F_r Jx, J'y\rangle\| = \left\|\left\langle \sum_t\sum_s F_r v_{st}a_t, \sum_m\sum_n v'_{nm}b_m\right\rangle\right\| \le$$

$$\le \sum_{t,s,n,m}\left(\sum_{h(n,m)\ge g(s,t)}\|\langle F_r v_{st}, v'_{nm}\rangle\| + \sum_{h(n,m)<g(s,t)}\|\langle F_r v_{st}, v'_{nm}\rangle\|\right) \le$$

$$\le \sum_{t,s,n,m}\frac{\varepsilon}{2^{h(n,m)}\cdot 2^{g(t,s)}} < \varepsilon,$$

since $h(n,m)>n+m$, $g(t,s)>t+s$ by the construction. From this we obtain

$$\|P'F_rP\| < \varepsilon,\qquad r=1,\dots,N. \qquad (7)$$

As it was shown in Lemma 2.2, it is sufficient to know how to construct a homotopy of picewise-linear map with the image in a finite polyhedron in GL^* with vertices $F_1,\dots,F_N$ into a map in a compact set $\{D(x)\}\subset\mathrm{GL}^*$, such that

$$PD(x)=D(x)P=P\quad \forall x\in S.$$

For this purpose we can apply a homotopy of Neubauer type (see Section 7). By (7) we have to take care only of that, we have an operator $H_0 : P'(l_2(A)) \to P(l_2(A))$, such that operators $H_0 P'$ and $H_0^{-1} P$ admit adjoint. Let us assume $H_0 = JJ'^*$. Then $H_0 P' = JJ'^* J' J'^* = JJ'^*$, where J'^* is an isomorphism $P'(l_2(A)) \to l_2(A)$, and $J : l_2(A) \cong P(l_2(A))$.

We have proved the following statement.

Theorem 4.2 [2] *Let A be a σ-unital C^*-algebra. Then $\mathrm{GL}^*(A)$ is contractible with the respect to the norm topology.* □

5 The case $A \subset \mathcal{K}$

Let algebra A be (for some faithful representation) a subalgebra of algebra $\mathcal{K}$ of compact operators on a separable Hilbert space H. Under these restrictions we can prove the following statement.

Lemma 5.1 *Let $a, b \in A$, $(f_1, f_2, \ldots) \in l_2'(A)$. Then*

$$\|a f_i b\| \to 0 \qquad (i \to \infty).$$

Proof: Since $a^* \in \mathcal{K}$, for any $\varepsilon > 0$ we can find a number $N = N(\varepsilon)$ and base $h_1, h_2, \ldots$ in H, such that

$$\|p_N' a^*\| < \frac{\varepsilon}{2 \cdot \sup \|f_i\|}, \qquad H_N = \mathrm{span}_{\mathbf{C}} \langle h_1, \ldots, h_N \rangle, \qquad H_N' = H_N^{\perp},$$

p_N and p_N' are the correspondent projections. Since [5] the partial sums of series $\sum_i f_i f_i^*$ form an increasing uniformly bounded sequence of positive operators in $\mathcal{B}(H)$, $f_i f_i^*$ is strong convergent to the zero operator. Hence, for any $h \in H$

$$\|f_i^* h\| = \langle f_i^* h, f_i^* h \rangle = \langle f_i f_i^* h, h \rangle \to 0.$$

Thus, f_i^* is strong convergent to 0. Let i_0 be so large, that

$$\|f_i^* p_N\| < \frac{\varepsilon}{2\|a\|}.$$

for $i > i_0$. Then

$$\|a f_i\| = \|f_i^* p_N a^*\| + \|f_i^* p_N' a^*\| < \frac{\varepsilon}{2\|a\|} \cdot \|a^*\| + \|f_i^*\| \frac{\varepsilon}{2 \cdot \sup \|f_i\|} \leq \varepsilon. \qquad \square$$

Let us remark, that similar properties for matrix elements themselves (which belong $\mathbf{LM}(\mathcal{K}) = \mathcal{B}(H)$) are not valid even for operators from have not GL^*. Moreover, the following example shows, that all matrix elements can have the norm 1.

Theorem 5.2 *The group $\mathrm{GL}\,(A)$ is contractible with the respect to the norm for $A \subset \mathcal{K}$.*

Proof: Since Lemma 5.1 is the analog of Lemma 3.3, the proof can be obtained by the literal repeating of the reasoning from Section 4. □

6 Some other cases

Definition 6.1 Let us tell, that C^*-algebra A has *property* (K), if for any functional $f : l_2(A) \to A$, any $\varepsilon > 0$ and any $a \in A$ it is possible to find a vector $x \in l_2(A)$, such that

$$\|f(x)\| < \varepsilon, \qquad \langle x, x\rangle = a^* a.$$

Definition 6.2 A C^*-algebra A has *property* (E), if for any functional $f = (f_1, \dots, f_n, \dots) \in l_2'(A)$ and any $\varepsilon > 0$ it is possible to find a another functional $g = (g_1, \dots, g_n \dots) \in l_2'(A)$ and a number $k \in \mathbf{Z}$, such that

$$\|f - g\| < \varepsilon, \qquad f_i = g_i, \quad i = k+1,\ k+2, \dots$$

and $g|_{L_k} : L_k \to A$ is *epimorphism*, where $L_n = \{ (a_1, \dots, a_n, 0, 0, \dots) \}$.

Example 6.3. Let A be the algebra of continuous functions on a smooth n-dimensional manifold M. Then A has the property (E) (with $k = n+1$).

