# Geometry and Topology of Operators on Hilbert $C^*$-Modules

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The main purpose of the present paper is the proof of a Dixmier-Douady type theorem for the Hilbert module $l_2(A)$ and also a new simple proof of the Kiper theorem for Hilbert modules (Cuntz-Higson theorem). The remaining results are preparatory.

In the first section we remind some general properties of algebras of the left, double and quasi multipliers. We also explain how to construct out of a Hilbert $C^*$-module (see [27]) some $W^*$-module possessing a number of useful properties.

In the second section algebra of multipliers of the algebra $\mathcal{K}(\mathcal{M})$ of $A$-compact operators in Hilbert $A$-module $\mathcal{M}$ is identified with the algebra $\mathrm{End}^*(\mathcal{M})$ of bounded $A$-operators on $\mathcal{M}$ admitting an adjoint [13]. Then the similar identifications [17] will be carried out for the algebra of left multipliers $\mathcal{K}(\mathcal{M})$ and the algebra $\mathrm{End}(\mathcal{M})$ of all bounded $A$-operatorors on $\mathcal{M}$, and also for the space of quasi multipliers of $\mathcal{K}(\mathcal{M})$ and the space $\mathrm{Hom}(\mathcal{M}, \mathcal{M}')$, where $\mathcal{M}'$ is the module of bounded $A$-functionals on $\mathcal{M}$. Obtained identifications allow to describe equivalent inner products on Hilbert modules [9, 17].

By describing an explicit form of various strict topologies in the context of our work, we prove weak contractibility of the group of invertible elements of $\mathrm{End}(l_2(A))$ with respect to the left strict topology for an arbitrary $\sigma$-unital algebra $A$.

As an illustration, a representation of spaces of operators in the Hilbert module $l_2(C(X))$ as sets of bounded operators in $l_2$, continuous in different topologies [9, 1], is considered in the third section.

In the fifth section we prove contractibility of the group of the invertible elements of $\mathrm{End}^*(l_2(A))$ with respect to the uniform topology [3]. Our proof is based on a generalization of Neubauer homotopy [26] obtained at the end of the section. It appears that the developed method allows to prove contractibility of the group of invertible elements $\mathrm{End}(l_2(A))$ for some classes of algebras.

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# 1 Multipliers and first structural results

## 1.1 Extension of Hilbert $C^*$-modules by the enveloping $W^*$-algebra

**Construction 1.1.1** Let $A$ be a $C^*$-algebra, $A^{**}$ be its enveloping $W^*$-algebra, $\mathcal{M}$ be a Hilbert $A$-module. Let us consider the algebraic tensor product (over the field $\mathbf{C}$) $\mathcal{M} \otimes A^{**}$. It is possible to equip this tensor product with a structure of a right $A^{**}$-module by the formula $(x \otimes a) \cdot b := x \otimes ab$, $x \in \mathcal{M}$, $a, b \in A^{**}$. Let us define an inner product

$$[\cdot,\cdot] : \mathcal{M} \otimes A^{**} \times \mathcal{M} \otimes A^{**} \longrightarrow A^{**}$$

by the equality

$$\left[\sum_{i=1}^{n} x_i \otimes a_i, \sum_{j=1}^{m} y_j \otimes b_j\right] = \sum_{i,j} a_i^* \langle x_i, y_j \rangle b_j,$$

where $x_i, y_j \in \mathcal{M}$, $a_i, b_j \in A^{**}$. Sesquilinearity and the properties $[z,w] = [w,z]^*$ and $[z, w \cdot a] = [z,w]a$ are obvious. To verify that this inner product is positive we need the following statement.

**Lemma 1.1.2** ([30], Lemma IV.3.2) *Let $B$ be a $C^*$-algebra , $c_{ij} \in B$, $i,j = 1,\dots,n$. A matrix $[c_{ij}] \in M_n(B)$ is positive iff $\sum_{i,j} b_i^* c_{ij} b_j \ge 0$ for any $b_1,\dots,b_n \in B$.* $\square$

Since for any $a_1,\dots,a_n \in A$

$$\sum_{i,j} a_i^* \langle x_i, x_j \rangle a_j = \left\langle \sum_{i=1}^{n} x_i \cdot a_i, \sum_{i=1}^{n} x_i \cdot a_i \right\rangle \ge 0,$$

the matrix $[\langle x_i, x_j\rangle] \in M_n(A)$ is positive, therefore the element $\sum_{i,j} b_i^* \langle x_i, x_j \rangle b_j$ is positive for all $b_i \in A^{**}$, hence $[z,z] \ge 0$ for all $z \in \mathcal{M} \otimes A^{**}$. Let us put

$$\mathcal{N} = \{z \in \mathcal{M} \otimes A^{**} : [z,z] = 0\},$$

then $\mathcal{N}$ is an $A^{**}$-submodule in $\mathcal{M} \otimes A^{**}$, and the quotient module $\mathcal{M} \otimes A^{**}/\mathcal{N}$ is a pre-Hilbert $A^{**}$-module. The Hilbert $A^{**}$-module obtained by the completion of $\mathcal{M} \otimes A^{**}/\mathcal{N}$ with respect to the norm given by the inner product $[\cdot,\cdot]$ we denote by $\mathcal{M}^{\#}$ and we call it the *extension* of the module $\mathcal{M}$ by the algebra $A^{**}$. The $W^*$-algebra $A^{**}$ contains the unit element and for any $x \in \mathcal{M}$, $a \in A$ we have $(x \cdot a) \otimes 1 - x \otimes a \in \mathcal{N}$, therefore the $A$-module map $x \longmapsto x \otimes 1 + \mathcal{N}$, $\mathcal{M} \longrightarrow \mathcal{M}^{\#}$. is well-defined. This map is an isometric inclusion, since $[x \otimes 1 + \mathcal{N}, y \otimes 1 + \mathcal{N}] = \langle x, y \rangle$.

Let us denote by $\mathrm{Hom}_A(\mathcal{M}, A^{**})$ the set of all bounded $A$-linear maps from $\mathcal{M}$ to $A^{**}$. Let us equip this set with a structure of a vector space over $\mathbf{C}$ by the formula $(\lambda\phi)(x) := \overline{\lambda}\phi(x)$, where $\lambda \in \mathbf{C}$, $x \in \mathcal{M}$, $\phi \in \mathrm{Hom}_A(\mathcal{M}, A^{**})$, and also with a structure of a right $A^{**}$-module by the formula $(\phi \cdot b)(x) := b^*\phi(x)$, $b \in A^{**}$. For a functional $f \in (\mathcal{M}^{\#})'$ we can define a map $f_R \in \mathrm{Hom}_A(\mathcal{M}, A^{**})$ as the restriction of $f$ onto $\mathcal{M}$, namely, $f_R(x) := f(x \otimes 1 + \mathcal{N})$. Obviously $\|f_R\| \le \|f\|$.

**Theorem 1.1.3** ([27]) *For any $C^*$-algebra $A$ and for any Hilbert $A$-module $\mathcal{M}$ the map $f \longmapsto f_R$ is an isometry of $(\mathcal{M}^{\#})'$ onto* $\mathrm{Hom}_A(\mathcal{M}, A^{**})$.

**Proof:** Let a matrix $[c_{ij}] \in M_n(A^{**})$ be such that $\sum_{i,j} a_i^* c_{ij} a_j \ge 0$ for any $a_1,\dots,a_n \in A$. Let us demonstrate that it is sufficient to prove positivity of the matrix $[c_{ij}]$. For this purpose it is sufficient to show that

$$\sum_{i,j} b_i^* c_{ij} b_j \ge 0 \qquad (64)$$

for any $b_1,\dots,b_n \in A^{**}$. Without loss of generality it is possible to suppose that the elements $b_i$ lie in the unit ball $B_1(A^{**})$ of the $W^*$-algebra $A^{**}$. Since the unit ball $B_1(A)$ of the $C^*$-algebra $A$ is dense in $B_1(A^{**})$ with respect to the strong* topology, it is possible to find nets $a_{i;\lambda} \in A$, $\lambda \in \Lambda$, converging with respect to the strong* topology to the elements $b_i \in A^{**}$. Then the net $\sum_{ij} a^*_{i;\lambda} c_{ij} a_{j;\lambda}$ converges to $\sum_{i,j} b^*_i c_{ij} b_j$ with respect to the weak topology, (see [30], §II.2) whence the inequality (64) follows.

We need to demonstrate that any map $\phi \in \mathrm{Hom}_A(\mathcal{M}, A^{**})$ with $\|\phi\| \leq 1$ can be extended up to a unique functional $f \in (\mathcal{M}^\#)'$ with $\|f\| \leq 1$. Let us consider the functional $f_0 : \mathcal{M} \otimes A^{**} \longrightarrow A^{**}$ given by the formula

$$f_0\left(\sum_{i=1}^n x_i \otimes a_i\right) = \sum_{i=1}^n \phi(x_i) a_i.$$

Obviously $f_0$ is an $A^{**}$-module map. Also for $a_i \in A$ one has

$$\begin{aligned}\sum_{i,j} a^*_i \phi(x_i)^* \phi(x_j) a_j &= \sum_{i,j} \phi(x_i \cdot a_i)^* \phi(x_j \cdot a_j) = \left(\phi\left(\sum_{i=1}^n x_i \cdot a_i\right)\right)^* \phi\left(\sum_{i=1}^n x_i \cdot a_i\right) \\ &\leq \left\langle \sum_{i=1}^n x_i \cdot a_i, \sum_{i=1}^n x_i \cdot a_i \right\rangle = \sum_{i,j} a^*_i \langle x_i, x_j \rangle a_j,\end{aligned}$$

therefore for any $b_i \in A^{**}$ the following inequality holds

$$\sum_{i,j} b^*_i \phi(x_i)^* \phi(x_j) b_j \leq \sum_{i,j} b^*_i \langle x_i, x_j \rangle b_j,$$

i.e.

$$f_0(z)^* f_o(z) \leq [z, z]$$

for all $z \in \mathcal{M} \otimes A^{**}$. Therefore the functional $f : \mathcal{M}^\# \longrightarrow A^{**}$, is well-defined by the formula

$$f\left(\sum_{i=1}^n x_i \otimes a_i + \mathcal{N}\right) := \sum_{i=1}^n \phi(x_i) a_i$$

and satisfies the inequality $f(y)^* f(y) \leq [y, y]$ for all $y \in \mathcal{M}^\#$, therefore $\|f\| \leq 1$. Hence $f \in (\mathcal{M}^\#)'$. It follows from the equality $f(x \otimes 1 + \mathcal{N}) = \phi(x)$ that $f$ is the extension for $\phi$. □

**Corollary 1.1.4** *Let $A$ be a $C^*$-algebra, $\mathcal{M}$ be a Hilbert $A$-module. Then an $A$-valued inner product on $\mathcal{M}$ can be extended up to an $A^{**}$-valued inner product on the set* $\mathrm{Hom}_A(\mathcal{M}, A^{**})$ *making this set a self-dual Hilbert $A^{**}$-module .* □

**Corollary 1.1.5** *Let $A$ be a $C^*$-algebra, $\mathcal{M}$ be a self-dual Hilbert $A$-module. Then the Hilbert $A^{**}$-module $\mathcal{M}^\#$ is self-dual too.* □

As one more corollary we shall present the following characterization of self-dual Hilbert modules.

**Theorem 1.1.6** ([8, 6]) *Let $A$ be a $C^*$-algebra. Then the following statements are equivalent:*

(i) *a Hilbert $A$-module $H_A$ is self-dual;*

(ii) *the $C^*$-algebra $A$ is finite-dimensional.*

**Proof:** Let us remark that both conditions of the theorem implies existence of a unit in the $C^*$-algebra $A$. Indeed, if $A$ is finite-dimensional then $1 \in A$. If the module $H_A$ is self-dual then the bounded $A$-module map $f : H_A \longrightarrow A$ defined by the formula $f(a) = a_1$, where $a = (a_i)$, $a_i \in A$, $i \in \mathbf{N}$, has to be the element of the module $H_A$. It means, in particular, that $\mathrm{End}_A(A) = \mathrm{End}^*_A(A)$ and so the identity mapping $A \longrightarrow A$ is identified with the unit element of $A$.

Let us use further the description of the dual module $H'_A$ as the set of all sequences $b = (b_i)$, $b_i \in A$ such that partial sums of the series $c_n = \sum_{i=1}^n b^*_i b_i$ are uniformly bounded. Self-duality means that for any increasing sequence $(c_n)$ of positive elements of the $C^*$-algebra $A$ boundedness is equivalent to convergence of this sequence with respect to the norm. But, if $C^*$-algebra is finite-dimensional, then the monotone bounded sequences are convergent, and it proves the implication (ii) $\Rightarrow$ (i). For the proof in the other direction we will pass to the Hilbert module $H_A^{\#} = H_{A^{**}}$ over the enveloping $W^*$-algebra $A^{**}$. This module is self-dual by Corollary 1.1.5. Since any monotone bounded sequence in the $W^*$-algebra $A^{**}$ is convergent, so any positive linear functional on $A^{**}$ is obliged to be normal, i.e. $(A^{**})_* = (A^{**})^*$. Let the $W^*$-algebra $A^{**}$ be infinite-dimensional. Then it contains an infinite collection of mutually orthogonal projections $p_k \in A^{**}$ such that $\sum_{k=1}^{\infty} p_k = 1$. Therefore there exists an inclusion of the commutative $W^*$-algebra of bounded sequences $l_\infty$ into $A^{**}$. Let $\varphi \in (l_\infty)^*$ be a positive linear functional on the algebra $l_\infty$. Let us extend it up to a positive linear functional $\overline{\varphi}$ on the greater algebra $A^{**}$. Under the assumption, $\overline{\varphi}$ is normal, therefore its restriction $\overline{\varphi}|_{l_\infty} = \varphi$ on the algebra $l_\infty$ is normal too. Hence we have obtained an incorrect statement $(l_\infty)_* = (l_\infty)^*$. This contradiction shows that the $W^*$-algebra $A^{**}$ is finite-dimensional, therefore $C^*$-algebra $A$ is finite-dimensional too and it proves the implication (i) $\Rightarrow$ (ii) . □

## 1.2 Multipliers and centralizers

While writing this section we used widely [29, 36]. Let $H$ be a Hilbert space, $\mathcal{B}(H)$ be the algebra of all bounded operators on $H$, $A$ be a $C^*$-algebra.

**Definition 1.2.1** A two-sided closed ideal $J \subset A$ is called *essential,* if $J \cap J' \neq \emptyset$ for any nonzero ideal $J' \subset A$.

**Remark 1.2.2** An ideal $J \subset A$ is essential if and only if

$$J^\perp := \{a \in A \mid aJ = 0\} = 0.$$

**Definition 1.2.3** A representation $\rho : A \to \mathcal{B}(H)$ is called *non-degenerate,* if for any $h \in H$ there exists an element $a \in A$ such that $\rho(a)h \neq 0$.

**Remark 1.2.4** For an arbitrary representation $\rho$ we can take its restriction onto the orthogonal complement $H'$ to the invariant subspace $H_\rho^0 := \{h \in H \mid \rho(A)h = 0\}$, which is invariant too. The new representation $\rho' : A \to \mathcal{B}(H')$ will be non-degenerate. Thus, roughly speaking, we lose nothing when we restrict ourselves to consideration of only non-degenerate representations.

**Lemma 1.2.5** *A representation is non-degenerate if and only if $\rho(A)(H)$ is dense in $H$.*

**Proof:** Let a representation be non-degenerate and $h \perp \rho(A)(H)$, i. e. for any $f \in H$ and any $a \in A$

$$0 = \langle h, \rho(a)f \rangle = \langle \rho(a^*)h, f \rangle$$

holds, whence $\rho(b)h = 0$ for any $b \in A$. Hence $h = 0$.

Conversely, let $\overline{\rho(A)(H)} = H$, $h \in H$ be an arbitrary nonzero vector. Without loss of generality it is possible to suppose that $\|h\| = 1$. Since $\rho(A)(H)$ is dense, one can find $g \in H$ and $a \in A$ such that $\|h - \rho(a)g\| < 1/2$. Then $\|\rho(a)g\| > 1/2$

$$1/4 > \langle h - \rho(a)g, h - \rho(a)g \rangle = 1 - \langle g, \rho(a^*)h \rangle - \langle \rho(a^*)h, g \rangle + 1/4,$$

$$\langle g, \rho(a^*)h \rangle + \langle \rho(a^*)h, g \rangle > 1, \qquad \rho(a^*)h \neq 0. \qquad \square$$

**Definition 1.2.6** Let $\rho : A \to \mathcal{B}(H)$ be a faithful nondegenerate representation, so we can assume $A \subset \mathcal{B}(H)$. An operator $x \in \mathcal{B}(H)$ is called a (two-sided) *multiplier* of $A$, if for any $a \in A$

$$xa \in A, \qquad ax \in A.$$

Let us denote by $\mathbf{M}(A)$ the set of all multipliers. It is obvious that they form an involutive unital algebra.

**Remark 1.2.7** Thus, until we prove Theorem 1.2.11, the definition of multipliers depends on the choice of a (nondegenerate faithful) representation.

**Proposition 1.2.8** *The set $\mathbf{M}(A)$ is a unital $C^*$-algebra,*

$$A \subset \mathbf{M}(A) \subset A^{**},$$

*$A$ is an essential ideal in $\mathbf{M}(A)$. If $A$ is without unit, then $A^+ \subset \mathbf{M}(A)$.*

**Proof:** Three statements are nontrivial: 1) that it is closed with respect to the norm, 2) that the ideal is essential, and 3) that there exists an inclusion into the second adjoint.

1) Let $x_i \to x$ with respect to the norm, $x_i \in \mathbf{M}(A)$, $x \in \mathcal{B}(H)$. Then $x_i a \to xa$ and $ax_i \to ax$ for any $a$. Since $A$ is closed, $xa \in A$ and $ax \in A$ (for any $a$), i. e. by definition, $x \in \mathbf{M}(A)$.

2) Let $J$ be an ideal in $\mathbf{M}(A)$ and $J \cap A = 0$ and $x \in J$ be an arbitrary element. Then $xa \in A$ (since $x$ is a multiplier) and $xa \in J$ (since $a \in A \subset \mathbf{M}(A)$ and $J$ is an ideal) for any $a \in A$. Therefore $xa \in J \cap A = 0$, $xa = 0$ for any $a \in A$. Then $x = 0$ by Lemma 1.2.5.

3) Since $A^{!!} \hookrightarrow A^{!!}_u \cong A^{**}$ (cf. the remark after Theorem [19, Theor. 3.1.3]), it is sufficient to prove that $\mathbf{M}(A) \subset A^{!!}$. For this purpose, first of all, let us remark that for any $x \in \mathbf{M}(A)$ and for any weakly converging net $a_\lambda \in A$ we have

$$x \; w\text{–}\lim_{\lambda \in \Lambda} a_\lambda = w\text{–}\lim_{\lambda \in \Lambda} (x a_\lambda) \in [A]_w,$$

since $xa_\lambda \in A$, and $[A]_w = A^{!!}$ by the nondegeneracy of the representation, where $[A]_w$ is the weak closure of $A$ in $\mathcal{B}(H)$ and where $w$–lim denotes the limit with respect to the weak topology. Hence $xA^{!!} = x[A]_w \subset [A]_w = A^{!!}$. Since $1 \in A^{!!}$, $x \in A^{!!}$. $\square$

Another definition was historically the first:

**Definition 1.2.9** A pair $(L, R)$ of maps

$$L : A \to A, \qquad R : A \to A, \qquad R(a)b = aL(b) \text{ для всех } a, b \in A.$$

is called a *double centralizer* of $A$ Let us denote the set of all double centralizers of $A$ by $\mathbf{DC}(A)$.