For the proof of the following theorem we need

Lemma 6.4 *Let $\mathcal{M}$ be a Hilbert module, $x \in \mathcal{M}$, $\langle x, x\rangle \geq a \geq 0$, $\|a\| \leq 1$. Then one can find an element $y = xb$, $\|b\| \leq 1$, such that $\langle y, y\rangle = a^2$.*

Proof: Let us put

$$\gamma := \langle x, x\rangle, \qquad b := \lim_{n\to\infty} \left(\gamma + \frac{1}{n}\right)^{-1/2} a.$$

This (norm) limit exists, as

$$\left[\left(\gamma + \frac{1}{n}\right)^{-1/2} - \left(\gamma + \frac{1}{m}\right)^{-1/2}\right] a^2 \left[\left(\gamma + \frac{1}{n}\right)^{-1/2} - \left(\gamma + \frac{1}{m}\right)^{-1/2}\right] \leq$$

$$\leq \left[\left(\gamma + \frac{1}{n}\right)^{-1/2} - \left(\gamma + \frac{1}{m}\right)^{-1/2}\right]^2 \gamma^2 \to 0,$$

since for any non-negative z holds

$$\frac{z^2}{z + \frac{1}{n}} - \frac{z^2}{z + \frac{1}{m}} = \frac{\frac{1}{m} z^2 - \frac{1}{n} z^2}{(z + \frac{1}{n})(z + \frac{1}{n})} = \left(\frac{1}{m} - \frac{1}{n}\right) \frac{z^2}{(z + \frac{1}{n})(z + \frac{1}{n})} \leq \frac{1}{m} - \frac{1}{n}.$$

Also $\|b\| \leq 1$, as

$$a\left(\gamma + \frac{1}{n}\right)^{-1} a \leq a^{1/2} \gamma \left(\gamma + \frac{1}{n}\right)^{-1} a^{1/2} \leq a \leq 1.$$

The condition $\langle y, y\rangle = a^2$ is obvious now. □

Theorem 6.5 *The property (E) implies the property (K).*

Proof: We can suppose $\|a\| = 1$. Let us consider an arbitrary functional $f = (f_1, \dots) \in l'_2(A)$ and $\varepsilon > 0$. Let g and k be as in the condition (E) with the respect to $\varepsilon/2$. Let us put $f' := f|_{L_k^\perp}$. Since $L_k^\perp \cong l_2(A)$, by (E) there exists a functional $g' : L_k^\perp \to A$, such that

$$\|f' - g'\| < \varepsilon/2, \qquad f'_i = g'_i = g_i, \quad i = k'+1,\ k'+2, \dots$$

and $g'|_{L_k^\perp \cap L_{k'}}$ is an epimorphism. Then the functional

$$h := \begin{cases} g & \text{on } L_k; \\ g' & \text{on } L_k^\perp, \end{cases}$$

satisfies to conditions: $\|f - h\| < \varepsilon$, h is an epimorphism on L_k and $L_k^\perp \cap L_{k'}$ separately. Without loss of generality it is possible to suppose, that $\|h\| = 1$. Let $x \in L_k$ and $y \in L_k^\perp \cap L_{k'}$ be such that $h(x) = h(y) = a$. Then $h(x - y) = 0$, and by [18]

$$a^* a = \langle h(x), h(x)\rangle \le \langle x, x\rangle, \qquad a^* a = \langle h(y), h(y)\rangle \le \langle y, y\rangle.$$

By Lemma 6.4 it is possible to find b, such that $\|b\| \le 1$ and $z = (x - y)b$ satisfies $\langle z, z\rangle = a^2$. Thus $h(z) = h((x-y)b) = 0$, and as $\|z\| = 1$, $\|f(z)\| < \varepsilon$. □

Remark 6.6 Let i and i' be enclosures admitting adjoint and respecting inner product, and for the correspondent projections $q = ii^*$ and $q' = i'i'^*$ we have $\|qq'\| < \varepsilon$, $\|q'q\| < \varepsilon$. Let us remark, that $qq' = ii^*i'i'^*$, where i is an isometric enclosure and i'^* is an epimorphism with norm 1. Therefore, the indicated inequalities are equivalent to $\|i^*i'\| < \varepsilon$, $\|i'^*i\| < \varepsilon$. Then the map $I := (i, i') : l_2(A) \oplus l_2(A) \to l_2(A)$ is also an enclosure, admitting adjoint $I^*(x) = (i^*(x), i'^*(x))$. Really, I^*, given by this formula, is continuous and

$$\langle I(x,y), z\rangle = \langle i(x) + i'(y), z\rangle = \langle x, i^*(z)\rangle + \langle y, i'^*(z)\rangle = \langle (x,y), I^*(z)\rangle.$$

Also,

$$I^*I(x,y) = (i^*(ix + i'y), i'^*(ix + i'y)) = (x,y) + (i^*i'y, i'^*ix),$$

so that

$$\|\mathrm{Id} - I^*I\| < 2\varepsilon \tag{8}$$

and I^*I is invertible. Therefore, I is an enclosure. Let us remark, that for this reasoning we need to have $\varepsilon < 1/2$.