**Proposition 1.2.10** *Let* $(L, R) \in \mathbf{DC}(A)$. *Then*

(i) $L(ab) = L(a)b$ *and* $R(ab) = aR(b)$*;*

(ii) $L$ *and* $R$ *are linear;*

(iii) $L$ *and* $R$ *are bounded, and* $\|L\| = \|R\|$.

*With respect to the norm*

$$\|(L, R)\| := \|L\| = \|R\|$$

*and to the actions*

$$(L_1, R_1) + (L_2, R_2) := (L_1 + L_2, R_1 + R_2), \qquad z(L, R) = (zL, zR), \quad z \in \mathbf{C},$$

$$(L_1, R_1)(L_2, R_2) := (L_1 L_2, R_2 R_1),$$

$$(L, R)^* := (R^*, L^*), \qquad L^*(a) := (L(a^*))^*, \qquad R^*(a) := (R(a^*))^*, \quad a \in A,$$

$\mathbf{DC}(A)$ *is a normed involutive algebra.*

**Proof:** 1) Let $a$ and $b$ be elements of $A$, $z \in \mathbf{C}$ and $e_\alpha$ $(\alpha \in \mathcal{A})$ be an approximate unit of $A$. Then

$$e_\alpha L(ab) = R(e_\alpha)ab = e_\alpha L(a)b, \qquad L(ab) = L(a)b,$$

$$e_\alpha L(za + zb) = R(e_\alpha)(za + zb) = R(e_\alpha)za + R(e_\alpha)zb = zR(e_\alpha)a + zR(e_\alpha)b =$$
$$= ze_\alpha L(a) + ze_\alpha L(b) = e_\alpha(z(L(a) + L(b))), \qquad L(za + zb) = z(L(a) + L(b)).$$

2) Thus, $L$ is a linear operator on the Banach space $A$ and for the proof of its continuity it is sufficient to prove that the graph is closed. Let $a_n \longrightarrow a$ and $L(a_n) \longrightarrow b$. Then for any $v \in A$

$$\|v(L(a) - b)\| \le \|vL(a) - vL(a_n)\| + \|vL(a_n) - vb\| = \|R(v)(a - a_n)\| + \|vL(a_n) - vb\| \le$$
$$\le \|R(v)\| \cdot \|a - a_n\| + \|v\| \cdot \|L(a_n) - b\| \longrightarrow 0.$$

Thus, $vL(a) = vb$, whence, since $v$ was taken arbitrarily, we obtain $b = L(a)$. We have proved that the graph is closed, so $L$ is continuous. Properties of $R$ can be verified similarly.

3) Let us compare $\|L\|$ and $\|R\|$:

$$\|L\|^2 = \sup_{\|a\|=1} \|L(a)\|^2 = \sup_{\|a\|=1} \|L(a)^* L(a)\| = \sup_{\|a\|=1} \|R(L(a)^*)a\|$$

$$\leq \sup_{\|a\|=1} \|R\| \cdot \|L(a)^*\| \cdot \|a\| \leq \sup_{\|a\|=1} \|R\| \cdot \|L\| \cdot \|a\|^2 = \|R\| \cdot \|L\|,$$

whence $\|L\| \leq \|R\|$. The similar calculation gives the opposite estimate.

The remaining statements are obvious, it is necessary to verify only that

$$R_2(R_1(a))b = R_1(a)L_2(b) = aL_1(L_2(b)). \qquad \Box$$

**Theorem 1.2.11** *The map*

$$\mu : \mathbf{M}(A) \to \mathbf{DC}(A), \qquad x \mapsto (L_x, R_x), \qquad L_x(a) = xa, \quad R_x(a) = ax,$$

*is an isometric $*$-isomorphism between $\mathbf{M}(A)$ and $\mathbf{DC}(A)$. Therefore $\mathbf{DC}(A)$ is a $C^*$-algebra and $\mathbf{M}(A)$ does not depend on the choice of a nondegenerate representation.*

**Proof:** First of all $aL_x(b) = axb = R_x(a)b$, so that $(L_x, R_x)$ really lies in $\mathbf{DC}(A)$. The linearity of the map is obvious. Also,

$$L_{xy}(a) = (xy)a = x(ya) = L_x(ya) = L_x(L_y(a)), \quad R_{xy}(a) = a(xy) = (ax)y = R_y(ax) = R_y(R_x(a)),$$

$$(L_{xy}, R_{xy}) = (L_xL_y, R_yR_x) = (L_x, R_x)(L_y, R_y),$$

thus, $\mu$ is a homomorphism of algebras. It is involutive:

$$(L_x)^*(a) = (L_xa^*)^* = (xa^*)^* = ax^* = R_{x^*}(a), (R_x)^*(a) = (R_xa^*)^* = (a^*x)^* = x^*a = L_{x^*}(a).$$

As $\|xa\| \leq \|x\|\,\|a\|$, so $\|L_x\| \leq \|x\|$. Conversely, since the representation is non-degenerate, $xe_\alpha \longrightarrow$ with respect to the strong topology, where $e_\alpha$ is an approximate unit of $A$. Indeed, for any $a \in A$ we have the convergence of $e_\alpha a \longrightarrow a$ with respect to the norm, whence $xe_\alpha a \longrightarrow xa$ with respect to the norm. From the nondegeneracy we obtain the strong convergence $xe_\alpha \longrightarrow x$ on the dense set $AH$, and, by boundedness $\|xe_\alpha\| \leq \|x\|$ the strong convergence takes place everywhere on $H$. Let $\varepsilon > 0$ be taken arbitrarily. Let us choose $h \in H$ such that $\|h\| = 1$ and $\|x\| \leq \|xh\| + \varepsilon/2$. Since $x = s\text{–}\lim_\alpha xe_\alpha$, $xe_\alpha h \longrightarrow xh$ and one can find $\alpha$ such that $\|xh - xe_\alpha h\| < \varepsilon/2$. Then $\|x\| \leq \|xe_\alpha\| + \varepsilon$. Therefore

$$\|L_x\| \geq \|L_x(e_\alpha)\| = \|xe_\alpha\| \geq \|x\| - \varepsilon,$$

and since $\varepsilon$ is arbitrary, $\|L_x\| \geq \|x\|$. So $\mu$ is an isometry.

It remains to demonstrate that $\operatorname{Im}\mu = \mathbf{DC}(A)$. Let us consider an arbitrary element $(L, R) \in \mathbf{DC}(A)$. Then the sets $L(e_\alpha)$ and $R(e_\alpha)$ are bounded and, by the weak compactness of the unit ball $\mathcal{B}(H)$, they have points of accumulation with respect to the weak topology $x_L \in A^{!!}$ and $x_R \in A^{!!}$, respectively. Passing, if necessary, to sub-nets, we can suppose without loss of generality that

$$x_L = w\text{–}\lim_{\mathcal{A}} L(e_\alpha), \qquad x_R = w\text{–}\lim_{\mathcal{A}} R(e_\alpha).$$

Then $x_R = x_L$. Indeed, for any $a$ and $b$ from $A$ the following relations hold

$$ax_Lb = a\, w\text{–}\lim_{\mathcal{A}} L(e_\alpha)b = a\, w\text{–}\lim_{\mathcal{A}} L(e_\alpha b) = aL(b) = R(a)b,$$

$$ax_Rb = a\, w\text{–}\lim_{\mathcal{A}} R(e_\alpha)b = w\text{–}\lim_{\mathcal{A}} R(ae_\alpha)b = R(a)b,$$

whence $x_L = x_R$ (cf. the proof of item 2 of Propositin 1.2.8). Let us denote $x := x_L = x_R$. Then

$$L(a) = w\text{–}\lim_{\mathcal{A}} L(e_\alpha a) = xa = L_x(a),$$

$$R(a) = w\text{–}\lim_{\mathcal{A}} R(ae_\alpha) = ax = R_x(a),$$

in particular, $x \in \mathbf{M}(A)$. The equalities demonstrate that $\mu(x) = (L, R)$. $\Box$

**Example 1.2.12** 1). The equality $\mathbf{M}(A) = A$ holds if and only if $A$ is unital.

2). For a commutative algebra $A = C_0(X)$ the following equality holds

$$\mathbf{M}(C_0(X)) = C_b(X) \cong C(\beta X),$$

where $C_b(X)$ is the algebra of all bounded functions with uniform convergence, and $\beta X$ is the Stone-Čech compactification of $X$.

3). For the algebra $\mathcal{K} = \mathcal{K}(H)$ of compact operators one has $\mathbf{M}(\mathcal{K}(H)) = \mathcal{B}(H)$.

The proof of 1) and 2) can be found, for example, in [36], and 3) will be proved below in a more general situation (Theorem 2.1.1).

**Definition 1.2.13** Let $\rho : A \to \mathcal{B}(H)$ be a faithful non-degenerate representation, so we can assume $A \subset \mathcal{B}(H)$. An operator $x \in \mathcal{B}(H)$ is called a *left multiplier* of $A$, if for every $a \in A$

$$xa \in A.$$

Let us denote by $\mathbf{LM}(A)$ the set of left multipliers. It is obvious that they form a unital algebra. Similarly one defines *right multipliers* $\mathbf{RM}(A)$.

An operator $x \in \mathcal{B}(H)$ is called *quasi-multiplier* of $A$, if for every $a,\ b \in A$

$$axb \in A.$$

Let us denote by $\mathbf{QM}(A)$ the set of all quasi-multipliers. It is obvious that they form an involutive linear space.

**Definition 1.2.14** A linear map $\lambda : A \to A$ is called a *left centralizer,* if

$$\lambda(ab) = \lambda(a)b, \qquad \text{for each } a,\ b \in A.$$

Similarly one defines a *right centralizer* . Let us denote the spaces of left and right centralizers by $\mathbf{LC}(A)$ and $\mathbf{RC}(A)$.

**Definition 1.2.15** A linear map $q : A \times A \to A$ is called a *quasi-centralizer,* if

$$\lambda_a \in \mathbf{LC}(A),\ \text{где}\ \lambda_a : b \mapsto q(a,b), \qquad \rho_b \in \mathbf{RC}(A),\ \text{где}\ \rho_b : a \mapsto q(a,b), \qquad \text{for any } a,\ b \in A.$$

In other words,

$$q(ca, bd) = cq(a,b)d, \qquad \text{for any } a,\ b,\ c,\ d \in A.$$

**Lemma 1.2.16** *Let $\rho \in \mathbf{RC}(A)$, then $\rho^* \in \mathbf{LC}(A)$.*

**Proof:** Let us remind that $\rho^*$ is defined as follows: $\rho^*(a) := (\rho(a^*))^*$. Then

$$\rho^*(ab) = (\rho((ab)^*))^* = (\rho(b^*a^*))^* = (b^*\rho(a^*))^* = \rho^*(a)b. \qquad \square$$

**Lemma 1.2.17** [29, Lemma 3.12.2] *Each right centralizer, each left centralizer and each quasi-centralizer is bounded.*

**Proof:** Let $\rho \in \mathbf{RC}(A)$. Let it be unbounded, i. e. there exists a sequence $x_n \in A$ such that $\|x_n\| < 1/n$ and $\|\rho(x_n)\| > n$. Then the element $a := \sum_n x_n^* x_n$ is well-defined. By Proposition [29, Prop. 1.4.5] (see. also [19, Prop. 1.1.5]), let us define for each $x_n$ an element $u_n \in A$ such that $\|u_n\| \le \|a^{1/6}\|$ and $x_n = u_n a^{1/3}$. Then

$$\|\rho(x_n)\| = \|u_n \rho(a^{1/3})\| \le \|\rho(a^{1/3})\| \cdot \|a^{1/6}\|.$$

We have obtained a contradiction, hence $\rho$ is bounded. In a similar way one can prove that any left centralizer is bounded too. Thus, $q \in \mathbf{QC}(A)$ is continuous separately in each variable as a map $A \times A \to A$. By the principle of uniform boundedness such operator is continuous in both variables (see [5]). $\square$

**Proposition 1.2.18** [29, Prop. 3.12.3] *Let $A \to \mathcal{B}(H)$ be a non-degenerate faithful representation. Then there exists a bijective isometric linear correspondence between left, right and quasi-multipliers and, correspondently, left, right and qasi-centralizers. In the first two cases it is a homomorphism of algebras, in the third it is a homomorphism of involutive spaces.*

**Proof:** The correspondence for the left and right multipliers, and also its properties, actually were already described in Theorem 1.2.11. Let $q \in \mathbf{QC}(A)$ and $x \in A^{!!}$ be an accumulation point with respect to the weak topology of the bounded directed net $\{q(e_\alpha, e_\alpha)\}$, where $\{e_\alpha\}$ is an approximate unit for $A$. Passing, if necessary, to a sub-net, we can, as well as before, suppose that $x = w\text{–}\lim_\alpha q(e_\alpha, e_\alpha)$. Then for any $a,\ b \in A$

$$A \ni q(a,b) = \lim_\alpha q(ae_\alpha, e_\alpha b) = \lim_\alpha aq(e_\alpha, e_\alpha)b = axy,$$

holds, so that $x \in \mathbf{QM}(A) \subset A^{!!}$. The necessary properties can be verified exactly as the similar ones in 1.2.11. $\square$

**Proposition 1.2.19** *Let $A$ be a (closed two-sided $*$-) ideal of a $C^*$-algebra $B$. Then there exists a unique homomorphism $\gamma : B \to \mathbf{M}(A)$ identical on $A$.*

**Proof:** Let us put $\gamma(b) := (L_b, R_b)$, i. e. $L_b(a) = ba$, $R_b(a) = ab$, where we identify $\mathbf{M}(A) = \mathbf{DC}(A)$. Since $A \subset B$ is an ideal, $ba \in A$ and $ab \in A$, so that $\gamma(b) \in \mathbf{DC}(A)$. Thus, obviously $\gamma|_A : A \hookrightarrow \mathbf{M}(A)$.

Let us assume that besides $\gamma$ there exists a homomorphism $\delta : B \to \mathbf{M}(A)$ possessing the demanded properties. Then for any $b \in B$ and $a \in A$

$$\delta(b)a = \delta(b)\delta(a) = \delta(ba) = ba, \qquad \gamma(b)a = \gamma(b)\gamma(a) = \gamma(ba) = ba,$$

i. e. $\gamma(b)$ and $\delta(b)$ coincide as multipliers of $A$, so $\delta = \gamma$. $\square$

**Corollary 1.2.20** *Let $\rho : A \to \mathcal{B}(H)$ be a faithful representation of $A$ and $A \subset B$ be an ideal. Then there exists a representation of $B$ extending $\rho$.*

**Proposition 1.2.21** *Let $A$ and $B$ be some $C^*$-algebras and let $\varphi : A \to B$ be a* surjective *morphism. Then $\varphi$ can be extended up to a morphism $\varphi'' : \mathbf{M}(A) \to \mathbf{M}(B)$ and induces a morphism $\overline{\varphi} : \mathbf{M}(A)/A \to \mathbf{M}(B)/B$, which completes the following diagram up to a commutative one:*

$$\begin{array}{ccccc} A & \longrightarrow & \mathbf{M}(A) & \xrightarrow{\pi} & \mathbf{M}(A)/A \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi''} & & \downarrow{\scriptstyle\overline{\varphi}} \\ B & \longrightarrow & \mathbf{M}(B) & \xrightarrow{\pi} & \mathbf{M}(B)/B. \end{array}$$

*If $\varphi$ is an isomorphism then $\varphi''$ and $\overline{\varphi}$ are isomorphisms too.*

**Proof:** Let $(L, R) \in \mathbf{DC}(A)$. Let us define $\widehat{L}, \widehat{R} : B \to B$ by putting

$$\widehat{L}(b) := \varphi(L(a)), \qquad \widehat{R}(b) := \varphi(R(a)), \qquad b \in B,\ b = \varphi(a).$$

Let us demonstrate that these maps are well-defined. Let $e_\alpha$ be an approximate unit of the algebra $A$ and $b = \varphi(a) = \varphi(a')$. Then

$$\varphi(L(a) - L(a')) = \lim_\alpha \varphi(e_\alpha L(a) - e_\alpha L(a')) = \lim_\alpha \varphi(R(e_\alpha))\, \varphi(a - a') = 0.$$

A similar equality holds for right multipliers. Since for $b_1 = \varphi(a_1)$ and $b_2 = \varphi(a_2)$

$$\widehat{R}(b_1)b_2 = \varphi(R(a_1))\,\varphi(a_2) = \varphi(R(a_1)a_2) = \varphi(a_1 L(a_2)) = \varphi(a_1)\,\varphi(L(a_2)) = \beta_1 \widehat{L}(b_2),$$

then $(\widehat{L}, \widehat{R}) \in \mathbf{DC}(A)$. Let us define $\varphi'' : \mathbf{DC}(A) \to \mathbf{DC}(B)$ as $\varphi''(L, R) = (\widehat{L}, \widehat{R})$. Then it is a $*$-morphism of algebras extending $\varphi$. This map induces a map of quotients. Indeed, if $(x - y) \in A$, $x$, $y \in \mathbf{M}(A)$, then $\varphi''(x - y) = \varphi(x - y) \in B$. We have obtained the desired commutative diagram.

If now $\varphi$ is an isomorphism, $\varphi''(L, R) = (\varphi \circ L \circ \varphi^{-1}, \varphi \circ R \circ \varphi^{-1})$, so that $\varphi''$ is an isomorphism, the inverse map is defined by $(\widehat{L}, \widehat{R}) \mapsto (\varphi^{-1} \circ \widehat{L} \circ \varphi, \varphi^{-1} \circ \widehat{R} \circ \varphi)$. By the five-lemma $\overline{\varphi}$ is also an isomorphism. $\square$

**Remark 1.2.22** The homomorphism $\varphi''$ coincides with the canonical extension of $\varphi$ to $A^{!!}$ restricted onto $\mathbf{M}(A)$.

# 2 Operators on Hilbert modules as multipliers

## 2.1 Multipliers of $A$-compact operators

**Theorem 2.1.1** [13] *Let $\mathcal{M}$ be an arbitrary Hilbert $A$-module. Let us define a map*

$$\phi : \mathrm{End}^*_A(\mathcal{M}) \to \mathbf{DC}(\mathcal{K}(\mathcal{M})), \qquad T \mapsto (T_1, T_2), \quad T_1(\theta_{x,y}) := \theta_{Tx,y}, \quad T_2(\theta_{x,y}) := \theta_{x,T^*y}.$$

*Then $\phi$ defines an isomorphism $\mathrm{End}^*_A(\mathcal{M}) \cong \mathbf{DC}(\mathcal{K}(\mathcal{M}))$.*

**Proof:** First of all, let us remark that

$$\theta_{Tx,y} z = Tx\langle y, z\rangle = T\circ\theta_{x,y}(z), \qquad \theta_{x,T^*y} z = x\langle T^*y, z\rangle = x\langle y, Tz\rangle = \theta_{xy}\circ T(z),$$

so that $T_1$ and $T_2$ can be defined in equivalent way (and for all compact operators simultaneously) by the formulas

$$T_1(k) := T\circ k, \qquad T_2(k) := k\circ T, \qquad k\in\mathcal{K}(\mathcal{M}).$$

From these equalities we obtain at once that $T_1$ and $T_2$ are well-defined as maps $\mathcal{K}(\mathcal{M})\to\mathcal{K}(\mathcal{M})$ (since $\mathcal{K}(\mathcal{M})\subset \mathrm{End}^*_A(\mathcal{M})$ is a two-sided ideal) and are bounded by the norm $\|T\|$. As

$$T_2(k_1)k_2 = k_1Tk_2 = k_1T_1(k_2), \qquad k_1,\, k_2\in\mathcal{K}(\mathcal{M}),$$

so $(T_1,T_2)\in\mathbf{DC}(\mathcal{K}(\mathcal{M}))$. Since

$$(TS)_1(k) = TSk = T_1(S_1k), \qquad (TS)_2(k) = kTS = S_2(T_2k),$$

$\phi$ is a homomorphism of algebras. It respects the involution:

$$T_1^*(\theta_{x,y}) = (T_1(\theta_{x,y}^*))^* = (T\theta_{y,x})^* = \theta_{x,y}T^*, \quad T_1^* = (T^*)_2,$$

$$T_2^*(\theta_{x,y}) = (T_2(\theta_{x,y}^*))^* = (\theta_{y,x}T)^* = T^*\theta_{x,y}, \quad T_2^* = (T^*)_1.$$

The map $\phi$ is algebraically injective. Indeed, let $T_1 = 0$ and $T_2 = 0$. Then for any $x\in\mathcal{M}$ $0 = T_1(\theta_{x,Tx})(Tx) = Tx\langle Tx, Tx\rangle$ holds, whence $\langle Tx, Tx\rangle^3 = 0$ and $Tx = 0$. Hence $T = 0$.

To prove that $\phi$ is an epimorphism, let us construct an inverse continuous map $\psi$. Let $(T_1,T_2)$ be an element of $\mathbf{DC}(\mathcal{K}(\mathcal{M}))$ and $x\in\mathcal{M}$. Let us consider the limits

$$T(x) := \lim_{n\to\infty} T_n(x), \qquad T_n(x) := T_1(\theta_{x,x})(x)[\langle x,x\rangle + 1/n]^{-1}, \tag{65}$$

$$T^*(x) = \lim_{n\to\infty} T_n^*(x), \qquad T_n^*(x) := [T_2(\theta_{x,x})]^*(x)[\langle x,x\rangle + 1/n]^{-1}. \tag{66}$$

Let us prove their existence. By Theorem 1.2.11 $(T_1,T_2) = (L_F, R_F)$, where $F\in\mathbf{M}(\mathcal{K}(\mathcal{M}))$. Then

$$(T_1(k))^*T_1(k) = (Fk)^*Fk = k^*F^*Fk \le \|F\|^2k^*k = \|T_1\|^2k^*k,$$

$$T_2(k)(T_2(k))^* = kF(kF)^* = kFF^*k^* \le \|F\|^2kk^* = \|T_2\|^2kk^*,$$

where the inequalities are the operator inequalities of elements from $\mathcal{K}(\mathcal{M})^{**}$. Then

$$\langle T_n(x) - T_m(x), T_n(x) - T_m(x)\rangle$$

$$= \{[\langle x,x\rangle + 1/n]^{-1} - [\langle x,x\rangle + 1/m]^{-1}\}\langle T_1(\theta_{x,x})(x), T_1(\theta_{x,x})(x)\rangle\{[\langle x,x\rangle + 1/n]^{-1} - [\langle x,x\rangle + 1/m]^{-1}\}$$

$$\le \{[\langle x,x\rangle + 1/n]^{-1} - [\langle x,x\rangle + 1/m]^{-1}\}\langle (T_1(\theta_{x,x}))^*T_1(\theta_{x,x})(x), x\rangle\{[\langle x,x\rangle + 1/n]^{-1} - [\langle x,x\rangle + 1/m]^{-1}\}$$

$$\le \{[\langle x,x\rangle + 1/n]^{-1} - [\langle x,x\rangle + 1/m]^{-1}\}\|T_1\|^2\langle (\theta_{x,x}\theta_{x,x})(x), x\rangle\{[\langle x,x\rangle + 1/n]^{-1} - [\langle x,x\rangle + 1/m]^{-1}\}$$

$$= \|T_1\|^2\{[\langle x,x\rangle + 1/n]^{-1} - [\langle x,x\rangle + 1/m]^{-1}\}\langle \theta_{x,x}x\langle x,x\rangle, x\rangle\{[\langle x,x\rangle + 1/n]^{-1} - [\langle x,x\rangle + 1/m]^{-1}\}$$

$$= \|T_1\|^2\langle x,x\rangle^3\{[\langle x,x\rangle + 1/n]^{-1} - [\langle x,x\rangle + 1/m]^{-1}\}^2.$$

Thus, the Cauchy criterion of convergence for (65) is the same, as for the limit from [19, Lemma 1.3.9], therefore the convergence is proved. The convergence of (66) can be proved similarly. We have obtained the maps $T$ and $T^*$, defined everywhere on $\mathcal{M}$. Also by [19, Lemma 1.3.9]

$$\langle x, T^*y\rangle = \lim_{n\to\infty}\left\langle x, [T_2(\theta_{y,y})]^*(y)\cdot[\langle y,y\rangle + 1/n]^{-1}\right\rangle$$

$$= \lim_{n\to\infty}\left\langle T_2(\theta_{y,y})\theta_{x,x}(x)[\langle x,x\rangle + 1/n]^{-1}, y\cdot[\langle y,y\rangle + 1/n]^{-1}\right\rangle$$

$$= \lim_{n\to\infty}\left\langle \theta_{y,y}T_1(\theta_{x,x})(x)[\langle x,x\rangle + 1/n]^{-1}, y\cdot[\langle y,y\rangle + 1/n]^{-1}\right\rangle$$

$$= \lim_{n\to\infty}\left\langle T_1(\theta_{x,x})(x)[\langle x,x\rangle + 1/n]^{-1}, \theta_{y,y}(y)\cdot[\langle y,y\rangle + 1/n]^{-1}\right\rangle = \langle Tx, y\rangle.$$

Hence, by [27, 13] (see. also [19, Lemma 2.1.1]) $T$, $T^* \in \mathrm{End}^*_A(\mathcal{M})$. It is necessary to verify that $\phi(T) = (T_1, T_2)$. Let us denote

$$x_n := \left(\langle x, x\rangle + \frac{1}{n}\right)^{-1},$$

and remark that by [19, Lemma 1.3.9]

$$\theta_{x,y}(z) = x\langle y, z\rangle = \lim_{n\to\infty} x\langle x, x\rangle x_n \langle y, z\rangle = \lim_{n\to\infty} \theta_{x,x}\theta_{xx_n,y}(z). \tag{67}$$

Therefore

$$T_1(\theta_{x,y})(z) = \lim_{n\to\infty} T_1(\theta_{x,x}\theta_{xx_n,y})(z) = \lim_{n\to\infty} T_1(\theta_{x,x})\theta_{xx_n,y}(z)$$
$$= \lim_{n\to\infty} T_1(\theta_{x,x})xx_n\langle y, z\rangle = T(x)\langle y, z\rangle = T\,\theta_{x,y}(z).$$

A similar reasoning for $T_2$ completes the proof. □

It is easy to obtain the following extension of this theorem.

**Theorem 2.1.2** [17, Theorem 1.5] *There exists an isometric isomorphism of Banach algebras*

$$\phi : \mathrm{End}_A(\mathcal{M}) \longrightarrow \mathbf{LM}(\mathcal{K}(\mathcal{M}))$$

*extending the homomorphism $\phi$ from the theorem* 2.1.1.

**Proof:** As usual, unitalizing, if necessary, we can suppose the algebra $A$ to be unital. Let us define $\phi$, as before, by the formula

$$\phi(T)(k) = Tk, \qquad k \in \mathcal{K}(\mathcal{M}),$$

so that it extends $\phi$ from Theorem 2.1.1. Then the calculations presented in the proof of 2.1.1 for $T_1$ show that $\phi$ is an algebraically injective homomorphism of algebras and $\|\phi\| \le 1$. To prove that it is an epimorphism, let us define a continuous inverse map for $\psi$ similarly to 2.1.1:

$$\psi(S)(x) := \lim_{n\to\infty} T_n(x), \qquad T_n(x) := S(\theta_{x,x})(x)[\langle x, x\rangle + 1/n]^{-1}, \qquad S \in \mathbf{LM}(\mathcal{K}(\mathcal{M})), \quad x \in \mathcal{M}.$$

By the same reasons as in Theorem 2.1.1, the limit exists. Let us show that it defines an $A$-homomorphism. The boundedness of the operator $\psi(S) : \mathcal{M} \to \mathcal{M}$ and continuity of $\psi$ can be verifed as follows. For any $x \in \mathcal{M}$, $\|x\| \le 1$, we have (see [27, 3.11] and [19, 3.4.1]) $x = u\langle x, x\rangle^{1/2}$, where $u \in (\mathcal{M}^{\#})'$ and for any $\beta > 0$ one has $u\langle x, x\rangle^{\beta} \in \mathcal{M}$. Put

$$y := u\langle x, x\rangle^{3\alpha - 1/2} \in \mathcal{M}, \qquad z := u\langle x, x\rangle^{1/2-\alpha} \in \mathcal{M}, \quad \frac{1}{4} < \alpha < \frac{1}{2},$$

so that

$$\langle x, y\rangle = \langle x, x\rangle^{3\alpha}, \qquad \langle z, z\rangle = \langle x, x\rangle^{1-2\alpha}.$$

Then

$$\theta_{z,z}\theta_{z,y}(v) = z\langle z, z\rangle\langle y, v\rangle = u\langle x, x\rangle^{1/2-\alpha}\langle x, x\rangle^{1-2\alpha+3\alpha-1}\langle x, v\rangle = \theta_{x,x}(v).$$

For any $n = 1, 2, \ldots$

$$\langle T_n x, T_n x\rangle = [\langle x, x\rangle + 1/n]^{-1}\,\langle S(\theta_{x,x})(x), S(\theta_{x,x})(x)\rangle\,[\langle x, x\rangle + 1/n]^{-1}$$
$$= [\langle x, x\rangle + 1/n]^{-1}\,\langle S(\theta_{z,z})\theta_{z,y}(x), S(\theta_{z,z})\theta_{z,y}(x)\rangle\,[\langle x, x\rangle + 1/n]^{-1}$$
$$= [\langle x, x\rangle + 1/n]^{-1}\,\langle y, x\rangle^*\langle S(\theta_{z,z})z, S(\theta_{z,z})z\rangle\langle y, x\rangle\,[\langle x, x\rangle + 1/n]^{-1}$$
$$\le \|S(\theta_{z,z})\|^2[\langle x, x\rangle + 1/n]^{-1}\,\langle y, x\rangle^*\langle z, z\rangle\langle y, x\rangle\,[\langle x, x\rangle + 1/n]^{-1}$$
$$= \|S(\theta_{z,z})\|^2[\langle x, x\rangle + 1/n]^{-1}\,\langle x, x\rangle^{4\alpha+1}\,[\langle x, x\rangle + 1/n]^{-1},$$

whence, while $n \longrightarrow \infty$, we obtain

$$\langle \psi(S)x, \psi(S)x\rangle \le \|S\|^2\langle x, x\rangle^{4\alpha-1}, \qquad 1/4 < \alpha < 1/2.$$

In the limit for $\alpha \longrightarrow 1/2$ we get the estimate

$$\langle \psi(S)x, \psi(S)x\rangle \leq \|S\|^2 \langle x, x\rangle.$$

Thus, $\|\psi(S)\| \leq \|S\|$ and by [27] (see also [19, 2.1.4]) $\psi(S)$ is an $A$-homomorphism. Hence $\psi(S) \in \mathrm{End}_A(\mathcal{M})$ and $\|\psi\| \leq 1$.

Let us show that $\phi\psi = \mathrm{Id}_{\mathbf{LM}}$. For this purpose it is sufficient to verify that $S(\theta_{x,y}) = \psi(S)\circ\theta_{x,y} = \theta_{\psi(S)x,y}$. Using the formula (67) and denotation from it, we obtain

$$S(\theta_{x,y})(z) = \lim_{n\to\infty} S(\theta_{x,x}\theta_{xx_n,y})(z) = \lim_{n\to\infty} S(\theta_{x,x})\theta_{xx_n,y}(z)$$

$$= \lim_{n\to\infty} S(\theta_{x,x})xx_n\langle y, z\rangle = \psi(S)(x)\langle y, z\rangle = \psi(S)\circ\theta_{x,y}(z).$$

As $\|\phi\|$, as well as $\|\psi\|$, does not exceed 1, so $\phi$ is an isometry. □

## 2.2 Quasi-multipliers of $A$-compact operators

In this section we present a modified proof of the theorem 1.6 of [17]. Let us remark that this theorem and similar statements about the left and double multipliers can be deduced from some general results about multipliers (see [28]).

**Theorem 2.2.1** *Let $\mathcal{M}$ be a Hilbert $A$-module. Then the map $\phi$ from Theorem 2.1.2 can be extended up to an isometric involutive isomorphism*

$$\phi : \mathrm{End}_A(\mathcal{M}, \mathcal{M}') \longrightarrow \mathbf{QM}(\mathcal{K}(\mathcal{M})).$$

**Proof:** The formula

$$\phi(T)(\theta_{x',y'}, \theta_{x,y}) := \theta_{x', y\cdot(T(x)(y'))}, \qquad x,\ y,\ x',\ y' \in \mathcal{M}, \quad T \in \mathrm{End}_A(\mathcal{M}, \mathcal{M}'),$$

obviously is bilinear, thus, it defines a map on a dense subset in $\mathcal{K}(\mathcal{M}) \times \mathcal{K}(\mathcal{M})$ with values in $\mathcal{K}(\mathcal{M})$.

Let us estimate the norm of this operator. Let $x = u\langle x, x\rangle^{1/2}$ be the polar decomposition of $x$ in $(\mathcal{M}^{\#})'$. Let $w_\varepsilon := u\langle x, x\rangle^{\varepsilon}$, where $0 < \varepsilon < 1/2$. Let us remind that the structure of a right module on $\mathcal{M}'$ is defined by $(\varphi a)(y) = a^*\varphi(y)$ for $a \in A$, $\varphi \in \mathcal{M}'$ and $y \in \mathcal{M}$. By [27] (see also [19, 2.1.4]), we obtain

$$\|\theta_{x',y(T(x)(y'))}(z)\|^2 = \|\langle z, y\rangle (T(x)(y'))\langle x', x'\rangle (T(x)(y'))^*\langle y, z\rangle\|$$

$$= \|\langle z, y\rangle\langle x, x\rangle^{\frac{1}{2}-\varepsilon}[T(w_\varepsilon)(y')]\langle x', x'\rangle [T(w_\varepsilon)(y')]^*\langle x, x\rangle^{\frac{1}{2}-\varepsilon}\langle y, z\rangle\|$$

$$\leq \|\langle x', x'\rangle^{1/2}[T(w_\varepsilon)(y')]^*\|^2 \cdot \|\langle x, x\rangle^{\frac{1}{2}-\varepsilon}\langle y, z\rangle\|^2$$

$$= \|\langle x', x'\rangle^{1/2}[T(w_\varepsilon)(y')]^*[T(w_\varepsilon)(y')]\langle x', x'\rangle^{1/2}\| \cdot \|\langle x, x\rangle^{\frac{1}{2}-\varepsilon}\langle y, z\rangle\|^2$$

$$\leq \|T(w_\varepsilon)\|^2 \cdot \|\langle x', x'\rangle^{1/2}\langle y', y'\rangle\langle x', x'\rangle^{1/2}\| \cdot \|\langle x, x\rangle^{\frac{1}{2}-\varepsilon}\langle y, z\rangle\langle z, y\rangle\langle x, x\rangle^{\frac{1}{2}-\varepsilon}\|$$

$$\leq \|T(w_\varepsilon)\|^2 \cdot \|\langle x', x'\rangle^{1/2}\langle y', y'\rangle^{1/2}\|^2 \cdot \|\langle x, x\rangle^{\frac{1}{2}-\varepsilon}\langle y, y\rangle^{1/2}\| \cdot \|z\|^2.$$

Passing to the limit $\varepsilon \longrightarrow 0$, we obtain $\|w_\varepsilon\| \longrightarrow 1$ and

$$\|\theta_{x',y(T(x)(y'))}(z)\| \leq \|T\| \cdot \|\langle x', x'\rangle^{1/2}\langle y', y'\rangle^{1/2}\| \cdot \|\langle x, x\rangle^{1/2}\langle y, y\rangle^{1/2}\| \cdot \|z\|.$$

Therefore $\|\phi(T)(\theta_{x',y'}, \theta_{x,y})\| \leq \|T\| \cdot \|\theta_{x',y'}\| \cdot \|\theta_{x,y}\|$. Thus, $\phi(T) : \mathcal{K}(\mathcal{M}) \times \mathcal{K}(\mathcal{M}) \to \mathcal{K}(\mathcal{M})$. As

$$\phi(T)(k'\theta_{x',y'}, \theta_{x,y}k) = \phi(T)(\theta_{k'x',y'}, \theta_{x,k^*y}) = \theta_{k'x', k^*y\cdot(T(x)(y'))} = k'\theta_{x',y\cdot(T(x)(y'))}k, \tag{68}$$

so we have $\phi(T) \in \mathbf{QC}(\mathcal{K}(\mathcal{M}))$. Moreover, the previous calculation shows that $\phi$ is continuous: $\|\phi\| \leq 1$.

Let us show the algebraic injectivity of $\phi$, i. e. that $\mathrm{Ker}\,\phi = 0$. Let $T \neq 0$. It means that there exists a vector $x \in \mathcal{M}$ such that $T(x)$ is a nonzero functional, $T(x)(y') \neq 0$ for some (nonzero) $y' \in \mathcal{M}$. Then

$$(T(x)(y'))^*T(x)(y') \leq \|T(x)\|^2\,\langle y', y'\rangle, \qquad \langle y', y'\rangle \geq \|T(x)\|^{-2}(T(x)(y'))^*T(x)(y'),$$

Therefore

$$\langle y'\cdot(T(x)(y'))^*(T(x)(y')),y'\cdot(T(x)(y'))^*(T(x)(y'))\rangle$$
$$=(T(x)(y'))^*(T(x)(y'))\langle y',y'\rangle(T(x)(y'))^*(T(x)(y'))\ge\|T(x)\|^{-2}\{\,(T(x)(y'))^*T(x)(y')\,\}^3\neq 0$$

and

$$\langle\phi(T)(\theta_{y'\cdot(T(x)(y'))^*(T(x)(y')),y'},\theta_{x,y'\cdot(T(x)(y'))^*})[y'\cdot(T(x)(y'))^*(T(x)(y'))],$$
$$\phi(T)(\theta_{y'\cdot(T(x)(y'))^*(T(x)(y')),y'},\theta_{x,y'\cdot(T(x)(y'))^*})[y'\cdot(T(x)(y'))^*(T(x)(y'))]\rangle$$
$$=\langle\theta_{y'\cdot(T(x)(y'))^*(T(x)(y')),y'\cdot(T(x)(y'))^*(T(x)(y'))}[y'\cdot(T(x)(y'))^*(T(x)(y'))],$$
$$\theta_{y'\cdot(T(x)(y'))^*(T(x)(y')),y'\cdot(T(x)(y'))^*(T(x)(y'))}[y'\cdot(T(x)(y'))^*(T(x)(y'))]\,\rangle$$
$$=\langle y'\cdot(T(x)(y'))^*(T(x)(y'))\cdot\langle y'\cdot(T(x)(y'))^*(T(x)(y')),y'\cdot(T(x)(y'))^*(T(x)(y'))\rangle,$$
$$y'\cdot(T(x)(y'))^*(T(x)(y'))\cdot\langle y'\cdot(T(x)(y'))^*(T(x)(y')),y'\cdot(T(x)(y'))^*(T(x)(y'))\rangle\,\rangle$$
$$=\langle y'\cdot(T(x)(y'))^*(T(x)(y')),y'\cdot(T(x)(y'))^*(T(x)(y'))\rangle^3\neq 0.$$

Thus, $\phi(T)\neq 0$ and the algebraic injectivity of $\phi$ is proved.