Theorem 6.7 *Let algebra A have the property (K). Then the group* $\mathrm{GL}(A)$ *is norm contractible.*

Proof: As above, it is necessary to prove a statement, similar to Lemma 3.3. In the present situation we argue as follows. Let F_1 be the first row (i. e., a functional) of matrix F with the respect to the standard decomposition $l_2(A)$. Let us remark, that any vector from $l_2(A)$ with any beforehand given exactness δ belongs to L_n for a sufficient large $n = n(\varepsilon)$. Hence, applying the property (K), it is possible at once to suppose, that $x \in L_n$. Really, let $f(x) < \varepsilon/2$, $\langle x, x\rangle = a \le 1$, $\|f\| = 1$. Let us find a number n, such that $\|(1 - p_n)x\| < \varepsilon/4$, $x' := p_n x$. Then $\langle x', x'\rangle \le \langle x, x\rangle = a$ and

$$\|\alpha\| \le \frac{\varepsilon}{4}, \qquad \text{if} \quad \alpha := \left(\langle x, x\rangle - \langle x', x'\rangle\right)^{1/2}.$$

Let us put $y := x' + e_{n+1}\alpha$. Then $\langle y, y\rangle = a$, $y \in L_{n+1}$ and

$$\|f(y)\| \leq \|f(x)\| + \|f(x - x')\| + \|f(x' - y)\| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.$$

By applying the property (K) infinitely many times with constants, decreasing as geometrical progression, we can find a sequence of vectors $x_i \in l_2(A)$, satisfying to conditions

$$x_i \in M_i, \qquad l_2(A) = M_1 \oplus M_2 \oplus \ldots, \qquad M_i = \{(0, \ldots, 0, a_{k(i)}, \ldots, a_{k(i+1)-1}, 0, \ldots)\}, \tag{9}$$

$$\langle x_i, x_i\rangle = \alpha_i, \qquad \alpha_i \text{ — approximate unit for } A, \tag{10}$$

$$\|F_1(x_i)\| < \frac{\varepsilon}{2} \cdot \frac{1}{2^i}. \tag{11}$$

Let us remark, that for $k > k(i)$: $\quad \omega_i = \alpha_k^{1/2} d(i, k)$, $\|d(i, k)\| \leq 1$. Therefore, similar to reasonings above, the map

$$J_1 : l_2(A) \to l_2(A), \qquad (a_1, a_2, \ldots) \mapsto \sum_j \sum_i x_{k(i,j)} d(i, k(i, j)) a_j,$$

where

$$k(1, 1); k(1, 2), k(2, 1); k(1, 3), k(2, 2), k(3, 1); \ldots$$

is some increasing sequence, will be an enclosure admitting an adjoint and preserving the inner product. If we denote $H_1 := \operatorname{Im} J_1$, then by (11)

$$\|F_1|_{H_1}\| < \frac{\varepsilon}{2}.$$

Let G_1 be the orthogonal complement to the image of the first copy of A under J_1. Let $m(2) > m(1) := 1$ be so large, that $\|(1 - p_{m(2)})Fy(1)\| < \varepsilon/2$, where $y_1 := J_1(\alpha_1^{1/2}, 0, \ldots)$. Let us denote by F_2 the restriction of the $m(2)$-th row of the matrix F on $G_1 \cong l_2(A)$, and let us find by the same algorithm a new enclosure J_2, such that its image equals to H_2 and there exists a correspondent submodule $G_2 \subset H_2$, and

$$\|F_2|_{H_2}\| < \frac{\varepsilon}{2^2}.$$

Let $m(3) > m(2)$ be so large, that

$$\|(1 - p_{m(3)})Fy_i\| < \frac{\varepsilon}{2^3 \cdot 2^i}, \quad i = 1, 2, \qquad y_2 := J_2(\alpha_2^{1/2}, 0, \ldots),$$

$$\|(1 - p_{m(3)})y_i\| < \frac{\varepsilon}{2^3 \cdot 2^i}, \quad i = 1, 2.$$

And so on. We obtain sequences $m(j)$ and y_i such, that

$$\|(1 - p_{m(j)})Fy_i\| < \frac{\varepsilon}{2^j \cdot 2^i}, \qquad i = 1, \ldots, j - 1, \tag{12}$$

$$\|(1 - p_{m(j)})y_i\| < \frac{\varepsilon}{2^j \cdot 2^i}, \qquad i = 1, \ldots, j - 1, \tag{13}$$

$$\|Q_{m(j)}Fy_i\| < \frac{\varepsilon}{2^j \cdot 2^i}, \qquad j = 1, \ldots, i, \tag{14}$$

Again, using ω_j, we can arrange an enclosure J of the module $l_2(A)$ on a submodule H of the linear span of y_i and an enclosure J' of the module $l_2(A)$ on the submodule $H' := \bigoplus_j E_{m(j)}$. Since these modules are ε-ortogonal, there exist mutually vanishing projectors p and p' on them. More precisely, let us remark first of all, that the enclosure J admits adjoint. Really, the image of each vector $(a_1, a_2, \ldots)$ under J_1 is a sum of the form

$$\sum_j \sum_i v_{ij} a_j, \qquad \langle v_{ij}, v_{ij} \rangle = \omega_i^2, \qquad v_{ij} \in M_{k(i,j)}.$$

For construction of the higher J_s the correspondent v_{ij}^s will lay again in direct sums of modules M_r, and for v_{i1}^s these sets are not intersecting. We can apply Lemma 3.2. The operator J will is defined by the formula

$$J : (a_1, a_2, \ldots) \mapsto \sum_s \sum_i v_{i1}^s a_s, \qquad \sum_i v_{i1}^s a_s = y_s \mu_s a_s. \tag{15}$$

Hence, there are the orthoprojections q and q' on H and H', correspondently. Let us remark, that from this reasoning we can make the following refinement. We, in particular, have shown, that for any J_s and any m there exists no more than one r, such that $Q_m J_s Q_r \neq 0$. Therefore, throwing out if necessary, a finite number of canonical summands in $l_2(A)$ and restricting J_s on the remaining module, we can suppose, that

$$Q_{m(j)} J_s = 0, \qquad j = 1, \ldots, s-1, \tag{16}$$

$$Q_{m(j)} y_i = 0, \qquad j = 1, \ldots, i, \tag{17}$$

Also, $\|qq'\| < \varepsilon$, $\|q'q\| < \varepsilon$. Really, let us consider a vector of the form

$$x = \sum_s \sum_i v_{i1}^s a_s = \sum_s y_s \mu_s a_s, \qquad \|\sum_s a_s^* a_s\| \leq 1.$$