To prove that the mapping is a surjection and an isometry, it is sufficient to define, as in the previous theorem, a map

$$\psi:\mathbf{QC}(\mathcal{K}(\mathcal{M}))\longrightarrow\mathrm{End}(\mathcal{M},\mathcal{M}'),\qquad\phi\psi(S)=S,\qquad\|\psi\|\le 1.$$

For any $S\in\mathbf{QC}(\mathcal{K}(\mathcal{M}))$ and any $k\in\mathcal{K}(\mathcal{M})$ the map $S(k,\,.\,):\mathcal{K}(\mathcal{M})\to\mathcal{K}(\mathcal{M})$ is a left centralizer, or, in terms of multipliers, for any $S\in\mathbf{QM}(\mathcal{K}(\mathcal{M}))$ and any $k\in\mathcal{K}(\mathcal{M})$ the element $kS\in(K(\mathcal{M}))^{!!}$ is a left multiplier. Then the map $\psi:\mathbf{LM}(\mathcal{K}(\mathcal{M}))\to\mathrm{End}(\mathcal{M})$ from the previous theorem is applicable to it. For making a difference between the mappings we shall denote the mappings obtained in the previous theorem by $\phi'$ and $\psi'$. To define $\psi$, let us put for each $x,\,y\in\mathcal{M}$

$$(\psi(S)(x))(y):=\lim_{n\to\infty}T_n(x,y),\qquad T_n(x,y):=\langle\psi'(\theta_{y,y}S)(x),y\rangle\left(\langle y,y\rangle+\frac{1}{n}\right)^{-1}.\tag{69}$$

We have to verify the following:

**1.** The existence of limit in (69).

**2.** The linearity over $A$ and $\mathbf{C}$ of this expression in $x$ and $y$.

**3.** That the estimate $\|(\psi(S)(x))(y)\|\le\|S\|\,\|x\|\,\|y\|$ holds.

**4.** The identity $\phi\psi(S)=S$.

Let $y=u\cdot\langle y,y\rangle^{1/2}$, $u\in(\mathcal{M}^{\#})'$ and let us put

$$z_1=u\cdot\langle y,y\rangle^{3\alpha-\frac{1}{2}},\qquad z_2=u\cdot\langle y,y\rangle^{\frac{1}{2}-\alpha},\qquad\frac{1}{4}<\alpha<\frac{1}{2}.$$

Then $y=z_1\cdot\langle y,y\rangle^{1-3\alpha}$ and $y=z_2\cdot\langle y,y\rangle^{\alpha}$, therefore $z_1\in\mathcal{M}$ and $z_2\in\mathcal{M}$, as $1-3\alpha<1/4<1/2$ and $\alpha<1/2$. For $n=1,2,\dots$ we have (similarly to the proof of Theorem 2.1.2)

$$\langle\psi'(\theta_{y,y}S)(x),y\rangle\left[\langle y,y\rangle+\frac{1}{n}\right]^{-1}=\langle\psi'(\theta_{z_1,z_2}\theta_{z_2,z_2}S)(x),y\rangle\left[\langle y,y\rangle+\frac{1}{n}\right]^{-1}$$
$$=\langle\psi'(\theta_{z_1,z_2})\psi'(\theta_{z_2,z_2}S)(x),y\rangle\left[\langle y,y\rangle+\frac{1}{n}\right]^{-1}=\langle\theta_{z_1,z_2}\psi'(\theta_{z_2,z_2}S)(x),y\rangle\left[\langle y,y\rangle+\frac{1}{n}\right]^{-1}$$
$$=\langle\psi'(\theta_{z_2,z_2}S)(x),z_2\rangle\,\langle z_1,y\rangle\left[\langle y,y\rangle+\frac{1}{n}\right]^{-1}=\langle\psi'(\theta_{z_2,z_2}S)(x),z_2\rangle\,\langle y,y\rangle^{3\alpha}\left[\langle y,y\rangle+\frac{1}{n}\right]^{-1}.$$

It is possible to deduce from this equality three corollaries. At first, for $y$ with $\|y\| \le 1$

$$\langle T_n(x,y), T_n(x,y)\rangle \le \|\psi'(\theta_{z_2,z_2}S)(x)\|^2 \left[\langle y,y\rangle + \frac{1}{n}\right]^{-1} \langle y,y\rangle^{3\alpha}\langle z_2,z_2\rangle\,\langle y,y\rangle^{3\alpha}\left[\langle y,y\rangle + \frac{1}{n}\right]^{-1}$$

$$\le \|\theta_{z_2,z_2}\|^2\|S\|^2\|x\|^2\langle y,y\rangle^{6\alpha+(1-2\alpha)}\left[\langle y,y\rangle + \frac{1}{n}\right]^{-2} \le \|y\|^{4(1-2\alpha)}\|S\|^2\|x\|^2\langle y,y\rangle^{1+4\alpha}\left[\langle y,y\rangle + \frac{1}{n}\right]^{-2}$$

$$\le \|y\|^{4(1-2\alpha)}\|S\|^2\|x\|^2\langle y,y\rangle^{2}\left[\langle y,y\rangle + \frac{1}{n}\right]^{-2} \le (\|S\|\,\|x\|)^2$$

and the item **3** is proved. Secondly, by fixing some $\alpha$, we have

$$\langle T_n(x,y) - T_m(x,y), T_n(x,y) - T_m(x,y)\rangle$$

$$\le \|y\|^{4(1-2\alpha)}\|S\|^2\|x\|^2\langle y,y\rangle^{1+4\alpha}\left[\left(\langle y,y\rangle + \frac{1}{n}\right)^{-1} - \left(\langle y,y\rangle + \frac{1}{m}\right)^{-1}\right]^2 \longrightarrow 0,$$

since $4\alpha > 1$. The item **1** is proved. Finally,

$$\langle T_n(x,y), T_n(x,y)\rangle \le \|y\|^{4(1-2\alpha)}\|S\|^2\|x\|^2\langle y,y\rangle^{1+4\alpha}\left[\langle y,y\rangle + \frac{1}{n}\right]^{-2}$$

$$\le \|y\|^{4(1-2\alpha)}\|S\|^2\|x\|^2\langle y,y\rangle^{4\alpha-1} \longrightarrow \|S\|^2\|x\|^2\langle y,y\rangle \qquad (\alpha \longrightarrow \frac{1}{2}),$$

and it gives **2** by [27] (see also [19, 2.1.4]), as linearity in $x$ is obvious.

To prove **4** it is sufficient to verify for elementary compact operators that

$$S(\theta_{x',y'},\theta_{x,y}) = \theta_{x',y\cdot(\psi(S)(x)(y'))}, \qquad S \in \mathbf{QC}(\mathcal{K}(\mathcal{M})), \quad x,\,y,\,x',\,y' \in \mathcal{M}.$$

Let

$$x = u\cdot\langle x,x\rangle^{1/2}, \quad y' = u'\cdot\langle y',y'\rangle^{1/2}, \quad v := u\cdot\langle x,x\rangle^{1/6} \in \mathcal{M}, \quad v' := u'\cdot\langle y',y'\rangle^{1/6} \in \mathcal{M}, \quad u,\,u' \in (\mathcal{M}^{\#})',$$

so that for $w := u\langle x,x\rangle^{1/3} \in \mathcal{M}$ and any $z \in \mathcal{M}$ we have

$$\theta_{v,v}\theta_{w,w}(z) = v\cdot\langle x,x\rangle^{\frac{1}{6}+\frac{1}{3}-\frac{1}{6}}\langle u\cdot\langle x,x\rangle^{1/2}, z\rangle = x\cdot\langle x,z\rangle = \theta_{x,x}(z), \tag{70}$$

while $\theta_{w,w}(x) = x\cdot\langle x,x\rangle^{-\frac{1}{6}-\frac{1}{6}+1} = x\cdot\langle x,x\rangle^{2/3}$. Therefore, if $T \in \mathbf{LM}(\mathcal{K}(\mathcal{M}))$ then for $w' := u\cdot\langle x,x\rangle^{1/12}$

$$\psi'(T)(x) = \lim_{n\to\infty}(T(\theta_{x,x}))(x)\left[\langle x,x\rangle + \frac{1}{n}\right]^{-1} = \lim_{n\to\infty}(T(\theta_{v,v}\theta_{w,w}))(x)\left[\langle x,x\rangle + \frac{1}{n}\right]^{-1}$$

$$= \lim_{n\to\infty}(T(\theta_{v,v}))x\cdot\langle x,x\rangle^{2/3}\left[\langle x,x\rangle + \frac{1}{n}\right]^{-1} = \lim_{n\to\infty}(T(\theta_{v,v}))w'\cdot\langle x,x\rangle^{\frac{5}{12}+\frac{2}{3}}\left[\langle x,x\rangle + \frac{1}{n}\right]^{-1}$$

$$= (T(\theta_{v,v}))w'\cdot\langle x,x\rangle^{\frac{13}{12}-1} = (T(\theta_{v,v}))v.$$

Similarly to (70), we obtain that

$$\theta_{y',y'} = \theta_{w'',w''}\,\theta_{v',v'}, \qquad w'' := v'\langle y',y'\rangle^{1/3} \in \mathcal{M}.$$

Then, after putting $w_* := v'\langle y',y'\rangle^{1/12} \in \mathcal{M}$, we have

$$\lim_{n\to\infty}\langle\psi'(\theta_{y',y'}S)(x), y'\rangle\,[\langle y',y'\rangle + 1/n]^{-1} = \lim_{n\to\infty}\langle\psi'(\theta_{w'',w''}\theta_{v',v'}S)(x), y'\rangle\,[\langle y',y'\rangle + 1/n]^{-1}$$

$$= \lim_{n\to\infty}\langle\theta_{w'',w''}\psi'(\theta_{v',v'}S)(x), y'\rangle\,[\langle y',y'\rangle + 1/n]^{-1} = \lim_{n\to\infty}\langle w''\cdot\langle w'',\psi'(\theta_{v',v'}S)(x)\rangle, y'\rangle\,[\langle y',y'\rangle + 1/n]^{-1}$$

$$= \lim_{n\to\infty} \langle \psi'(\theta_{v',v'}S)(x), w''\rangle \langle w'', y'\rangle [\langle y', y'\rangle + 1/n]^{-1}$$

$$= \lim_{n\to\infty} \langle \psi'(\theta_{v',v'}S)(x), w_*\rangle \langle y', y'\rangle^{\frac{1}{3}-\frac{1}{12}-\frac{1}{6}+1} [\langle y', y'\rangle + 1/n]^{-1}$$

$$= \lim_{n\to\infty} \langle \psi'(\theta_{v',v'}S)(x), w_*\rangle \langle y', y'\rangle^{\frac{13}{12}} [\langle y', y'\rangle + 1/n]^{-1}$$

$$= \langle \psi'(\theta_{v',v'}S)(x), w_*\rangle \langle y', y'\rangle^{\frac{1}{12}} = \langle \psi'(\theta_{v',v'}S)(x), v'\rangle,$$

whence

$$\theta_{x',y\cdot(\psi(S)(x)(y'))} = \lim_{n\to\infty} \theta_{x',y\cdot\langle\psi'(\theta_{y',y'}S)(x),(y')\rangle[\langle y',y'\rangle+1/n]^{-1}}$$

$$= \theta_{x',y\cdot\langle\psi'(\theta_{v',v'}S)(x),v'\rangle} = \theta_{x',y\cdot\langle(\theta_{v',v'}S\theta_{v,v})(v),v'\rangle} = \theta_{x',y\cdot\langle S(\theta_{v',v'},\theta_{v,v})(v),v'\rangle}.$$

In the last expression we used the presentation of quasi-multipliers in the form of quasi-centralizers. On the other hand, $y' = v'\cdot\langle v', v'\rangle$, $x = v\cdot\langle v, v\rangle$

$$S(\theta_{x',y'}, \theta_{x,y}) = S(\theta_{x',v'}\theta_{v',v'}, \theta_{v,v}\theta_{v,y}) = \theta_{x',v'}S(\theta_{v',v'}, \theta_{v,v})\theta_{v,y}$$

$$= \theta_{x',v'}\theta_{S(\theta_{v',v'},\theta_{v,v})v,y} = \theta_{x',y\cdot\langle S(\theta_{v',v'},\theta_{v,v})v,v'\rangle}.$$

The item **4** is proved, and finishes the proof of the theorem. □

## 2.3 Inner products on Hilbert $C^*$-modules

Let $\mathcal{M}$ be a module over $C^*$-algebra $A$, on which two sesquilinear maps $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ are defined in such a way that with respect to each of these maps the module $\mathcal{M}$ is a pre-Hilbert one.

**Definition 2.3.1** Two inner products $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ are called *equivalent*, if the norms definied by these inner products are equivalent.

Let us remark that if the inner products are equivalent then, if the module $\{\mathcal{M},,\langle\cdot,\cdot\rangle_1\}$ is Hilbert then the module $\{\mathcal{M},,\langle\cdot,\cdot\rangle_2\}$ is Hilbert too.

Let us consider at first the case of different inner products on self-dual Hilbert modules.

**Proposition 2.3.2** ([8]) *Let $\mathcal{M}$ be a self-dual Hilbert $A$-module over a $C^*$-algebra $A$ with the inner product $\langle\cdot,\cdot\rangle_1$. If $\langle\cdot,\cdot\rangle_2$ is another inner product equivalent to the given one then there exists a unique invertible positive operator $S \in \mathrm{End}^*_A(\mathcal{M})$ such that $\langle x, y\rangle_1 = \langle Sx, Sy\rangle_2$ for all $x, y \in \mathcal{M}$.*

**Proof:** Let us consider for $x \in \mathcal{M}$ a functional on the module $\mathcal{M}$ defined by the formula $y \longmapsto \langle x, y\rangle_2$. As the module $\mathcal{M}$ is self-dual, so there exists an element $Bx \in \mathcal{M}$ such that

$$\langle x, y\rangle_2 = \langle Bx, y\rangle_1$$

for all $y \in \mathcal{M}$. The map $x \longmapsto Bx$ is an $A$-homomorphism. Let us denote by $\|\cdot\|_i$ the norm defined by the inner product $\langle\cdot,\cdot\rangle_i$, $i = 1, 2$. By the assumption there exist constants $k, l > 0$ such that for all $x \in \mathcal{M}$

$$\|x\|_1 \le k\,\|x\|_2 \le l\,\|x\|_1 .$$

Then

$$\|Bx\|_1^2 = \|\langle Bx, Bx\rangle_1\| = \|\langle x, Bx\rangle_2\| \le \|x\|_2\,\|Bx\|_2 \le \frac{l^2}{k^2}\,\|x\|_1\,\|Bx\|_1 ,$$

therefore $\|Bx\|_1 \le \frac{l^2}{k^2}\|x\|_1$, i. e. the map $B$ is bounded. The equality $\langle Bx, x\rangle_1 = \langle x, x\rangle_2 \ge 0$ means that the operator $B$ is positive with respect to the initial inner product. The inequality

$$\|x\|_1^2 \le k^2\,\|x\|_2^2 = k^2\,\|\langle x, x\rangle_2\| = k^2\,\|\langle Bx, x\rangle_1\| \le k^2\,\|Bx\|_1\,\|x\|_1$$

shows that the estimate $\|Bx\|_1 \ge \frac{1}{k^2}\|x\|_1$ holds, from which we obtain by [24] (cf. the proof of Theorem 2.3.3 from [19]) the invertibility of the operator $B$. To complete the proof it remains to put $S = B^{-1/2}$. □

**Proposition 2.3.3** ([9]) *Let $\mathcal{M}$ be a Hilbert module with an inner product $\langle\cdot,\cdot\rangle_1$. Let $\langle\cdot,\cdot\rangle_2$ be another inner product equivalent to the initial one. Then the map $\mathcal{M}\longrightarrow\mathcal{M}'$ given by the formula $x\longmapsto\langle x,\cdot\rangle_2$, $x\in\mathcal{M}$, defines an invertible positive element in $\mathbf{QM}(\mathcal{K}(\mathcal{M}))\subset\mathcal{K}(\mathcal{M})^{**}$. Conversely, any invertible positive element in $\mathbf{QM}(\mathcal{K}(\mathcal{M}))\subset\mathcal{K}(\mathcal{M})^{**}$ (which can be identified with an element $T$ of the set $\mathrm{End}_A(\mathcal{M},\mathcal{M})$) defines an inner product $\langle x,y\rangle=T(x)(y)$, $x,y\in\mathcal{M}$.*

**Proof:** Let $\phi:\mathrm{End}_A(\mathcal{M},\mathcal{M}')\longrightarrow\mathbf{QM}(\mathcal{K}(\mathcal{M}))$ be the isometric isomorphism defined in Theorem 2.2.1, $\theta_{x,y}\phi(T)\theta_{z,t}=\theta_{x,t\cdot T(z)(y)}$, $x,y,z,t\in\mathcal{M}$. The map $x\longmapsto\langle x,\cdot\rangle_2$ is bounded, therefore it defines a map $T:\mathcal{M}\longrightarrow\mathcal{M}'$ and the element $\phi(T)\in\mathbf{QM}(\mathcal{K}(\mathcal{M}))$ is defined by the equality

$$\theta_{x,y}\phi(T)\theta_{z,t}=\theta_{x,t\cdot\langle z,y\rangle_2}$$

(let us remark that the elementary operators of the form $\theta_{x,y}$ are considered with respect to the initial inner product $\langle\cdot,\cdot\rangle_1$). Then for $s\in\mathcal{M}$

$$\langle\theta_{x,x\cdot\langle y,y\rangle_2}(s),s\rangle_1=\langle x\cdot\langle x\cdot\langle y,y\rangle_2,s\rangle_1,s\rangle_1=\langle x\cdot\langle y,y\rangle_2,s\rangle_1^*\langle x,s\rangle_1=\langle x,s\rangle_1^*\langle y,y\rangle_2\langle x,s\rangle_1\geq 0.$$

As linear combinations of elementary operators are dense in the algebra $\mathcal{K}(\mathcal{M})$, so we obtain that the operator $\phi(T)$ is positive. Let us show that it is invertible. Let us pass for this purpose to the Hilbert module $\mathcal{M}^{\#}$ over the enveloping $W^*$-algebra $A^{**}$. Both inner products can be extended to the module $\mathcal{M}^{\#}$ and to the self-dual module $(\mathcal{M}^{\#})'$. By Proposition 2.3.2 there exists an invertible operator $S\in\mathrm{End}^*_{A^{**}}((\mathcal{M}^{\#})')$ such that these extensions of inner product are related by $\langle x,y\rangle_1=\langle Sx,Sy\rangle_2$ for all $x,y\in(\mathcal{M}^{\#})'$. But the image of the operator $\phi(T)$ under the inclusion $\mathbf{QM}(\mathcal{K}(\mathcal{M}))\subset\mathrm{End}^*_{A^{**}}((\mathcal{M}^{\#})')$ obviously coincides with the product $S^*S$. Since the operator $S$ is invertible, the spectrum of the operator $\phi(T)$ is separated from zero, therefore $\phi(T)$ is invertible. In the opposite direction the statement can be proved similarly. □

**Corollary 2.3.4** ([9]) *Let $\mathcal{M}$ be a Hilbert $C^*$-module with an inner product $\langle\cdot,\cdot\rangle_1$. The following conditions are equivalent:*

(i) *any other inner product $\langle\cdot,\cdot\rangle_2$ equivalent to the initial one is defined by an invertible operator $S\in\mathrm{End}_A(\mathcal{M})$ and is given by the formula $\langle x,y\rangle_2=\langle Sx,Sy\rangle_1$, $x,y\in\mathcal{M}$;*

(ii) *each positive invertible quasi-multiplier $T\in\mathbf{QM}(\mathcal{K}(\mathcal{M}))$ can be decomposed into a product $T=S^*S$ for some invertible left multiplier $S\in\mathbf{LM}(\mathcal{K}(\mathcal{M}))$.* □

**Theorem 2.3.5** ([9], see also [2]) *Let $\mathcal{M}$ be a countably generated Hilbert $C^*$-module with an inner product $\langle\cdot,\cdot\rangle_1$. Then for any inner product $\langle\cdot,\cdot\rangle_2$ equivalent to the initial one there exists an invertible operator $S\in\mathrm{End}_A(\mathcal{M})$ such that $\langle x,y\rangle_2=\langle Sx,Sy\rangle_1$.*

**Proof:** Under the supposition the $C^*$-algebra $\mathcal{K}(\mathcal{M})$ is $\sigma$-unital, therefore it contains a strictly positive element $H\in\mathcal{K}(\mathcal{M})$. It is sufficient to show that each positive invertible quasi-multiplier admits a decomposition $T=S^*S$ with some left multiplier $S$. Let us put $K=(HTH)^{1/2}\in\mathcal{K}(\mathcal{M})$, $V_n=K\left(H^2+\frac{1}{n}\right)^{-1}H\in\mathcal{K}(\mathcal{M})$. Then $\|V_n\|\leq\|T\|^{1/2}$ and the sequence $(V_nH)$ converges to $K$ with respect to the norm. Then for any $K'\in H\cdot\mathcal{K}(\mathcal{M})$ the sequence $(V_nK')$ is norm-convergent to $KK'$. Since $H\cdot\mathcal{K}(\mathcal{M})$ is dense in $\mathcal{K}(\mathcal{M})$, we conclude that the sequence $(V_n)$ converges with respect to the left strict topology to some element $S\in\mathbf{LM}(\mathcal{K}(\mathcal{M}))$ and $SH=K$. Therefore $HS^*SH=K^*K=HTH$ and finally $S^*S=T$. □

As we can see from the following example of nontrivial inner product, the requirement for Hilbert modules to be countably generated is essential for Theorem 2.3.5.

**Example 2.3.6** ([2, 9]) Let $H$ be a *non-separable* Hilbert space. Let us consider the space

$$X=\{x\in\mathcal{B}(H):1/2\leq x\leq 1\}$$

equipped with the weak topology and the standard Hilbert $C(X)$-module $H_{C(X)}$. Let us show that there exist inner products on $H_{C(X)}$ equivalent to the standard one, but not admitting representations of the form $\langle S\cdot,S\cdot\rangle$ for any operator $S\in\mathrm{End}_{C(X)}(H_{C(X)})$. For this purpose it will be sufficient to find a quasi-multiplier $T \in \mathbf{QM}(\mathcal{K} \otimes C(X))$ not representable in the form $T = S^*S$, $S \in \mathbf{LM}(\mathcal{K} \otimes C(X))$. Let us use the identification of $\mathbf{LM}(\mathcal{K} \otimes C(X))$ (resp. $\mathbf{QM}(\mathcal{K} \otimes C(X))$) with the set of bounded maps from $X$ to $\mathcal{B}(H)$ continuous with respect to the strong (resp. weak) topology (it is discussed in detail in the next Section). Let us define a new inner product on the module $H_{C(X)}$ by the formula

$$\langle y, z \rangle_o (x) = \langle y, x(x) \rangle (x), \tag{71}$$

where $y, z \in H_{C(X)}$, $x \in X$. It is easy to see that $1/2\langle y, y\rangle \leq \langle y, y\rangle_o \leq \langle y, y\rangle$. This inner product defines a positive invertible quasi-multiplier $T$. Suppose that $T = S^*S$ for some $S \in \mathbf{LM}(\mathcal{K} \otimes C(X))$. Let us show that it is possible to choose a separable infinite-dimensional Hilbert space $H_T \subset H$ such that $T(x)H_T \subset H_T$ and $T^{-1}(x)H_T \subset H_T$ for all $x \in X$. Let $\{e_1, \ldots, e_k, \ldots\}$ be a basis of some separable subspace $H_0 \subset H$. By the compactness of $X$ the sets $T(X)e_k$ and $T^{-1}(X)e_k$ are compact subsets in $H$ for each number $k$, therefore they generate a separable Hilbert subspace $H_1 \subset H$ such that $T(X)H_0 \subset H_1$, $T^{-1}(X)H_0 \subset H_1$. Further on we find by induction separable subspaces $H_n \subset H$ such that $T(X)H_n \subset H_{n+1}$, $T^{-1}(X)H_n \subset H_{n+1}$. Finally let us put $H_T := (\cup_n H_n)^{-}$, i. e. the closure of the union of all $H_n$.

Let us denote by $X_0 \subset X$ the subset

$$X_0 = \left\{ \begin{pmatrix} 3/4 & r \\ r^* & 3/4 \end{pmatrix} : \ r : H_T \longrightarrow H_T^\perp, \ \|r\| \leq 1/4, \ r \text{ is linear} \right\}.$$

The restriction of the operator $S$ onto the subspace $X_0$ has the form

$$S|_{X_0} = \begin{pmatrix} s_1 & s_2 \\ 0 & s_3 \end{pmatrix}, \ s_1^* s_1 = 3/4, \ s_2^* s_2 + s_3^* s_3 = 3/4, \ s_1^* s_2 = r$$

with respect to the decomposition $H = H_T \oplus H_T^\perp$. Since the subspace $H_T$ is invariant under the action of $T^{-1}$, the operator $s_1 \in \mathcal{B}(H_T)$ is invertible, and the operator $\frac{2}{\sqrt{3}} s_1$ is unitary. Since the map $u \longmapsto u^*$ is continuous with respect to the strong topology on the group of unitary elements, we conclude that $s_1^*$ is continuous on $X_0$. Therefore the map $r = s_1^* s_2$ is also strong continuous as a map from $X_0$ to $\mathcal{B}(H_T^\perp, H_T)$. Thus, the assumption of possibility of decomposition $T = S^*S$ implies that arbitrary weak continuous bounded (by the number $1/4$) linear map $r : H_T \longrightarrow H_T^\perp$ turns to be strong continuous. But, as the strong and the weak topologies on the ball of radius $1/4$ in $\mathcal{B}(H_T^\perp, H_T)$ do not coincide, so the obtained contradiction shows that the inner product (71) is not related to the standard inner product on the module $H_{C(X)}$ by any invertible bounded operator $S \in \mathrm{End}_A(H_{C(X)})$.

If one considers different equivalent inner products on Hilbert $C^*$-module, the problem on whether an operator admits an adjoint, depends on the concrete inner product. By $\mathrm{End}_A^{*(i)}(\mathcal{M})$ (resp. $\mathcal{K}^{(i)}(\mathcal{M})$) we denote the $C^*$-algebra of operators admitting adjoint (resp. compact operators) with respect to the inner product $\langle \cdot, \cdot \rangle_i$, $i = 1, 2$. The adjoint operator for the operator $T$ with respect to this inner product we denote by $T^*_{(i)}$.

**Proposition 2.3.7** ([9]) *Let $\mathcal{M}$ be a Hilbert $A$-module over $C^*$-algebra $A$ with the inner product $\langle \cdot, \cdot \rangle_1$. Let $S \in \mathrm{End}_A(\mathcal{M})$ be an invertible operator defining the inner product $\langle \cdot, \cdot \rangle_2 = \langle S \cdot, S \cdot \rangle_1$. Then the operator $S$ admits an adjoint with respect to the first inner product if and only if it admits an adjoint with respect to the second one.*

*If $S$ admits an adjoint then the sets $\mathrm{End}_A^{*(1)}$ and $\mathrm{End}_A^{*(2)}$, $\mathcal{K}^{(1)}(\mathcal{M})$ and $\mathcal{K}^{(2)}(\mathcal{M})$ coincide.*

**Proof:** Let the operator $S$ admit an adjoint with respect to the inner product $\langle \cdot, \cdot \rangle_1$. Then for all $x, y \in \mathcal{M}$

$$\langle Sx, y \rangle_2 = \langle S^2 x, Sy \rangle_1 = \langle Sx, S(S^{-1} S_1^* S) y \rangle_1 = \langle x, (S^{-1} S^*_{(1)} S) y \rangle_2$$

and the operator $S^*_{(2)} = S^{-1} S^*_{(1)} S$ is an adjoint to $S$ with respect to the second inner product. The converce statement can be proved similarly.

Let us assume now that $S \in \mathrm{End}_A^* \mathcal{M})$. Let $B \in \mathrm{End}_A^{*(1)}(\mathcal{M})$. For all $x, y \in \mathcal{M}$ we have the following equality

$$\langle Bx, y \rangle_2 = \langle SBx, Sy \rangle_1 = \langle Sx, S(S^{-1}(S^{-1})^*_{(1)} B^*_{(1)} S^*_{(1)} S) y \rangle_1 = \langle x, (S^{-1}(S^{-1})^*_{(1)} B^*_{(1)} S^*_{(1)} S) y \rangle_2,$$

therefore $B^*_{(2)} = S^{-1}(S^{-1})^*_{(1)} B^*_{(1)} S^*_{(1)} S$, i. e. $B \in \mathrm{End}_A^{*(2)}(\mathcal{M})$. The statement about compact operators can be proved in the same way. □

However, if the inner product is defined with the help of an operator $S$ which does not admit an adjoint then operators admitting an adjoint with respect to one of the equivalent inner products need not admit an adjoint with respect to the other one. Thus a problem arises if a functional on $\mathcal{M}$ can be represented as an inner product by elements from $\mathcal{M}$. More precisely, we define the set $F \subset \mathcal{M}'$ by the equality

$$F = \bigcup_{\beta \in \mathbf{B};\, y \in \mathcal{M}} \langle y, \cdot \rangle_\beta,$$

where $\mathbf{B}$ is the set of all inner products $\langle \cdot, \cdot \rangle_\beta$ equivalent to the initial one. The functional $f \in \mathcal{M}'$ will be called *representable,* if $f \in F$. We study the set $F$ for the standard Hilbert module $H_A$. Let us denote the extension of the initial inner product from the module $H_A$ to $H_A^\# = H_{A^{**}}$ and to its adjoint module $H'_{A^{**}}$ still by $\langle \cdot, \cdot \rangle$. It is obvious that $H'_A \subset H'_{A^{**}}$.

**Proposition 2.3.8** *If $f \in H'_A$ is representable then in the module $H_A$ there exists such element $z$, for which the following operator inequality*

$$\alpha \langle z, z \rangle \le \beta \langle f, f \rangle \le \langle f, z \rangle \le \gamma \langle f, f \rangle \le \delta \langle z, z \rangle \tag{72}$$

*holds for some positive constants $\alpha$, $\beta$, $\gamma$, $\delta$.*

**Proof:** By the theorem 2.3.5 any inner product equivalent to the given one has the form $\langle x, y \rangle_\beta = \langle Sx, Sy \rangle$, where $S \in \mathrm{End}_A(H_A)$ is an invertible bounded operator. If $f$ is representable then $f = S^*Sz$ for some $S$ and some $z \in H_A$, the operator $\langle f, z \rangle \in A$ is positive, and we have

$$\langle f, z \rangle = \langle S^*Sz, z \rangle = \langle Sz, Sz \rangle = \langle z, z \rangle_\beta.$$

Since $S$ is invertible, there exist such positive numbers $a$ and $b$ that

$$a\langle z, z \rangle \le \langle z, z \rangle_\beta \le b \langle z, z \rangle,$$

therefore

$$a\langle z, z \rangle \le \langle f, z \rangle \le b \langle z, z \rangle. \tag{73}$$

Let us estimate now $\langle f, f \rangle = \langle S^*Sz, S^*Sz \rangle$. Since $a^2 \le (S^*S)^2 \le b^2$, we have

$$a^2 \langle z, z \rangle \le \langle f, f \rangle \le b^2 \langle z, z \rangle. \tag{74}$$

Combining (73) and (74), we obtain the estimate (72). □

Let us call a functional $f \in H'_A$ *non-singular,* if there exists in $H_A$ an element $z$ such that the spectrum of the element $\langle f, z \rangle \in A$ is separated from the origin (then it is possible to assume that the operator inequality $\langle f, z \rangle \ge c > 0$ holds for some number $c$). The following example shows that there exist singular functionals with the property $\langle f, f \rangle = 1$.

**Example 2.3.9** Let $A = L^\infty([0;1])$. Let us define $f \in H'_A$ as a sequence of functions $f = (f_k(t))$,

$$f_k(t) = \begin{cases} 1, & t \in \left[1/2^k; 1/2^{k-1}\right], \\ 0, & \text{for remaining } t. \end{cases}$$

The property $\langle f, f \rangle = 1$ is obvious. Let us show that the spectrum of the operator $\langle f, z \rangle$ is not separated from zero for all $z = (z_k) \in H_A$. As the series $\sum_{k=1}^\infty z_k^* z_k$ is norm convergent, so for any $\varepsilon > 0$ there exists a number $n$ such that $\| \sum_{k=n+1}^\infty z_k^* z_k \| < \varepsilon$. But then for $t < 1/2^k$ the estimate $|f_k(t) z_k(t)| < \varepsilon$ holds. Hence $f$ is singular. The condition (72) for it is false, therefore $f$ is not representable.

**Proposition 2.3.10** *Let $\mathcal{M}$ be a Hilbert $C^*$-module and let $f \in \mathcal{M}'$ be a non-singular functional. Then it is representable.*

**Proof:** Non-singularity and the Cauchy–Bunyakovskii inequality for the Hilbert modules give us the estimate $0 < c \le \langle f, z\rangle \le \|f\| \langle z, z\rangle^{1/2}$, from which it follows that the module $\operatorname{Span}_A z$ is isomorphic to $A$. Let us show that the decomposition into a (non-orthogonal) direct sum $\mathcal{M} = \operatorname{Span}_A z \widetilde{\oplus} \operatorname{Ker} f$ holds. If $x \in \mathcal{M}$, then put $a = \langle f, z\rangle^{-1} \cdot \langle f, x\rangle$; $y = x - za$. Then $x = za + y$ and $y \in \operatorname{Ker} f$. The uniqueness of this decomposition is obvious. Let us denote $\mathcal{M}_1 = \operatorname{Span}_A z$ and $\mathcal{M}_2 = \operatorname{Ker} f$ and let us choose with the help [34] (see, also [19, Corollary 2.8.15]) a new inner product in such a way that the submodules $\mathcal{M}_1$ and $\mathcal{M}_2$ become orthogonal. Then $z \perp \operatorname{Ker} f$. Put $z' = z \cdot \langle z, z\rangle_\beta^{-1} \cdot \langle f, z\rangle$. Then $\langle f, x\rangle = \langle z', x\rangle_\beta$, i.e. $f$ is representable. □

**Proposition 2.3.11** *Let a $C^*$-algebra $A$ be such that invertible elements are dense in it. Then the representable functionals are dense in $H'_A$ with respect to the initial norm.*

**Proof:** It is sufficient to verify that non-singular functionals are dense in $H'_A$. If $f = (f_i) \in H'_A$ then it is possible to find in $A$ an invertible element $g_1$ such that $\|g_1 - f_1\| < \varepsilon$. By putting $g = (g_1, f_2, f_3, \ldots) \in H'_A$ and by taking $z = e_1 = (1, 0, 0, \ldots) \in H'_A$, we obtain that $\langle g, z\rangle$ is invertible and $\|g - f\| < \varepsilon$. □

Situation with representability of functionals in the general case is more complicated. Let us consider the following

**Example 2.3.12** Let $A$ be a $C^*$-algebra of bounded operators in an infinite-dimensional Hilbert space. As it is shown in [9] (see also [19, Example 2.5.6]), there exists an isomorphism of Hilbert modules $S : A \longrightarrow H'_A$. Let $a \in A$, $f = S(a)$. Then the condition $\langle f, x\rangle = 0$ can be written as

$$\langle S(a), x\rangle = \langle S(a), S(S^{-1}x)\rangle = \langle a, S^{-1}x\rangle = a^* \cdot S^{-1}x = 0. \tag{75}$$

If $a \in A$ is invertible then it follows from (72) that $S^{-1}x = 0$, i. e. $\operatorname{Ker} f = 0$. But, if $f$ was representable, $\langle f, \cdot\rangle = \langle z, \cdot\rangle_\beta$ with $z \in H_A$, then the kernel of $f$ could not vanish. Therefore the functional $f = S(1_A)$ is not representable, moreover, it possesses an open neighbourhood consisting also only of non-representable functionals.

To finish this section we show how the averaging theorem [33] (see also [19, 2.8.12] can be generalized from compact groups to amenable ones in the case of Hilbert $W^*$-modules. We do it for group $\mathbf{Z}$, but the idea of the proof is suitable for arbitrary amenable groups.

**Theorem 2.3.13** [18] *Let $\mathcal{M}$ be a Hilbert module over a $W^*$-algebra $\mathcal{A}$, $T : \mathcal{M} \longrightarrow \mathcal{M}$ be an operator, all* integer *degrees of which are uniformly bounded, $\|T^n\| \le C$, $n \in \mathbf{Z}$. Then there exists an inner product $\langle \cdot, \cdot\rangle_\beta$ equivalent to the initial one and such that the operator $T$ is unitary with respect to it.*

**Proof:** For any normal linear functional $\phi \in \mathcal{A}_*$, where $\mathcal{A}_*$ is the pre-dual Banach space for $\mathcal{A}$, let us define a function $f_{x,y}$ on the group $\mathbf{Z}$ by the equality

$$f_{x,y}(n) = \phi(\langle T^n x, T^n y\rangle),$$

where $x, y \in \mathcal{M}$. By the assumption this function is bounded. Let us put

$$\phi_{x,y} = \lim_{n\to\infty} \frac{1}{2n+1} \sum_{k=-n}^{n} f_{x,y}(k).$$

By fixing $x$ and $y$, we obtain a linear bounded map

$$a_{x,y} : \mathcal{A}_* \longrightarrow \mathbf{C}; \qquad \phi \longmapsto \phi_{x,y}.$$

This map is an element of $(\mathcal{A}_*)^* = \mathcal{A}$. Let us define a new inner product on the module $\mathcal{M}$ by the equality $\langle x, y\rangle_\beta = a_{x,y} \in \mathcal{A}$. Let us verify that it is well-defined. Its sesquilinearity is obvious. If $\phi \in \mathcal{A}_*$ is a state then $f_{x,x}(n) \ge 0$, hence $\phi(\langle x, x\rangle_\beta) = \phi_{x,x} \ge 0$. Suppose that $\langle x, x\rangle_\beta = 0$ for some $x \in \mathcal{M}$. Then $\phi_{x,x} = 0$. But, as

$$\langle x, x\rangle = \langle T^{-k}(T^k x), T^{-k}(T^k x)\rangle \le C^2 \langle T^k x, T^k x\rangle,$$

so we have $\frac{1}{C^2} f_{x,x}(0) \leq f_{x,x}(n)$ and

$$\frac{1}{2n+1} \sum_{k=-n}^{n} f_{x,x}(k) \geq \frac{1}{C^2} f_{x,x}(0),$$

Hence $\phi_{x,x} \geq \frac{1}{C^2} f_{x,x}(0)$ and by the assumption $f_{x,x}(0) = 0$, i.e. $\phi(\langle x, x\rangle) = 0$ for an arbitrary state $\phi$. But then $\langle x, x\rangle = 0$, hence $x = 0$. Therefore $\langle \cdot, \cdot\rangle_\beta$ is an inner product. The property $\langle Tx, Ty\rangle_\beta = \langle x, y\rangle_\beta$ is obvious, therefore the operator $T$ is unitary. The equivalence of $\langle \cdot, \cdot\rangle$ and $\langle \cdot, \cdot\rangle_\beta$ follows immediately from the estimate

$$\frac{1}{C^2}\langle x, x\rangle \leq \langle T^k x, T^k x\rangle \leq C^2 \langle x, x\rangle,$$

which is valid for all $k$. □

# 3 Theorem of Dixmier and Douady for $l_2(A)$

## 3.1 Strict topology

**Definition 3.1.1** Let $A \hookrightarrow \mathcal{B}(H)$ be a non-degenerate faithful representation of a $C^*$-algebra $A$. By *strict topology* on $\mathcal{B}(H)$ we call the topology satisfying one of the following (obviously, equivalent) conditions

(i) it is the weakest topology, for which the maps

$$r_a : \mathcal{B}(H) \to \mathcal{B}(H), \quad r_a : x \mapsto xa, \qquad l_a : \mathcal{B}(H) \to \mathcal{B}(H), \quad l_a : x \mapsto ax, \qquad x \in \mathcal{B}(H),\ a \in A$$

are continuous,

(ii) it is the topology generated by the system of seminorms

$$\{\nu_a^R, \nu_a^L\}_{a\in A}, \qquad \nu_a^R(x) := \|xa\|, \qquad \nu_a^L(x) := \|ax\|.$$

Usually this topology is denoted by $\beta$ in view of the analogy with the Stone–Čech compactification (cf. 1.2.12). For example, by $[X]_\beta$ we denote the maximal ideals space of the closure of the algebra $C(X)$ in $\mathcal{B}(H)$ with respect to the strict topology, and the corresponding limit we denote by $\beta$–lim.