It is necessary to show, that $\|q'x\| < \varepsilon$. It follows from (13, 17):

$$\|q'x\| = \left\| \sum_j Q_{m(j)} \sum_s \sum_i v_{i1}^s a_s \right\| \leq \sum_s \left\| \sum_{j>s} Q_{m(j)} \left(\sum_i v_{i1}^s a_s \right) \right\| + \sum_s \sum_{j \leq s} \left\| Q_{m(j)} \left(\sum_i v_{i1}^s a_s \right) \right\| \leq$$

$$\leq \sum_s \left\| (1 - p_{m(s)}) y_s \mu_s a_s \right\| + \sum_s \sum_{j \leq s} 0 \leq \sum_s \frac{\varepsilon}{2^s} = \varepsilon.$$

Since the projections q and q' are self-adjoint, we obtain and second estimation.

Then by Remark 6.6 $H \tilde{\oplus} H'$ is the image of an enclosure, admitting adjoint, and by [15] the decomposition $l_2(A) = H \tilde{\oplus} H' \oplus (H^\perp \cap H'^\perp)$ takes place. Let us denote by p and p' projections on H and H' correspondent to this decomposition, so that $pp' = p'p = 0$. Thus

$$\|p - q\| < 3\varepsilon, \qquad \|p' - q'\| < 3\varepsilon, \qquad \|p\| < 1 + 3\varepsilon < 2, \qquad \|p'\| < 1 + 3\varepsilon < 2. \tag{18}$$

Really, let $x \in H \tilde{\oplus} H'$, $\|x\| = 1$, so that $x = II^* Iy$, and by (8) $\|Iy\| \leq 2(1 + \varepsilon)$,

$$\|(p-q)x\| = \|(p-q)(ii^* Iy + i'i'^* Iy)\| = \|(p-q)(q+q')Iy\| = \|-qq'Iy\| \leq 2\varepsilon(1+\varepsilon) < 3\varepsilon.$$

Besides, $\|p'Fp\| < 7\|F\|\varepsilon$. In fact,

$$\|p'Fp\| = \|(p'-q')Fp + q'Fp\| < 3\varepsilon\|F\| + \|q'Fp\|,$$

and by (18) it is sufficient to prove, that for $x \in H$, $\|x\| \leq 1$, holds $\|q'Fx\| < 2\varepsilon$. Any such vector x can be presented as

$$\sum_s \sum_i v_{i1}^s a_s = \sum_s y_s \mu_s a_s, \qquad \|\sum_s a_s^* a_s\| \leq 1.$$

Then

$$\|q'Fx\| = \left\| \sum_j Q_{m(j)} \sum_s \sum_i F v_{i1}^s a_s \right\| \leq$$

$$\leq \sum_s \sum_i \left\| \sum_{j>s} Q_{m(j)} F v_{i1}^s \right\| + \sum_s \sum_{j \leq s} \left\| Q_{m(j)} F \left(\sum_i v_{i1}^s a_s \right) \right\| \leq$$

$$\leq \sum_s \left\| (1 - p_{m(s)}) F y_s \mu_s a_s \right\| + \sum_s \sum_{j \leq s} \frac{\varepsilon}{2^j \cdot 2^s} \leq \sum_s \frac{\varepsilon}{2^s} + \varepsilon = 2\varepsilon.$$

Let us remark, that similar statement we can receive not only for one operator F (actually for two: F and Id), but for a finite collection (vertices of a simplicial complex): $F^{(1)}, \ldots, F^{(N)}$. For this purpose it is necessary to conduct reasonings for $F = F^{(1)}$ with a constant ε and to receive projections P_1 and P_1'. Then apply algorithm To $P_1' F^{(2)} P_1$ and receive projections P_2' and P_2, such that

$$P_1' P_2' = P_2' P_1' = P_2', \quad P_1 P_2 = P_2 P_1 = P_2, \quad P_2 P_1 = P_1 P_2 = 0,$$

$$\|P_2' F^{(1)} P_2\| < \varepsilon, \quad \|P_2' F^{(2)} P_2\| < \varepsilon.$$

And so on. This completes the proof, since now it is possible to apply the Neubauer homotopy. □

7 Neubauer type homotopy

In this section we describe, how to modify the homotopy from [17] for our purposes. Though we work with completely other objects, the construction in [17] is so universal, that proofs can be transferred almost without modifications.

Lemma 7.1 *Let $\mathcal{M}$ be a Hilbert A-module , X be a topological space, $T : X \to \mathcal{G} = \mathcal{G}(\mathcal{M})$ be a continuous map, and P and P' be projections from $\mathcal{E} = \mathcal{E}(\mathcal{M})$, such that*

$$PP' = P'P = 0, \qquad H_0 : P'\mathcal{M} \cong P\mathcal{M}, \qquad H_0 P' \in \mathcal{E}, \quad H_0^{-1} P \in \mathcal{E},$$

$$P'T(x)P = 0 \qquad \forall\, x \in X.$$

Then there is a homotopy $T \sim D$ in $\mathcal{G}$, such that

$$PD(x) = D(x)P = P \quad \forall\, x \in X.$$

Proof: Le us put $Q := \mathrm{Id} - P$, $Q' := \mathrm{Id} - P'$,

$$\mathcal{P}(x) := T(x)PT(x)^{-1}Q', \qquad \mathcal{Q}(x) := Q' - \mathcal{P}(x).$$