**Proposition 3.1.2** *The set* $\mathbf{M}(A)$ *is strictly closed,*

$$[A]_\beta \subset [\mathbf{M}(A)]_\beta = \mathbf{M}(A).$$

**Proof:** Let the net $\{x_\alpha\}_{\alpha\in\mathcal{A}} \subset \mathbf{M}(A)$ be strictly convergent to $x \in \mathcal{B}(H)$. Then for any $a \in A$ there exist the norm-limits

$$L(a) := \lim_\alpha x_\alpha a, \qquad R(a) := \lim_\alpha a x_\alpha$$

defining maps the $L,\ R : A \to A$. As

$$aL(b) = a\ \lim_\alpha (x_\alpha b) = \lim_\alpha (a x_\alpha b) = \left(\lim_\alpha (a x_\alpha)\right) b = R(a)b,$$

so the pair $(L, R)$ is an element of $\mathbf{DC}(A)$, i. e. a double centralizer. Identifying double centralizers with multipliers, by Theorem 1.2.11 we obtain $y \in \mathbf{M}(A)$. Then $x_\alpha \xrightarrow{\beta} y$. Indeed,

$$ya - x_\alpha a = L(a) - x_\alpha a \longrightarrow 0, \qquad ay - ax_\alpha = R(a) - ax_\alpha \longrightarrow 0$$

with respect to the norm. So, $\mathbf{M}(A)$ is $\beta$-closed. □

**Lemma 3.1.3** *The net* $\{e_\alpha\}_{\alpha\in\mathcal{A}}$ *is an approximate unit for* $A$ *if and only if* $\mathbf{M}(A) \ni 1 = \beta\text{–}\lim_\alpha e_\alpha$.

**Proof:** The result immediately follows from the definition. □

**Proposition 3.1.4** (i) *The conjugation in* $\mathbf{M}(A)$ *is* $\beta$*-continuous.*

(ii) *Multiplication in* $\mathbf{M}(A)$ *by a fixed element is* $\beta$*-continuous.*

(iii) *Multiplication in* $\mathbf{M}(A)$ *is* $\beta$*-continuous on bounded sets.*

**Proof:** Let us consider an arbitrary element $x \in \mathbf{M}(A)$ and let $x_\alpha \xrightarrow{\beta} x$. Then

$$x_\alpha a \xrightarrow{\|\cdot\|} xa, \qquad a x_\alpha \xrightarrow{\|\cdot\|} ax \quad \text{for any } a \in A,$$

whence, after conjugation,

$$b x_\alpha^* \xrightarrow{\|\cdot\|} b x^*, \qquad x_\alpha^* b \xrightarrow{\|\cdot\|} x^* b \quad \text{for any } b \in A, \qquad (b = a^*),$$

i. e. $x_\alpha^* \xrightarrow{\beta} x^*$.

Let now $y \in \mathbf{M}(A)$ be a fixed element and $x_\alpha \xrightarrow{\beta} x$. Then

$$\|(x_\alpha y)a - (xy)a\| = \|x_\alpha(ya) - x(ya)\| \longrightarrow 0, \qquad \|a(x_\alpha y) - a(xy)\| \le \|a x_\alpha - ax\| \cdot \|y\| \longrightarrow 0$$

for any $a \in A$. It means that $x_\alpha y \xrightarrow{\beta} xy$.

Let now $x_\alpha \xrightarrow{\beta} x$, $\|x_\alpha\| < c_x$ for any $\alpha \in \mathcal{A}$, and $y_\gamma \xrightarrow{\beta} y$, $\|y_\gamma\| < c_y$ for any $\gamma \in \Gamma$. Then for any $a \in A$ and any $\varepsilon > 0$ there exists a pair $(\alpha_0, \gamma_0)$ such that for any pair $(\alpha, \gamma) > (\alpha_0, \gamma_0) \in \mathcal{A} \times \Gamma$ (i. e. for $\alpha > \alpha_0$ and $\gamma > \gamma_0$) one has

$$\|(x_\alpha y_\gamma)a - (xy)a\| \le \|x_\alpha y_\gamma a - x_\alpha y a\| + \|x_\alpha y a - xya\| \le c_x \cdot \|y_\gamma a - ya\| + \|x_\alpha(ya) - x(ya)\| < \varepsilon,$$

$$\|a(x_\alpha y_\gamma) - a(xy)\| \le \|a x_\alpha y_\gamma - a x y_\gamma\| + \|a x y_\gamma - axy\| \le \|a x_\alpha - ax\| \cdot c_y + \|(ax)y_\gamma - (ax)y\| < \varepsilon.$$

It means that $(xy) = \beta\text{--}\lim_{(\alpha,\gamma)\in\mathcal{A}\times\Gamma} (x_\alpha y_\gamma)$. □

**Theorem 3.1.5** *The algebra of multipliers* $\mathbf{M}(A)$ *coincides with the* $\beta$*-closure of* $A$ *in* $\mathcal{B}(H)$*,*

$$\mathbf{M}(A) = [A]_\beta.$$

**Proof:** By Proposition 3.1.2 it is sufficient to prove that $\mathbf{M}(A) \subset [A]_\beta$. Let $\{e_\alpha\}_{\alpha\in\mathcal{A}}$ be an approximate unit for $A$. By Lemma 3.1.3 $e_\alpha \xrightarrow{\beta} 1 \in \mathbf{M}(A)$, as it is bounded. Since $e_\alpha \xrightarrow{\beta} 1$, we have by item (ii) of the previous lemma that for each $x \in \mathbf{M}(A)$ the net $A \ni x e_\alpha \xrightarrow{\beta} x \cdot 1 = x$. □

**Definition 3.1.6** Let $A \hookrightarrow \mathcal{B}(H)$ be a non-degenerate faithful representation of a $C^*$-algebra $A$. We call by *left strict* topology on $\mathcal{B}(H)$ the topology satisfying one of the following equivalent conditions:

(i) it is the weakest topology, for which the maps

$$r_a : \mathcal{B}(H) \to \mathcal{B}(H), \quad r_a : x \mapsto xa, \qquad x \in \mathcal{B}(H),\ a \in A$$

are continuous,

(ii) it is the topology generated by the system of seminorms

$$\{\nu_a^R\}_{a\in A}, \qquad \nu_a^R(x) := \|xa\|.$$

For the left strict topology it is possible to prove the following analog of the Theorem 3.1.5.

**Theorem 3.1.7** *The algebra of left multipliers* $\mathbf{LM}(A)$ *coincides with the closure of* $A$ *in* $A^{!!}$ *with respect to the left strict topology.*

**Proof:** This statement can be obtained by the same way as in 3.1.5, it is sufficient to take only a "half" of the argument. □

For Hilbert modules it is natural to consider the following two topologies on the space of bounded homomorphisms.

**Definition 3.1.8** Let $\mathcal{M}$ be a Hilbert $A$-module. The *strong module* topology on $\mathrm{End}(\mathcal{M})$ is the topology generated by the system of seminorms

$$\{s^h\}_{h\in\mathcal{M}}, \qquad s^h(x) := \|x(h)\|, \quad x \in \mathrm{End}(\mathcal{M}),$$

and the $*$-*strong module* topology on $\mathrm{End}^*(\mathcal{M})$ is the topology generated by the system of seminorms

$$\{s^h; s^h_*\}_{h\in\mathcal{M}}, \qquad s^h(x) := \|x(h)\|, \quad s^h_*(x) := \|x^*(h)\| \quad x \in \mathrm{End}^*(\mathcal{M}).$$

**Proposition 3.1.9** *The strong topology is not weaker than the $*$-strong module topology on $\mathcal{K}(\mathcal{M})$ (hence by Theorems* 3.1.5 *and* 2.1.1 *everywhere on* $\mathrm{End}^*(\mathcal{M})$ *).*

*The left strong topology is not weaker than the strong module topology on $\mathcal{K}(\mathcal{M})$ ( hence by Theorems* 3.1.7 *and* 2.1.2 *everywhere on* $\mathrm{End}(\mathcal{M})$ *).*

*The corresponding topologies coincide on bounded sets in $\mathcal{K}(\mathcal{M})$.*

**Proof:** We shall check equivalence of appropriate seminorms. We have

$$s^h(x) = \|x(h)\| = \|xk(g)\| \le \|xk\| \cdot \|g\| = \|g\| \cdot \nu^R_k(x)$$

for some $k \in \mathcal{K}(\mathcal{M})$, $g \in \mathcal{M}$ (see [19, Lemma 2.2.3]) and

$$s^h_*(x) = \|x^*(h)\| = \|x^*k(g)\| \le \|x^*k\| \cdot \|g\| = \|k^*x\| \cdot \|g\| = \|g\| \cdot \nu^L_{k^*}(x)$$

for some $k \in \mathcal{K}(\mathcal{M})$, $g \in \mathcal{M}$. Conversely, let $k \in \mathcal{K}(\mathcal{M})$ be an arbitrary element and $x_\alpha \to 0$ with respect to the strong module topology, being bounded: $\|x_\alpha\| < c$. Then for any $\varepsilon > 0$ there exist vectors $h_1, \ldots, h_n$ and $g_1, \ldots, g_n$ from $\mathcal{M}$ such that

$$\left\| k - \sum_{i=1}^n \theta_{h_i,g_i} \right\| < \frac{\varepsilon}{c},$$

and $\alpha_0$ big enough so that for $\alpha > \alpha_0$

$$\|x_\alpha(h_i)\| < \frac{\varepsilon}{n \cdot \|g_i\|}, \qquad i = 1, \ldots, n.$$

Then for these $\alpha$

$$\nu^R_k(x_\alpha) = \|x_\alpha k\| \le \frac{\varepsilon}{c} \cdot \|x_\alpha\| + \sum_{i=1}^n \|x_\alpha \theta_{h_i,g_i}\| \le c \cdot \frac{\varepsilon}{c} + \sum_{i=1}^n \|x_\alpha h_i\| \cdot \|g_i\| \le \varepsilon + n \cdot \frac{\varepsilon}{n \cdot \|g_i\|} \cdot \|g_i\| < 2\varepsilon.$$

Similarly, if $x_\alpha \to 0$ in the $*$-strong module topology then we can additionally require that for $\alpha > \alpha_0$

$$\|x^*_\alpha(g_i)\| < \frac{\varepsilon}{n \cdot \|h_i\|}, \qquad i = 1, \ldots, n.$$

Then

$$\nu^L_k(x_\alpha) = \|kx_\alpha\| \le \frac{\varepsilon}{c} \cdot \|x_\alpha\| + \sum_{i=1}^n \|\theta_{h_i,g_i} x_\alpha\| \le c \cdot \frac{\varepsilon}{c} + \sum_{i=1}^n \|h_i\| \cdot \|x^*_\alpha g_i\| \le \varepsilon + \sum_{i=1}^n \frac{\varepsilon}{n \cdot \|g_i\|} \|g_i\| = 2\varepsilon. \quad \square$$

**Proposition 3.1.10** *Let $A$ and $B$ be $C^*$-algebras and $\psi : \mathbf{M}(A) \to \mathbf{M}(B)$ be a morphism such that $B \subset \psi(A)$. Then $\psi$ is strictly continuous. In particular, the extension $\varphi''$ from Proposition* 1.2.21 *is the extension by continuity, thereby, is unique.*

**Proof:** Let $x_\alpha \xrightarrow{\beta} x$ in $\mathbf{M}(A)$ and $b$ be an arbitrary element of $B$. Then $b = \psi(a)$ for some $a \in A$. The nets $x_\alpha a$ and $a x_\alpha$ converge with respect to the norm in $A$. The map $\psi$, being a morphism of $C^*$-algebras, does not increase the norm, therefore $\psi(\chi_\alpha a) = \psi(x_\alpha) b$ and $\psi(a x_\alpha) = b\psi(x_\alpha)$ are the Cauchy nets, so they converge with respect to the norm. Since $b$ was arbitrary, it means that $\psi(x_\alpha) \xrightarrow{\beta} y \in \mathbf{M}(B)$. Thus, for any $b' \in B$, $b' = \psi(a')$

$$yb' = \beta\text{–}\lim_\alpha \psi(x_\alpha)b' = \lim_\alpha \psi(x_\alpha)b' = \lim_\alpha \psi(x_\alpha a') = \psi(xa') = \psi(x)b'$$

holds. Thus, $y = \psi(x)$. □

The similar theory can be developed for quasi-multipliers, if one gives the following definition.

**Definition 3.1.11** Let $A \hookrightarrow \mathcal{B}(H)$ be a non-degenerate faithful representation of a $C^*$-algebra $A$. *Quasi-strict topology* on $\mathcal{B}(H)$ is the topology satisfying one of the following equivalent conditions:

(i) it is the weakest topology for which the maps

$$Q_{ab} : \mathcal{B}(H) \to \mathcal{B}(H), \quad Q_{ab} : x \mapsto axb, \qquad x \in \mathcal{B}(H),\ a,\ b \in A$$

are continuous.

(ii) it is the topology generated by the system of seminorms

$$\{\nu_{ab}\}_{a,\, b\in A}, \qquad \nu_{ab}(x) := \|axb\|.$$

## 3.2 Proof of the main theorem

Let us realise $l_2(A)$ as the completion of the algebraic tensor product $H \otimes A = L^2([0,1])\o A$ completed with respect to the $A$-inner product $\langle f\otimes\gamma, g\otimes\beta\rangle = \langle f, g\rangle\,\gamma^*\beta$. We suppose here that the inner product on $L^2([0,1])$ is linear in the second entry.

**Lemma 3.2.1** [4, p. 250]. *There exists for each $t \in [0,1]$ a closed linear subspace $H_t \subset H$ and for each $t \in (0,1]$ a linear isometry $U_t : H_t \to H$ such that*

(i) *the orthogonal projection $P_t$ onto $H_t$ is strong continuous in $t \in [0,1]$,*

(ii) *the operators $U_t P_t$ and $U_t^{-1}$ are strong continuous in $t \in (0,1]$,*

(iii) $H_1 = H, \quad H_0 = 0, \quad U_1 = 1.$ □

Let us remind that in [4] the subspaces are defined in the following way:

$$H_t := \left\{ f \in L^2([0,1]) \,|\, f(x) = 0 \quad \text{for} \quad x \geq t \right\}.$$

**Lemma 3.2.2** *If $F_t \to F$, $\;t \to 0$ with respect to the strong topology in $B(H)$, being bounded, then $F_t \otimes \mathrm{Id}_A \to F \otimes \mathrm{Id}_A$ with respect to the left strict topology.*

**Proof:** It is sufficient to prove that

$$\|(F_t \otimes \mathrm{Id}_A - F \otimes \mathrm{Id}_A)\theta_{x,y}\| \to 0 \qquad (t \to 0),$$

where

$$\theta_{x,y}(z) = x\langle y, z\rangle, \quad x = \sum_{i=1}^N h_i x_i \otimes \beta_i, \quad x_i \in \mathbf{C},\ \beta_i \in A, \quad \|x\| = \|y\| = 1,$$

and $\{h_i\}$ is an orthogonal basis of $H$. Then for $z = \sum_i h_i z_i \otimes \mu_i$

$$\|(F_t \otimes \mathrm{Id}_A - F \otimes \mathrm{Id}_A)\,\theta_{x,y}(z)\| = \|\sum_{i=1}^N (F_t - F)h_i x_i \otimes \beta_i \langle y, z\rangle\|$$

is less then $\varepsilon$ if $t$ is so close to 0 that

$$\|(F_t - F)h_i x_i\| \cdot \|\beta_i\| < \frac{1}{N}\,\varepsilon. \qquad \square$$

**Lemma 3.2.3** *Let a set $G(t)$ be uniformly bounded (by a constant $C$), $G(t) \to G$ and $S(t) \to S$ $(t \to 0)$ in the left strict topology. Then $G(t)S(t) \to GS$ $(t \to 0)$ in the left strict topology.*

**Proof:** Let $k \in \mathcal{K}_A$ be an arbitrary operator. Then $Sk \in \mathcal{K}_A$ and

$$\|S(t)k - Sk\| \to 0, \qquad \|(G(t) - G)(Sk)\| \to 0 \qquad (t \to 0).$$

Hence

$$\begin{aligned} \|G(t)S(t)k - GSk\| &\leq \|(G(t) - G)Sk + G(t)(S(t) - S)k\| \\ &\leq \|(G(t) - G)Sk\| + C\|(S(t) - S)k\| \to 0 \quad (t \to 0). \end{aligned}$$

□

**Theorem 3.2.4** *The unitary group $\mathcal{U}$ of operators in $l_2(A)$ is contractible with respect to the left strict topology.*

**Proof:** For any $\mathbf{U} \in \mathcal{U}$ and $t \in (0, 1]$ we define

$$\Phi(\mathbf{U}, t) := (\mathrm{Id}_{\,l_2(A)} - P_t \otimes \mathrm{Id}_{\,A}) + (U_t^{-1} \otimes \mathrm{Id}_{\,A})\, \mathbf{U}\, (U_t \otimes \mathrm{Id}_{\,A})\, (P_t \otimes \mathrm{Id}_{\,A})$$

and

$$\Phi(\mathbf{U}, 0) := \mathbf{U}.$$

The operator $\Phi(\mathbf{U}, t)$, $t \in (0, 1]$ defines an identity mapping $H_t^\perp \otimes A$ and coincides with the restriction of the unitary map $(U_t^{-1} \otimes \mathrm{Id}_{\,A})\, \mathbf{U}\, (U_t \otimes \mathrm{Id}_{\,A})$ on $H_t \otimes A$. Therefore

$$\Phi(\mathbf{U}, t) \in \mathcal{U}, \qquad \Phi(\mathbf{U}, 1) = \mathbf{U}.$$

Thus, as $U_t^* = U_t^{-1}$, so all operators admit an adjoint.