Then $\mathcal{P}(x)$ is a projection on $T(x)P\mathcal{M}$ and there is the decomposition into projections $\mathrm{Id} = \mathcal{Q}(x) + \mathcal{P}(x) + P'$, and $\mathcal{Q}(x)$, $\mathcal{P}(x)$ and P' are mutual vanishing for each x. Really,

$$Q'T(x)P = (\mathrm{Id} - P')T(x)P = T(x)P, \quad PT(x)^{-1}Q'T(x)P = P,$$

$$T(x)P\mathcal{M} \subset \mathcal{P}(x)\mathcal{M} \subset T(x)P\mathcal{M},$$

$$\mathcal{P}(x)\mathcal{P}(x) = T(x)PT(x)^{-1}(\mathrm{Id} - P')T(x)PT(x)^{-1}(\mathrm{Id} - P') =$$

$$= T(x)PT(x)^{-1}T(x)PT(x)^{-1}(\mathrm{Id} - P') = T(x)PT(x)^{-1}(\mathrm{Id} - P') = \mathcal{P}(x),$$

$$\mathcal{Q}(x) + \mathcal{P}(x) + P' = Q' - \mathcal{P}(x) + P(x) + P' = \mathrm{Id}\,,$$

$$\mathcal{P}(x)P' = T(x)PT(x)^{-1}(\mathrm{Id} - P')P' = 0, \qquad P'\mathcal{P}(x) = P'T(x)PT(x)^{-1}(\mathrm{Id} - P') = 0,$$

Hence, $\mathcal{P}(x) + P'$ is a projection, whence $\mathcal{Q}(x) = \mathrm{Id} - (\mathcal{P}(x) + P')$ is a projection too.

Let us define

$$H = -H_0P' + H_0^{-1}P,$$

then, as $P'P = PP' = 0$, $P'H_0 = PH_0^{-1} = 0$ and

$$H^2 = (-H_0P' + H_0^{-1}P)(-H_0P' + H_0^{-1}P) = -(P' + P),$$

$$HP'H = (-H_0P' + H_0^{-1}P)P'(-H_0P' + H_0^{-1}P) = -H_0P'H_0^{-1}P = -H_0H_0^{-1}P = -P,$$

$$Q'HP = HP - P'HP = H_0^{-1}P - H_0^{-1}P = 0,$$

$$Q'HT(x)^{-1}\mathcal{P}(x) = Q'HT(x)^{-1}T(x)PT(x)^{-1}Q' = 0,$$

$$\mathcal{P}(x)T(x)HP' = T(x)PT(x)^{-1}Q'T(x)HP' = T(x)PT(x)^{-1}(1 - P')T(x)(-H_0P') =$$

$$= T(x)PT(x)^{-1}(1 - P')T(x)P(-H_0P') =$$

$$= T(x)PT(x)^{-1}T(x)P(-H_0P') = T(x)P(-H_0P') = T(x)HP'.$$

Let's assume

$$G(x) := HT(x)^{-1}\mathcal{P}(x) + T(x)HP',$$

Then

$$G(x)^2 = (HT(x)^{-1}\mathcal{P}(x) + T(x)HP')(HT(x)^{-1}\mathcal{P}(x) + T(x)HP') =$$

$$= HT(x)^{-1}\mathcal{P}(x)HT(x)^{-1}\mathcal{P}(x) + T(x)(-P)T(x)^{-1}\mathcal{P}(x)+$$

$$+HT(x)^{-1}\mathcal{P}(x)T(x)HP' + T(x)HP'T(x)HP' =$$

$$= HT(x)^{-1}T(x)PT(x)^{-1}Q'HT(x)^{-1}\mathcal{P}(x) + T(x)(-P)T(x)^{-1}T(x)PT(x)^{-1}Q'+$$

$$+HT(x)^{-1}T(x)HP' + T(x)HP'T(x)HP' =$$

$$= 0 - T(x)PT(x)^{-1}Q' - P' + T(x)(-H_0P')T(x)(-H_0P') = 0 - \mathcal{P}(x) - P' + 0 = -(P' + \mathcal{P}(x)),$$

$$G(x)\mathcal{Q}(x) = 0, \qquad \mathcal{Q}(x)G(x) = (Q' - \mathcal{P}(x))(HT(x)^{-1}\mathcal{P}(x) + T(x)HP') =$$

$$= Q'HT(x)^{-1}\mathcal{P}(x) + (\mathrm{Id} - P')T(x)(-PH_0P') - \mathcal{P}(x)HT(x)^{-1}\mathcal{P}(x) - \mathcal{P}(x)T(x)HP' =$$

$$= 0 + T(x)HP' - (T(x)PT(x)^{-1}Q')(-H_0P' + H_0^{-1}P)T(x)^{-1}(T(x)PT(x)^{-1}Q') - T(x)HP' =$$

$$= (T(x)PT(x)^{-1}Q'H_0[P'P]T(x)^{-1}Q')-$$
$$-T(x)PT(x)^{-1}[Q'P']H_0^{-1}PT(x)^{-1}(T(x)PT(x)^{-1}Q') = 0.$$