From Lemma 3.2.3 it is clear that $\Phi$ is continuous in $t \in (0, 1]$, and, similarly, in $(\mathbf{U}, t)$. Indeed, let $(\mathbf{U}', t') \in \mathcal{U} \times (0, 1]$ tend to $(\mathbf{U}, t) \in \mathcal{U} \times (0, 1]$. Then for any $k \in \mathcal{K}_A$

$$\begin{aligned} &\|\Phi(\mathbf{U}, t)k - \Phi(\mathbf{U}', t')k\| = \|(\mathrm{Id}_{\,l_2(A)} - P_t \otimes \mathrm{Id}_{\,A})k + (U_t^{-1} \otimes \mathrm{Id}_{\,A})\, \mathbf{U}\, (U_t \otimes \mathrm{Id}_{\,A})\, (P_t \otimes \mathrm{Id}_{\,A})k \\ &\quad -(\mathrm{Id}_{\,l_2(A)} - P_{t'} \otimes \mathrm{Id}_{\,A})k - (U_{t'}^{-1} \otimes \mathrm{Id}_{\,A})\, \mathbf{U}'\, (U_{t'} \otimes \mathrm{Id}_{\,A})\, (P_{t'} \otimes \mathrm{Id}_{\,A})k\| \\ &\leq \|(P_t \otimes \mathrm{Id}_{\,A} - P_{t'} \otimes \mathrm{Id}_{\,A})k\| + \|[(U_t^{-1} \otimes \mathrm{Id}_{\,A}) - (U_{t'}^{-1} \otimes \mathrm{Id}_{\,A})]\, \mathbf{U}\, (U_t \otimes \mathrm{Id}_{\,A})\, (P_t \otimes \mathrm{Id}_{\,A})k\| \\ &\quad +\|(U_{t'}^{-1} \otimes \mathrm{Id}_{\,A})\, [\mathbf{U} - \mathbf{U}']\, (U_t \otimes \mathrm{Id}_{\,A})\, (P_t \otimes \mathrm{Id}_{\,A})k\| \\ &\quad +\|(U_{t'}^{-1} \otimes \mathrm{Id}_{\,A})\, \mathbf{U}'\, [(U_t \otimes \mathrm{Id}_{\,A}) - (U_{t'} \otimes \mathrm{Id}_{\,A})]\, (P_t \otimes \mathrm{Id}_{\,A})k\| \\ &\quad +\|(U_{t'}^{-1} \otimes \mathrm{Id}_{\,A})\, \mathbf{U}'\, (U_{t'} \otimes \mathrm{Id}_{\,A})\, [(P_t \otimes \mathrm{Id}_{\,A}) - (P_{t'} \otimes \mathrm{Id}_{\,A})]k\| \\ &\leq \|(P_t \otimes \mathrm{Id}_{\,A} - P_{t'} \otimes \mathrm{Id}_{\,A})k\| + \|[(U_t^{-1} \otimes \mathrm{Id}_{\,A}) - (U_{t'}^{-1} \otimes \mathrm{Id}_{\,A})]\, k_1\| + \|[\mathbf{U} - \mathbf{U}']\, k_2\| \\ &\quad +\|[(U_t \otimes \mathrm{Id}_{\,A}) - (U_{t'} \otimes \mathrm{Id}_{\,A})]\, k_3\| + \|[(P_t \otimes \mathrm{Id}_{\,A}) - (P_{t'} \otimes \mathrm{Id}_{\,A})]k\| \to 0 \end{aligned}$$

by Lemma 3.2.2. Here $k_1$, $k_2$ and $k_3$ are fixed operators from $\mathcal{K}_A$. Let now $(\mathbf{U}', t') \in \mathcal{U} \times (0, 1]$ tend to $(\mathbf{U}, 0) \in \mathcal{U} \times [0, 1]$. Then $P_{t'} \to 0$ with respect to the strong topology, $P_{t'} \otimes \mathrm{Id}_{\,A} \to 0$ with respect to the left strict topology by Lemma 3.2.2. Therefore for any $k \in \mathcal{K}_A$

$$\|(U_{t'}^{-1} \otimes \mathrm{Id}_{\,A})\, \mathbf{U}\, (U_{t'} \otimes \mathrm{Id}_{\,A})\, (P_{t'} \otimes \mathrm{Id}_{\,A})\, k\| \leq \|(P_{t'} \otimes \mathrm{Id}_{\,A})\, k\| \to 0, \qquad \|\Phi(\mathbf{U}, t)k\| \to 0. \qquad \square$$

### 3.3 Some generalizations

Let us remark that in the proof of Theorem 3.2.4 we used only the boundedness of the set of invertible operators $\{\mathbf{U}\}$, but not the unitarity. Thus, actually we have proved the following statement.

**Theorem 3.3.1** *Every bounded set of invertible operators in Hilbert space $H$ is contractible in invertibles with respect to the strong topology.*

*Every bounded set of invertible operators from* GL *(resp.* GL$^*$*) is contractible in* GL *(resp.* GL$^*$*) with respect to the left strict topology.* □

**Lemma 3.3.2** *Let $S$ be a compact set and*

$$f : S \to B(H), \qquad s \mapsto F_s$$

*be continuous with respect to the strong topology. Then $\{\|F_s\|\}$ is bounded.*

**Proof:** As $S$ is compact, so $\{\|F_s x\|\}$ is bounded for any $x \in H$ by some $C(x)$. Therefore, by the uniform boundedness principle [5, II.3.21] there exists a constant $C$ such that

$$\|F_s x\| \le C, \qquad \forall \quad s \in S, \quad x \in B_1(H).$$

Therefore $\|F_s\| \le C$. □

**Lemma 3.3.3** *Let $S$ be a compact set and*

$$f : S \to \mathrm{End}_A l_2(A) = \mathbf{LM}(\mathcal{K}_A), \qquad s \mapsto F_s$$

*be continuous with respect to the left strict topology. Then $\{\|F_s\|\}$ is bounded.*

**Proof:** Let $x \in l_2(A)$ be an arbitrary element. Let us choose $k \in \mathcal{K}$ and $z$ so that $x = kz$ (see [19, Lemma 2.2.3]). Then $s \mapsto F_s x$ is continuous: we apply the definition of the left strict topology to the inequality

$$\|F_s x - F_t x\| = \|F_s kz - F_t kz\| \le \|F_s k - F_t k\| \, \|z\|$$

The proof is finished similarly to 3.3.2. □

Now from Theorem 3.3.1 by Lemma 3.3.2 and Lemma 3.3.3 we obtain the following statement.

**Theorem 3.3.4** *The group $G(H)$ of invertible operators in a Hilbert space $H$ is weakly contractible (i. e. the homotopy groups $\pi_i(G(H)) = 0$) with respect to the strong topology.*

*The group* GL *(resp.* GL$^*$*) is weakly contractible with respect to the left strict topology.* □

**Remark 3.3.5** We suppose that the results of this section in the part, concerning Hilbert spaces, were known earlier, but we have not found them published anywhere.

## 4 Multipliers and Hilbert modules. The commutative case

### 4.1 Description of modules

The following results describing the modules $l_2(C_0(X, A))$ and spaces of operators on them in terms of spaces of maps are obtained by combination and small modification of [7, 1].

**Definition 4.1.1** Let us denote by $C_0(X, \mathcal{M})$ the space of continuous maps $X \to \mathcal{M}$ tending to zero at infinity and by $A_0(X)$ the space of continuous maps $X \to A$, tending to zero at infinity.

Notice that $A_0(X)$ is a $C^*$-algebra with respect to the sup-norm and $C_0(X, \mathcal{M})$ is a Hilbert $A_0(X)$-module with the inner product given by $\langle f, g \rangle = \langle f(x), g(x) \rangle_{\mathcal{M}}$, where $f, g \in C_0(X, \mathcal{M})$, $x \in X$.

**Definition 4.1.2** Let us call a pair $(X, \mathcal{M})$, where $X$ is a locally compact Hausdorff topological space, and $\mathcal{M}$ is a Hilbert $A$-module, *compatible,* if the following conditions are satisfied:

(i) the map

$$j : \mathcal{M} \otimes_{\mathbf{C}} C_0(X) \to C_0(X, \mathcal{M}), \qquad j(m \otimes f)(x) := f(x)m, \qquad m \in \mathcal{M}, \quad f \in C_0(X)$$

is an isometric $A_0(X)$-module isomorphism,

(ii) let $\varphi \in C_0(X, \mathcal{M})$ be such that $\varphi(x_0) = 0$ for some $x_0 \in X$, and $F \in \mathrm{End}_{A_0(X)}(C_0(X, \mathcal{M}))$ be an arbitrary operator; then $(F\varphi)(x_0) = 0$.

**Remark 4.1.3** Here and further by a tensor product we always mean the projective tensor product (in the case of $C^*$-algebras called also spatial or minimal). However in (i) we use the tensor product with a commutative algebra which is nuclear, so all $C^*$-norms on this tensor product coincide. For more details see [25, §6.3].

**Remark 4.1.4** Generalizing [7], we shall prove in the following two lemmas that the pair $(X, l_2(A))$ is compatible (in [7] the case of compact $X$ is considered). Let us remark that a more weak compatibiluty of an arbitrary pair (namely, if in (ii) we replace End by End$^*$) follows from the results [1]. Thus, the results [7], which we will prove in Theorem 4.2.3 in the part, concerning the operators admitting an adjoint, can be deduced from the results of [1] using the identification of multipliers of $\mathcal{K}(\mathcal{M})$ with $\mathrm{End}^*_A(\mathcal{M})$.

**Lemma 4.1.5** *The map*

$$j : l_2(A_0(X)) \to C_0(X, l_2(A)), \qquad j(f)(x) := (f_1(x), f_2(x) \ldots), \quad f = (f_1, f_2, \ldots) \in l_2(A_0(X))$$

*is an isometry.*

**Proof:** It is obvious that $j$ is an isometric inclusion. Let us show that it is an epimorphism. Let $\varphi \in C_0(X, l_2(A))$ be an arbitrary element. Then, as

$$\|(\varphi(x))_i\| \le \|\varphi(x)\|, \qquad \|(\varphi(x))_i - (\varphi(y))_i\| \le \|\varphi(x) - \varphi(y)\|,$$

so we have $(\varphi(x))_i \in A_0(X)$. It is necessary to verify the convergence of the series $\sum_i (\varphi(x))_i^* (\varphi(x))_i$ with respect to the norm. Let $\varepsilon > 0$ be an arbitrary number and let $K \subset X$ be a compact set such that for any $y \in Y := X \setminus K$ the inequality $\|\varphi(y)\| < \varepsilon$ holds. For each point $x \in K$ we choose a number $n(x)$ such that

$$\sum_{i=n(x)}^{\infty} (\varphi(x))_i^* (\varphi(x))_i < \frac{\varepsilon}{2}.$$

Since the map

$$X \xrightarrow{\varphi} l_2(A) \xrightarrow{1-p_n} L_n^{\perp}$$

is continuous, we can find for each $x \in K$ an open neighbourhood $U_x$ in $K$ such that for each $z \in U_x$

$$\sum_{i=n(x)}^{\infty} (\varphi(z))_i^* (\varphi(z))_i < \varepsilon.$$

Due to compactness of $K$ we can choose a finite subcovering $U_{x_1}, \ldots, U_{x_s}$ and put $n := \max\{n_{x_1}, \ldots, n_{x_s}\}$. Then for any $m > n$

$$\sup_{x \in X} \left[ \sum_{i=n}^{m} (\varphi(x))_i^* (\varphi(x))_i \right] \le \max \left\{ \sup_{x \in Y} \left[ \sum_{i=n}^{m} (\varphi(x))_i^* (\varphi(x))_i \right] , \sup_{x \in K} \left[ \sum_{i=n}^{m} (\varphi(x))_i^* (\varphi(x))_i \right] \right\}$$

$$\le \max \left\{ \sup_{x \in Y} \|\varphi(x)\| , \max_{j=1}^{s} \sup_{x \in U_j} \left[ \sum_{i=n(x_j)}^{m} (\varphi(x))_i^* (\varphi(x))_i \right] \right\} \le \max \left\{ \varepsilon , \max_{j=1}^{s} \varepsilon \right\} = \varepsilon.$$

By the Cauchy criterion the series is convergent. □

## 4.2 Description of operators

**Lemma 4.2.1** *Let $\varphi \in C_0(X, l_2(A))$ be such that $\varphi(x_0) = 0$ for some $x_0 \in X$, and $F \in \mathrm{End}_{A_0(X)}(C_0(X, l_2(A)))$ be an arbitrary operator. Then $(F\varphi)(x_0) = 0$.*

**Proof:** By [27] (see also [19, 2.1.4]) $\langle F\varphi, F\varphi\rangle \le \|F\|^2\langle\varphi,\varphi\rangle$, so

$$\langle F\varphi(x_0), F\varphi(x_0)\rangle = \langle F\varphi, F\varphi\rangle(x_0) \le \|F\|^2\langle\varphi,\varphi\rangle(x_0) = \|F\|^2\langle\varphi(x_0),\varphi(x_0)\rangle = 0. \qquad \square$$

**Definition 4.2.2** Let us denote by $\mathcal{B}(X, \mathrm{End}_A(\mathcal{M}))$ (resp. $\mathcal{B}^*(X, \mathrm{End}^*_A(\mathcal{M}))$) the algebra of bounded continuous maps from $X$ to $\mathrm{End}_A(\mathcal{M}) = \mathbf{LM}(\mathcal{K}(\mathcal{M}))$ equipped with the left strict topology (resp. the algebra of bounded continuous maps from $X$ to $\mathrm{End}^*_A(\mathcal{M}) = \mathbf{M}(\mathcal{K}(\mathcal{M}))$ equipped with the strict topology). By Proposition 3.1.9 it is possible to consider the strong module (resp., $*$-strong module) topology instead of the left strict (resp. strict) topology. We equip these algebras $\mathcal{B}$ and $\mathcal{B}^*$ with the sup-norm.

**Theorem 4.2.3** *Let $(X, \mathcal{M})$ be a compatible pair. Then*

(i) *the Banach algebras $\mathrm{End}_{A_0(X)}(\mathcal{M} \otimes C_0(X))$ and $\mathcal{B}(X, \mathrm{End}_A(\mathcal{M})))$ are naturally isomorphic,*

(ii) *the $C^*$-algebras $\mathrm{End}^*_{A_0(X)}(\mathcal{M} \otimes C_0(X))$ and $\mathcal{B}^*(X, \mathrm{End}^*_A(\mathcal{M})))$ are naturally isomorphic.*

**Proof:** In correspondence with the condition (i) we can identify $\mathcal{M} \otimes_{\mathbf{C}} C_0(X) = C_0(X, \mathcal{M})$ and define the map

$$J : \mathcal{B}(X, \mathrm{End}_A(\mathcal{M}))) \to \mathrm{End}_{A_0(X)}(C_0(X, \mathcal{M})), \quad (J(D)\varphi)(x) := D(x)(\varphi(x)), \quad x \in X, \quad \varphi \in C_0(X, \mathcal{M}).$$

First of all, let us show that $J(D)\varphi \in C_0(X, \mathcal{M})$. For this purpose it is necessary to verify the continuity and vanishing at infinity. Since $\|D(x)\|$ is bounded, say, by a constant $C$, we can choose a compact set $K \subset X$ such that $\|\varphi(x)\| < \varepsilon/C$ outside $K$. We obtain that

$$\|(J(D)\varphi)(x)\| = \|D(x)(\varphi(x))\| < C \cdot \frac{\varepsilon}{C} = \varepsilon \qquad \text{outside of} \quad K.$$

Vanishing at infinity is established. To verify the continuity, we choose arbitrary $x \in X$ and $\varepsilon > 0$. Let us find an open neighbourhood $V_1$ of the point $x$ in $X$ such that for any $y \in V_1$ the estimate $\|\varphi(x) - \varphi(y)\| < \varepsilon/C$ is satisfied. By the definition of continuity with respect to the strong module topology, for (fixed element) $\varphi(x) \in \mathcal{M}$ there exists an open neighbourhood $V_2$ of the point $x$ in $X$ such that for any $y \in V_2$

$$\|D(x)(\varphi(x)) - D(y)(\varphi(x))\|_{\mathcal{M}} < \varepsilon$$

holds. Then for any $y \in U := V_1 \cap V_2$

$$\|(J(D)\varphi)(x) - (J(D)\varphi)(y)\|_{\mathcal{M}} = \|D(x)(\varphi(x)) - D(y)(\varphi(y))\|_{\mathcal{M}}$$

$$\le \|D(x)(\varphi(x)) - D(y)(\varphi(x))\|_{\mathcal{M}} + \|D(y)(\varphi(x)) - D(y)(\varphi(y))\|_{\mathcal{M}} < \varepsilon + C \cdot \frac{\varepsilon}{C} = 2\varepsilon$$

is valid. So continuity is checked out.

The linearity over $\mathbf{C}$ and $A_0(X)$ of the operator $J(D)$ is obvious. Since

$$\|J(D)\varphi\|_{C_0(X,\mathcal{M})} = \sup_{x\in X}\|D(x)(\varphi(x))\|_{\mathcal{M}} \le \sup_{x\in X}\|D(x)\|_{\mathrm{End}\,\mathcal{M}} \cdot \sup_{x\in X}\|\varphi(x)\|_{\mathcal{M}} \le C \cdot \|\varphi\|_{C_0(X,\mathcal{M})},$$

the operator $J(D)$ is bounded. Thereby the map $J$ is well-defined. It is obvious that it is $\mathbf{C}$-linear and

$$(J(DC)\varphi)(x) = DC(x)(\varphi(x)) = D(x)[C(x)(\varphi(x))] = D(x)[(J(C)\varphi)(x)] = (J(D)J(C)\varphi)(x),$$

so that $J$ is a homomorphism of algebras. Let us demonstrate its (algebraic) injectivity. Let for any $\varphi \in C_0(X, \mathcal{M})$ and any $x \in X$ the relation $(J(D)\varphi)(x) = 0$ be true, i. e. $D(x)\varphi(x) = 0$. Let us remark that passing to the one-point compactification $X^+$ of the space $X$, we can define on the normal space $X^+$ a continuous function $\chi^{x_0} : X^+ \to [0, 1]$, equal to 1 at the point $x_0$ and vanishing at $\infty$. Then for any $m \in \mathcal{M}$ the function $\chi_m^{x_0}(x) := m\chi^{x_0}(x)$ tends to zero at infinity. Since $0 = D(x_0)\chi_m^{x_0}(x_0) = D(x_0)m$ and $m$ and $x_0$ are arbitrary, the operator $D(x_0) = 0$ for any $x_0$, i. e. $D = 0$.

Let us remark that the above mentioned estimate, indicating the continuity of $J(D)$, gives also the inequality $\|J\| \leq 1$. Thus, the first part of the theorem will be proved, if we should manage to define a linear map

$$S : \mathrm{End}_{A_0(X)} C_0(X, \mathcal{M}) \to \mathcal{B}(X, \mathrm{End}_A(\mathcal{M}))), \qquad \|S\| \leq 1.$$

Let us put

$$(S(T)(x))(m) := (T\varphi)(x), \tag{76}$$

where $\varphi$ is a (non-uniquely defined) map $\varphi \in C_0(X, \mathcal{M})$, satisfying the condition $\varphi(x) = m$. Let us verify the independence of the definition of this non-unique choice of $\varphi$. Let $\varphi(x) = 0$. Then by the property (ii) for any operator $T$ one has $(T\varphi)(x) = 0$ and we have proved that (76) is well-defined. Once more the linearity of $S(T)(x)$ is obvious, so we should verify its boundedness. We have

$$\|(S(T)(x))(m)\| = \|(T\chi_m^x)(x)\| \leq \|T\|_{\mathrm{End}(C_0(X,\mathcal{M}))} \cdot \|\chi_m^x\|_{C_0(X,\mathcal{M})} = \|T\|_{\mathrm{End}(C_0(X,\mathcal{M}))} \cdot \|m\|_{\mathcal{M}}.$$

This estimate gives, at first, boundedness of $S(T)(x)$, secondly, the condition of boundedness not depending on $x$, and thirdly, the condition $\|S\| \leq 1$. To complete the proof of the first part of the theorem, it is necessary to verify the continuity of $S(T)(x)$ in $x$ with respect to the strong module topology. Let $x \in X$ be an arbitrary point, $\varepsilon > 0$ be an arbitrary number, $m \in \mathcal{M}$ be an arbitrary (fixed) element. Let us find an open neighbourhood $U$ of the point $x$ such that for any $y \in U$ and some map of the form $\chi_m^x$

$$\|(T\chi_m^x)(x) - (T\chi_m^x)(y)\| < \varepsilon$$

holds. Then for any $y \in U$

$$\|S(T)(x)m - S(T)(y)m\| = \|(T\chi_m^x)(x) - (T\chi_m^x)(y)\| < \varepsilon.$$

The first statement is proved.