Hence, for
$$U(s,x) := \mathcal{Q}(x) + (1-s)(\mathcal{P}(x)+P') + sG(x)$$
we obtain
$$U(s,x)^{-1} = \mathcal{Q}(x) + \frac{1}{s^2+(1-s)^2}[(1-s)(\mathcal{P}(x)+P') - sG(x)].$$
Therefore, $U(s,x)T(x)$ defines a homotopy in $\mathcal{G}$
$$U(0,x)T(x) = \mathrm{Id} \circ T(x) \sim U(1,x)\circ T(x).$$
Thus, as $\mathcal{P}(x)T(x)P = T(x)P$,
$$\begin{aligned} U(1,x)T(x)P &= \mathcal{Q}(x)\mathcal{P}(x)T(x)P + G(x)\mathcal{P}(x)T(x)P = \\ &= 0 + HT(x)^{-1}\mathcal{P}(x)T(x)P = HT(x)^{-1}T(x)P = HP. \end{aligned}$$
Since $H(P+P') = (P+P')H = H$, for
$$V(s) := QQ' + (1-s)(P+P') - sH$$
we have
$$V(s)^{-1} = QQ' + \frac{1}{s^2+(1-s)^2}[(1-s)(P+P') + sH].$$
Besides, $V(0) = QQ' + P + P' = \mathrm{Id}$. Therefore, the following homotopy is defined
$$R(x) := V(1)\,U(1,x)\,T(x) \sim U(1,x)\,T(x) \qquad \text{в} \qquad C(X,\mathcal{G}(\mathcal{M})),$$
and
$$\begin{aligned} R(x)P &= V(1)\,U(1,x)\,T(x)P = \\ &= V(1)\,HP = QQ'HP - H^2P = 0 + (P+P')\,P = P. \end{aligned}$$
Let us put
$$R(s,x) := R(x) - sPR(x)Q.$$
Let for some $e \in \mathcal{M}$ the equality $R(s,x)e = 0$ hold. Then
$$0 = R(s,x)e = R(x)(P+Q)e - sPR(x)Qe = Pe + R(x)Qe - sPR(x)Qe,$$
$$0 = QR(s,x)e = QR(x)Qe.$$
Let $f = PR(x)Qe$, so that $f = Pf$. Then
$$PR(x)(Qe - Pf) = f - Pf = 0, \qquad QR(x)Pf = 0.$$
Therefore, $R(x)(Qe - Pf) = 0$, $Qe = Pf = f = 0$ and $PR(s,x)e = Pe = 0$, $e = 0$. Also
$$R(x)\mathcal{M} = \mathcal{M}, \quad R(x)P = P, \quad QR(x)Q\mathcal{M} = QR(x)(1-P)\mathcal{M} = QR(x)\mathcal{M} = Q\mathcal{M}.$$
Therefore, with the respect to the decomposition $\mathcal{M} = P\mathcal{M}\tilde{\oplus}Q\mathcal{M}$ the operator $R(s,x)$ has the matrix
$$\begin{pmatrix} \mathrm{Id} & \star \\ 0 & QR(x)Q \end{pmatrix}, \qquad QR(x)Q\mathcal{M} = Q\mathcal{M},$$
hence, $R(s,x)$ is an epimorphism, and $R(s,x) \in \mathcal{G}(\mathcal{M})$ as an epimorphism without kernel. It is sufficient to put $D(x) := R(1,x)$. □

Lemma 7.2 *Let $\mathcal{M}$ be a Hilbert A-module , X be a compact set, $T : X \to \mathcal{G}(\mathcal{M})$ be a continuous map with $0 < \varepsilon < \min \|T(x)^{-1}\|^{-1}$,, and P and P' be such projections from $\mathcal{E} = \mathcal{E}(\mathcal{M})$, that*

$$\|P'T(x)P\| \le \varepsilon \quad \forall\, x \in X.$$

Then there exists a homotopy $S(s,x)$ in $\mathcal{G}$, such that

$$S(0,x) = T(x), \qquad P'S(1,x)P = 0 \quad \forall\, x \in X.$$

Proof: Let us put $S(s,x) := T(x) - sP'T(x)P$. Since

$$\|S(s,x) - T(x)\| \le \varepsilon,$$

$S(s,x) \in \mathcal{G}(\mathcal{M})$. $\square$

8 Dixmier-Douady type theorems

Let us realise $l_2(A)$ as the completion of the algebraic tensor product $H \otimes A = L^2([0,1]) ø A$ completed with respect to the A-inner product $\langle f \otimes \gamma, g \otimes \beta \rangle = \langle f, g \rangle\, \gamma^* \beta$. We suppose here that the inner product on $L^2([0,1])$ is linear in the second entry.

Lemma 8.1 [3, p. 250]. *There exists for each $t \in [0,1]$ a closed linear subspace $H_t \subset H$ and for each $t \in (0,1]$ a linear isometry $U_t : H_t \to H$ such that*

(i) *the orthogonal projection P_t onto H_t is strong continuous in $t \in [0,1]$,*

(ii) *the operators $U_t P_t$ and U_t^{-1} are strong continuous in $t \in (0,1]$,*

(iii) $H_1 = H, \quad H_0 = 0, \quad U_1 = 1.$ $\square$

Let us remind that in [3] the subspaces are defined in the following way:

$$H_t := \left\{ f \in L^2([0,1]) \,|\, f(x) = 0 \quad \text{for} \quad x \ge t \right\}.$$

Lemma 8.2 *If $F_t \to F, \quad t \to 0$ with respect to the strong topology in $B(H)$, being bounded, then $F_t \otimes \mathrm{Id}_A \to F \otimes \mathrm{Id}_A$ with respect to the left strict topology.*

Proof: It is sufficient to prove that

$$\|(F_t \otimes \mathrm{Id}_A - F \otimes \mathrm{Id}_A)\theta_{x,y}\| \to 0 \qquad (t \to 0),$$

where

$$\theta_{x,y}(z) = x\langle y, z\rangle, \quad x = \sum_{i=1}^{N} h_i x_i \otimes \beta_i, \quad x_i \in \mathbf{C},\ \beta_i \in A, \quad \|x\| = \|y\| = 1,$$

and $\{h_i\}$ is an orthogonal basis of H. Then for $z = \sum_i h_i z_i \otimes \mu_i$

$$\|(F_t \otimes \mathrm{Id}_A - F \otimes \mathrm{Id}_A)\, \theta_{x,y}(z)\| = \|\sum_{i=1}^{N} (F_t - F) h_i x_i \otimes \beta_i \langle y, z\rangle\|$$

is less then ε if t is so close to 0 that

$$\|(F_t - F)h_i x_i\| \cdot \|\beta_i\| < \frac{1}{N}\,\varepsilon. \qquad \square$$

Lemma 8.3 *Let a set $G(t)$ be uniformly bounded (by a constant C), $G(t) \to G$ and $S(t) \to S$ $(t \to 0)$ in the left strict topology. Then $G(t)S(t) \to GS$ $(t \to 0)$ in the left strict topology.*

Proof: Let $k \in \mathcal{K}_A$ be an arbitrary operator. Then $Sk \in \mathcal{K}_A$ and

$$\|S(t)k - Sk\| \to 0, \qquad \|(G(t)-G)(Sk)\| \to 0 \qquad (t \to 0).$$

Hence

$$\begin{aligned} \|G(t)S(t)k - GSk\| &\leq \|(G(t)-G)Sk + G(t)(S(t)-S)k\| \\ &\leq \|(G(t)-G)Sk\| + C\|(S(t)-S)k\| \to 0 \quad (t \to 0). \end{aligned}$$

□

Theorem 8.4 *The unitary group $\mathcal{U}$ of operators in $l_2(A)$ is contractible with respect to the left strict topology.*

Proof: For any $\mathbf{U} \in \mathcal{U}$ and $t \in (0,1]$ we define

$$\Phi(\mathbf{U}, t) := (\mathrm{Id}_{\,l_2(A)} - P_t \otimes \mathrm{Id}_{\,A}) + (U_t^{-1} \otimes \mathrm{Id}_{\,A})\, \mathbf{U}\, (U_t \otimes \mathrm{Id}_{\,A})\, (P_t \otimes \mathrm{Id}_{\,A})$$

and

$$\Phi(\mathbf{U}, 0) := \mathbf{U}.$$

The operator $\Phi(\mathbf{U}, t)$, $t \in (0,1]$ defines an identity mapping $H_t^{\perp} \otimes A$ and coincides with the restriction of the unitary map $(U_t^{-1} \otimes \mathrm{Id}_{\,A})\, \mathbf{U}\, (U_t \otimes \mathrm{Id}_{\,A})$ on $H_t \otimes A$. Therefore

$$\Phi(\mathbf{U}, t) \in \mathcal{U}, \qquad \Phi(\mathbf{U}, 1) = \mathbf{U}.$$

Thus, as $U_t^* = U_t^{-1}$, so all operators admit an adjoint.

From Lemma 8.3 it is clear that Φ is continuous in $t \in (0,1]$, and, similarly, in $(\mathbf{U}, t)$. Indeed, let $(\mathbf{U}', t') \in \mathcal{U} \times (0,1]$ tend to $(\mathbf{U}, t) \in \mathcal{U} \times (0,1]$. Then for any $k \in \mathcal{K}_A$

$$\begin{gathered} \|\Phi(\mathbf{U},t)k - \Phi(\mathbf{U}',t')k\| = \|(\mathrm{Id}_{\,l_2(A)} - P_t \otimes \mathrm{Id}_{\,A})k + (U_t^{-1} \otimes \mathrm{Id}_{\,A})\, \mathbf{U}\, (U_t \otimes \mathrm{Id}_{\,A})\, (P_t \otimes \mathrm{Id}_{\,A})k \\ -(\mathrm{Id}_{\,l_2(A)} - P_{t'} \otimes \mathrm{Id}_{\,A})k - (U_{t'}^{-1} \otimes \mathrm{Id}_{\,A})\, \mathbf{U}'\, (U_{t'} \otimes \mathrm{Id}_{\,A})\, (P_{t'} \otimes \mathrm{Id}_{\,A})k\| \\ \leq \|(P_t \otimes \mathrm{Id}_{\,A} - P_{t'} \otimes \mathrm{Id}_{\,A})k\| + \|[(U_t^{-1} \otimes \mathrm{Id}_{\,A}) - (U_{t'}^{-1} \otimes \mathrm{Id}_{\,A})]\, \mathbf{U}\, (U_t \otimes \mathrm{Id}_{\,A})\, (P_t \otimes \mathrm{Id}_{\,A})k\| \\ +\|(U_{t'}^{-1} \otimes \mathrm{Id}_{\,A})\, [\mathbf{U} - \mathbf{U}']\, (U_t \otimes \mathrm{Id}_{\,A})\, (P_t \otimes \mathrm{Id}_{\,A})k\| \\ +\|(U_{t'}^{-1} \otimes \mathrm{Id}_{\,A})\, \mathbf{U}'\, [(U_t \otimes \mathrm{Id}_{\,A}) - (U_{t'} \otimes \mathrm{Id}_{\,A})]\, (P_t \otimes \mathrm{Id}_{\,A})k\| \\ +\|(U_{t'}^{-1} \otimes \mathrm{Id}_{\,A})\, \mathbf{U}'\, (U_{t'} \otimes \mathrm{Id}_{\,A})\, [(P_t \otimes \mathrm{Id}_{\,A}) - (P_{t'} \otimes \mathrm{Id}_{\,A})]k\| \\ \leq \|(P_t \otimes \mathrm{Id}_{\,A} - P_{t'} \otimes \mathrm{Id}_{\,A})k\| + \|[(U_t^{-1} \otimes \mathrm{Id}_{\,A}) - (U_{t'}^{-1} \otimes \mathrm{Id}_{\,A})]\, k_1\| + \|[\mathbf{U} - \mathbf{U}']\, k_2\| \\ +\|[(U_t \otimes \mathrm{Id}_{\,A}) - (U_{t'} \otimes \mathrm{Id}_{\,A})]\, k_3\| + \|[(P_t \otimes \mathrm{Id}_{\,A}) - (P_{t'} \otimes \mathrm{Id}_{\,A})]k\| \to 0 \end{gathered}$$

by Lemma 8.2. Here k_1, k_2 and k_3 are fixed operators from $\mathcal{K}_A$. Let now $(\mathbf{U}', t') \in \mathcal{U} \times (0,1]$ tend to $(\mathbf{U}, 0) \in \mathcal{U} \times [0,1]$. Then $P_{t'} \to 0$ with respect to the strong topology, $P_{t'} \otimes \mathrm{Id}_{\,A} \to 0$ with respect to the left strict topology by Lemma 8.2. Therefore for any $k \in \mathcal{K}_A$

$$\|(U_{t'}^{-1} \otimes \mathrm{Id}_{\,A})\, \mathbf{U}\, (U_{t'} \otimes \mathrm{Id}_{\,A})\, (P_{t'} \otimes \mathrm{Id}_{\,A})\, k\| \leq \|(P_{t'} \otimes \mathrm{Id}_{\,A})\, k\| \to 0, \quad \|\Phi(\mathbf{U}, t)k\| \to 0. \quad \square$$

Let us remark that in the proof of Theorem 8.4 we used only the boundedness of the set of invertible operators $\{\mathbf{U}\}$, but not the unitarity. Thus, actually we have proved the following statement.

Theorem 8.5 *Every bounded set of invertible operators in Hilbert space H is contractible in invertibles with respect to the strong topology.*

Every bounded set of invertible operators from GL *(resp.* GL**) is contractible in* GL *(resp.* GL**) with respect to the left strict topology.* $\square$

Lemma 8.6 *Let S be a compact set and*

$$f : S \to B(H), \qquad s \mapsto F_s$$

be continuous with respect to the strong topology. Then $\{\|F_s\|\}$ is bounded.

Proof: As S is compact, so $\{\|F_s x\|\}$ is bounded for any $x \in H$ by some $C(x)$. Therefore, by the uniform boundedness principle [4, II.3.21] there exists a constant C such that

$$\|F_s x\| \leq C, \qquad \forall \quad s \in S, \quad x \in B_1(H).$$

Therefore $\|F_s\| \leq C$. $\square$

Lemma 8.7 *Let $x \in \mathcal{M}$ be an arbitrary element. Then there exists $z \in \mathcal{M}$ and $k = \theta_{u,v} \in \mathcal{K}(\mathcal{M})$ such that $x = kz$.*

Proof: Let us put

$$u := v := z := \lim_{\varepsilon \to 0} x\,(\varepsilon + \langle x, x \rangle^{1/3})^{-1}.$$

As $s^2(\varepsilon + s)^{-1}$ is uniformly convergent to s on bounded sets, so in order to prove that u is well-defined we should remark that for $t = \langle x, x \rangle$ one has

$$\begin{aligned}\langle x\,(\varepsilon + \langle x, x \rangle^{1/3})^{-1} - x\,(\mu + \langle x, x \rangle^{1/3})^{-1}, x\,(\varepsilon + \langle x, x \rangle^{1/3})^{-1} - x\,(\mu + \langle x, x \rangle^{1/3})^{-1} \rangle \\ = [(\varepsilon + t^{1/3})^{-1} - (\mu + t^{1/3})^{-1}]t[(\varepsilon + t^{1/3})^{-1} - (\mu + t^{1/3})^{-1}] \\ = [(\varepsilon + t^{1/3})^{-1} - (\mu + t^{1/3})^{-1}]^2 (t^{1/3})^4.\end{aligned}$$

The same argument shows that $x = kz$. $\square$

Lemma 8.8 *Let S be a compact set and*

$$f : S \to \mathrm{End}_A l_2(A) = \mathbf{LM}(\mathcal{K}_A), \qquad s \mapsto F_s$$

be continuous with respect to the left strict topology. Then $\{\|F_s\|\}$ is bounded.

Proof: Let $x \in l_2(A)$ be an arbitrary element. Let us choose $k \in \mathcal{K}$ and z so that $x = kz$ by Lemma 8.7]. Then $s \mapsto F_s x$ is continuous: we apply the definition of the left strict topology to the inequality

$$\|F_s x - F_t x\| = \|F_s kz - F_t kz\| \leq \|F_s k - F_t k\| \, \|z\|$$

The proof is finished similarly to 8.6. $\square$

Now from Theorem 8.5 by Lemma 8.6 and Lemma 8.8 we obtain the following statement.

Theorem 8.9 *The group $G(H)$ of invertible operators in a Hilbert space H is weakly contractible (i. e. the homotopy groups $\pi_i(G(H)) = 0$) with respect to the strong topology.*

The group GL *(resp.* GL**) is weakly contractible with respect to the left strict topology.*
$\square$

Remark 8.10 We suppose that the results of this section in the part, concerning Hilbert spaces, were known earlier, but we have not found them published anywhere.

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