Let now $J' = J|_{\mathcal{B}^*(X,\mathrm{End}_A^*(\mathcal{M})))}$. Then for any $D \in \mathcal{B}^*(X, \mathrm{End}_A^*(\mathcal{M})))$

$$\langle (J'(D)\varphi), \psi\rangle(x) = \langle (J'(D)\varphi)(x), \psi(x)\rangle = \langle (D(x)(\varphi(x)), \psi(x)\rangle = \langle \varphi(x), (D(x))^*(\psi(x))\rangle$$
$$= \langle \varphi(x), D^*(x)(\psi(x))\rangle = \langle \varphi(x), [J'(D^*)(\psi)](x)\rangle = \langle \varphi, J'(D^*)(\psi)\rangle(x),$$

so that $\mathrm{Im}\, J' \subset \mathrm{End}^*_{A_0(X)} C_0(X, \mathcal{M})$.

To complete the proof of the second part, we need to verify, at first, that

$$\mathrm{Im}\, S|_{\mathrm{End}^*_{A_0(X)} C_0(X,\mathcal{M})} \subset \mathcal{B}^*(X, \mathrm{End}_A^*(\mathcal{M}))),$$

i. e. that for each $x \in X$ the operator $S(T)(x)$ admits an adjoint, and secondly, that $S(T)(x)$ is continuous with respect to the $*$-strong module topology.

The first follows from the following calculation

$$\langle S(T)(x)m, m'\rangle = \langle (T\chi_m^x)(x), \chi_{m'}^x(x)\rangle = \langle (T\chi_m^x), \chi_{m'}^x\rangle(x) = \langle \chi_m^x, T^*\chi_{m'}^x\rangle(x)$$
$$= \langle \chi_m^x(x), (T^*\chi_{m'}^x)(x)\rangle = \langle m, S(T^*)(x)m'\rangle, \qquad m,\ m' \in \mathcal{M}.$$

Moreover, it implies that $S$ is involutive. Let now $x \in X$ be an arbitrary point, $\varepsilon > 0$ be an arbitrary number, $m \in \mathcal{M}$ be an arbitrary (fixed) element. Let us find an open neighbourhood $U$ of the point $x$ such that for any $y \in U$ and for some map of the form $\chi_m^x$

$$\|(T^*\chi_m^x)(x) - (T^*\chi_m^x)(y)\| < \varepsilon$$

holds. Then for any $y \in U$

$$\|S(T)^*(x)m - S(T)^*(y)m\| = \|(T^*\chi_m^x)(x) - (T^*\chi_m^x)(y)\| < \varepsilon. \qquad \square$$

**Corollary 4.2.4** *The defined above homomorphism $J$ realizes an isometric isomorphism of the groups*

$$\mathrm{GL}\,(A_0(X)) \cong \mathcal{B}_\bullet(X, \mathrm{GL}\,(A)), \qquad \mathrm{GL}^*(A_0(X)) \cong \mathcal{B}^*_\bullet(X, \mathrm{GL}^*(A)),$$

*where $\bullet$ indicates that we consider functions with bounded pointwise inverse.*

**Proof:** Since $J$ is a homomorphism of algebras, the statement immediately follows from its unitality. $\square$

Let us remark that it is necessary to be cautious while identifying different classes of operators in the standard Hilbert module over a commutative $C^*$-algebra with continuous sets of operators of same class on a Hilbert space. For example, though an operator of a finite rank on the Hilbert module $H_{C(X)}$ defines a continuous set of operators of finite rank on a compact space $X$, the inverse statement is not true, as can be seen from the following example, recently obtained by D.Kucerovsky [14]. It is interesting that this example is of a topological origin. Let us denote by $L_n$ the standard tautological vector bundle over the projective space $\mathbf{C}P(n)$. Necessary information about bundles and their characteristic classes can be find, for example, in the books [10, 11]. Let $\Gamma(L_n)$ be the Hilbert $C(\mathbf{C}P(n))$-module of sections of the bundle $L_n$.

**Lemma 4.2.5** ([14]) *Let $K$ be a compact operator with algebraically $n$-generated image on the Hilbert $C(\mathbf{C}P(n))$-module $\Gamma(L_n)$. Then there exists a point $x \in \mathbf{C}P(n)$ such that at this point one has $K(x) = 0$, where $K(x) \in C(\mathbf{C}P(n), \mathcal{K})$ is the set of compact operators defined by the operator $K$.*

**Proof:** The operator $K$ has the form $\sum_{k=1}^{n} s_k \langle r_k, \cdot \rangle$, where $s_k, r_k$ are continuous sections of the bundle $L_n$. Let $E = L_n \oplus \cdot \oplus L_n$ be the vector bundle equal to the direct sum of $n$ copies of the bundle $L_n$. Then $s_1 \oplus \ldots \oplus s_n$ is a section of the bundle $E$. Let us calculate the higher Chern class of the bundle $E$:

$$c_n(E) = c_n(L_n \oplus \cdot \oplus L_n) = c_1(L_n)^n \neq 0,$$

as $c_1(L_n) \neq 0$. But it means that any section of the bundle $E$ vanishes at some point. In particular, at some point $x \in \mathbf{C}P(n)$ the section $s_1 \oplus \ldots \oplus s_n$ vanishes, therefore all sections $s_i$, $i = 1, \ldots, n$ vanish at the point $x$. $\square$

**Example 4.2.6** ([14]) Let

$$X = \coprod_{n=1}^{\infty} \mathbf{C}P(n)$$

be the disjoint union of projective spaces, $X^+$ be the one-point compactification of the space $X$. Let us define a Hilbert $C(X^+)$-module $\mathcal{H}$ as a direct sum of spaces of sections of the bundles $L_n$, $\mathcal{H} = \oplus_{n=1}^{\infty} \Gamma(L_n)$. The module $\mathcal{H}$ is countably generated and, therefore, can be realized as an orthogonally complemented submodule of the standard Hilbert module $H_{C(X^+)}$. Let us define a compact operator $K$ on the module $\mathcal{H}$ by the formula

$$K(\oplus_{n=1}^{\infty} s_n) = \oplus_{n=1}^{\infty} \frac{1}{n} s_n,$$

where $s = \oplus_{n=1}^{\infty} s_n \in \mathcal{H}$. The operator $K$ defines a continuous set of operators of rank one over the space $X^+$, however, as the set $K(x)$ does not vanish at any point of $X$, so the operator $K$ is not an operator of a finite rank on the module $\mathcal{H}$ by Lemma 4.2.5. At the same time, as the set $K(x)$ is continuous, so $K \in \mathcal{K}(\mathcal{H})$. Extending the operator $K$ by zero, it is possible to obtain a compact operator on the module $H_{C(X^+)}$ possessing the same property.

# 5 Kuiper theorem for Hilbert modules

## 5.1 Preliminary notes

Let us denote, as well as earlier, through $\mathrm{End}_A l_2(A)$ the Banach algebra of all bounded $A$-homomorphisms of Hilbert $A$-module $l_2(A)$, and through $\mathrm{End}^*_A l_2(A)$ the $C^*$-algebra (cf. 2.1 of [19]) of operators, admitting adjoint. Let $\mathrm{GL}\,(A)$ and $\mathrm{GL}^*(A)$ denote the correspondent groups of invertible operators. The question about the contractibility of general linear groups is very important for $K$–theory to construct classifying spaces in terms of Fredholm operators. To this problem a series of papers is devoted: [21, 12, 32, 22]. The author used these results to construct the classifying spaces of $K^{p,q}(X; A)$ in [31]. In paper [3] J. Cuntz and N. Higson proved the contractibility of $\mathrm{GL}^*(A)$ for $A$ with strictly positive element (or, equivalent, with countable approximate unit = $\sigma$-unital).

In the present chapter, based on preprint [35], we give a simple proof of the theorem of Cuntz and Higson, distinguished from original, and based on generalization of a construction of homotopy from [26]. We also show, that the similar reasonings are aplicable to prove the contractibility $\mathrm{GL}\,(A)$ in some special cases, in particular, for $A$, being a subalgebra of algebra of compact operators in separable Hilbert space, and for $A = C_0(M)$, where $M$ is a finite-dimensional manifold.

We finish with the section with a detailed exposition of the modified Neubauer homotopy.

It is known, that the set of invertible operators in a Banach space is open with the respect to the topology of a norm, while the set of bounded $A$-homomorphisms is closed in the set of all endomorphisms. Thus, $\mathrm{GL}$ is an open set in a Banach space. The similar argument is valid for $\mathrm{GL}^*$. According to the Milnor theorem [20] such sets have the homotopy type of $CW$-compexes, and, therefore, by the theorem of Whitehead, strong and weak homotopy triviality are equivalent for them. We have proved the following statement.

**Lemma 5.1.1** *To prove the contractibility* $\mathrm{GL}$ *(resp.,* $\mathrm{GL}^*$ *) it is sufficient to verify the following. Let* $f: S \to \mathrm{GL}$ *be a continuous map of a sphere of arbitrary dimension. Then* $f$ *is homotopic to the map to the single point* $\mathrm{Id} \in \mathrm{GL}$ *. The similar statement holds for* $\mathrm{GL}^*$*.* □

Let us produce one more reduction. To consider simultaneously case $\mathrm{GL}$ and case $\mathrm{GL}^*$, we shall enter a common notation: $\mathcal{G} := \mathrm{GL}$ (resp., $\mathrm{GL}^*$), $\mathcal{E}(\mathcal{M}) := \mathrm{End}_A(\mathcal{M})$ (resp., $\mathrm{End}^*_A(\mathcal{M})$).

**Lemma 5.1.2** (a variant of the Atiyah theorem about small balls) *Let* $f: S \to \mathcal{G}$ *be a continuous map of a sphere of arbitrary finite dimension. Then* $f$ *is homotopic to a map* $f'$ *such that* $f'(S)$ *is a finite polyhedron in* $\mathcal{E}(l_2(A))$*, laying in* $\mathcal{G}$ *together with the homotopy.*

**Proof:** Let $\varepsilon > 0$ be such that $\varepsilon$-neighborhood of the compact set $f(S)$ lays in $\mathcal{G}$. Let us choose a fine simplicial subdivision of the sphere $S$, such that $diam(f(\sigma)) < \varepsilon/2$ for any simplex $\sigma$ of this subdivision. It is possible to do this, since $S$ is compact. Let $f'$ be a piecewise linear map, being the extension of the restriction $f$ to the 0-dimensional sceleton. Thus $diam(f'(\sigma)) \le diam(f(\sigma)) < \varepsilon/2$ for any $s$. For any point $s \in S$ there exists a vertex $s_i \in S$, such that $\|f(s) - f'(s_i)\| = \|f(s) - f(s_i)\| < \varepsilon/2$ and $\|f'(s) - f'(s_i)\| < \varepsilon/2$, hence the segment $[f(s), f'(s)] \subset \mathcal{G}$ for any point $s \in S$. Therefore, the linear homotopy $f_t(s) = tf'(s) + (1-t)f(s)$ is in $\mathcal{G}$. Passing to a subdivision of $f'(S)$, we obtain a structure of simplicial complex. □

**Remark 5.1.3** Let us remark, that this argument is not valid for other topologies, which we shall consider. For example, with the respect to the strong topology on operators in a Hilbert space, the sequence $\mathrm{Id}_n$ converges to $\mathrm{Id}$, where $\mathrm{Id}_n$ has the matrix $diag(1,\dots,1,0,0,\dots)$ (unit up to $n$-th place). So that with the respect to this topology the general linear group is not an open set.

One more step from the original work of Kuiper [15] is universal.

**Lemma 5.1.4** *Subset* $V \subset \mathcal{G}$*, defined as*

$$V = \{g \in \mathcal{G} \mid g|_{H'} = \mathrm{Id}_{H'},\, g(H_1) = H_1\},$$

*where*

$$l_2(A) = H' \oplus H_1, \qquad H' \cong H_1 \cong l_2(A),$$

*is contractible in* $\mathcal{G}$ *to* $1 \in \mathcal{G}$*.*

**Proof:** Let us represent $H'$ as

$$H' = H_2 \oplus H_3 \oplus \dots, \qquad H_i \cong l_2(A),$$

so that $l_2(A) = H_1 \oplus H_2 \oplus H_3 \oplus \dots$. The matrix of $g$ with the respect to this decomposition has the form

$$m(1,1) = u = g|_{H_1}, \quad m(i,i) = 1 \in \mathcal{E}(H_i),\ i > 1, \quad m(i,j) = 0,\ i \neq j,$$

$$g = diag(u,1,1,1,\dots) = diag(u, u^{-1}u, 1, u^{-1}u, 1, \dots).$$

We want so to define a homotopy $g_t \in \mathcal{G}$, $t \in [0,\pi]$, in such a way that

$$g_0 = g, \qquad g_{\pi/2} = diag(u, u^{-1}, u, u^{-1}, u, \dots), \qquad g_\pi = diag(1,1,1,\dots) = \mathrm{Id} \in \mathcal{G}.$$

For this purpose let us put for $t \in [0, \pi/2]$

$$m_t(1,1) = u,$$

for $i \geq 1$

$$\begin{pmatrix} m_t(2i,2i) & m_t(2i,2i+1) \\ m_t(2i+1,2i) & m_t(2i+1,2i+1) \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix},$$

$$m_t(r,s) = 0 \qquad \text{for remaining } r,\ s.$$

Let us put for $t \in [\pi/2, \pi]$

$$\begin{pmatrix} m_t(2i-1,2i-1) & m_t(2i-1,2i) \\ m_t(2i,2i-1) & m_t(2i,2i) \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix},$$

$$m_t(r,s) = 0 \qquad \text{for remaining } r,\ s. \qquad \square$$

**Lemma 5.1.5** *Subset $W \subset \mathcal{G}$, defined as*

$$W = \{g \in \mathcal{G} \mid g|_{H'} = \mathrm{Id}_{H'}\},$$

*where*

$$l_2(A) = H' \oplus H_1, \qquad H' \cong H_1 \cong l_2(A),$$

*is contractible inside $\mathcal{G}$ to*

$$V = \{g \in \mathcal{G} \mid g|_{H'} = \mathrm{Id}_{H'},\ g(H_1) = H_1\}.$$

**Proof:** With the respect to the decomposition $l_2(A) = H' \oplus H_1$ we define a homotopy by the formula

$$f_t(s) = \begin{pmatrix} 1 & \beta(s)(1-t) \\ 0 & \gamma(s) \end{pmatrix}.$$

$$F_t(s) = \begin{pmatrix} 1 & \beta(1-t) \\ 0 & \gamma \end{pmatrix}.$$

Let the operator $\begin{pmatrix} \varphi & \psi \\ \chi & \xi \end{pmatrix}$ be the inverse to $\begin{pmatrix} 1 & \beta \\ 0 & \gamma \end{pmatrix}$. Then

$$\begin{pmatrix} \varphi & \psi \\ \chi & \xi \end{pmatrix} \begin{pmatrix} 1 & \beta \\ 0 & \gamma \end{pmatrix} = \begin{pmatrix} \varphi & \varphi\beta + \psi\gamma \\ \chi & \chi\beta + \xi\gamma \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & \beta \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} \varphi & \psi \\ \chi & \xi \end{pmatrix} = \begin{pmatrix} \varphi + \beta\chi & \psi + \beta\xi \\ \gamma\chi & \gamma\xi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

whence

$$\varphi = 1, \qquad \chi = 0, \qquad \gamma\xi = \xi\gamma = 1,$$

$$\beta + \psi\gamma = 0, \qquad \psi + \beta\xi = 0,$$

and

$$\begin{pmatrix} 1 & \psi(1-t) \\ 0 & \xi \end{pmatrix} \begin{pmatrix} 1 & \beta(1-t) \\ 0 & \gamma \end{pmatrix} = \begin{pmatrix} 1 & \beta(1-t) + (1-t)\psi\gamma \\ 0 & \xi\gamma \end{pmatrix} = \begin{pmatrix} 1 & (1-t)\cdot 0 \\ 0 & 1 \end{pmatrix},$$

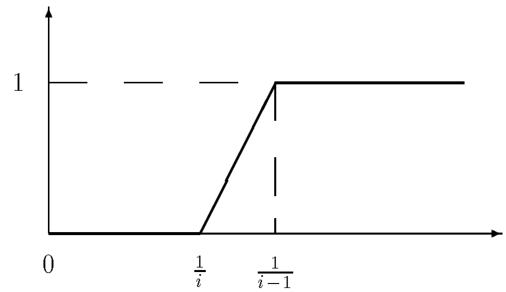
$$\begin{pmatrix} 1 & \beta(1-t) \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} 1 & \psi(1-t) \\ 0 & \xi \end{pmatrix} = \begin{pmatrix} 1 & \psi(1-t) + \beta\xi(1-t) \\ 0 & \gamma\xi \end{pmatrix} = \begin{pmatrix} 1 & (1-t)\cdot 0 \\ 0 & 1 \end{pmatrix}.$$

Hence, the homotopy lies in $\mathcal{G}$. $\square$

## 5.2 Technical lemmas

Let through $\mathcal{K}_A$ be denoted the $C^*$-algebra of $A$-compact operators on $l_2(A)$, through $\mathbf{LM}(\mathcal{K}_A) \cong \mathrm{End}_A l_2(A)$ the algebra of the left multipliers, through $\mathbf{M}(\mathcal{K}_A) \cong \mathrm{End}^*_A l_2(A)$ the $C^*$-algebra of multipliers and through $\mathbf{QM}(\mathcal{K}_A) \cong \mathrm{End}_A(l_2(A), l_2(A)')$ the space of quasi-multipliers (see [2, 17, 9, 13, 24, 27] and §§ 2.1 and 2.2).

Let $\alpha$ be a strictly positive element (see, e.g., § 1.1 of [19]) in $\sigma$-unital algebra $A$, $\alpha_i := \varphi_i(\alpha)$ be a countable approximate unit, where $\varphi_i$ has the graph




$\omega_i := (\alpha_i - \alpha_{i-1})^{1/2}$ для $i \geq 3$ и $\omega_2 = \alpha_2^{1/2}$, так что

$$\omega_j \alpha_i = \alpha_i \omega_j = 0, \quad j = i+2, i+3, \ldots, \qquad \omega_j \alpha_i = \alpha_i \omega_j = \omega_j, \quad j = 1, \ldots, i-1. \tag{77}$$

Since there is no unit in $A$, the notion of "standard base" $\{e_i\}$ of module $l_2(A)$ makes no sense. Nevertheless, it is possible to define properly elements $e_i\gamma$ for any $\gamma \in A$, namely,

$$e_i\gamma := (0, \ldots, 0, \gamma, 0, \ldots), \qquad \gamma \text{ at } i\text{-th place.}$$

Let us denote the correspondent orthoprojections on these one-dimensional submodules $E_i$ through $Q_i$.

**Lemma 5.2.1** *The injection $i : A \to l_2(A)$, defined by the formula*

$$x \mapsto \sum_i e_{k(i)} \omega_i x, \qquad k(1) < k(2) < k(3) < \ldots,$$

*remain the inner product and admits adjoint. In particular, the image* $\mathrm{Im}\, i$ *is defined by a selfadjoint projection of the form*

$$p = ii^*. \tag{78}$$

**Proof:** First of all,

$$\begin{aligned} \langle ix, iy \rangle &= \langle \textstyle\sum_i e_{k(i)} \omega_i x, \sum_i e_{k(i)} \omega_i y \rangle = \sum_i \langle e_{k(i)} \omega_i x, e_{k(i)} \omega_i y \rangle = \\ &= \textstyle\sum_i x^* \omega_i \omega_i y = x^* y = \langle x, y \rangle. \end{aligned}$$

Let us consider operator $t : l_2(A) \to A$ of the form

$$t(z) := \sum_i \langle e_{k(i)} \omega_i, z \rangle = \sum_i \omega_i z_{k(i)}.$$

This series satisfies to the Cauchy criterion: if number $m$ is so great, that

$$\sum_{i=m+1}^{\infty} z_i^* z_i < \delta,$$

then

$$\| \sum_{i=s}^{r} \omega_i z_{k(i)} \| \leq \| \sum_{i=s}^{r} \omega_i^2 \|^{1/2} \cdot \| \sum_{i=s}^{r} z_{k(i)}^* z_{k(i)} \|^{1/2} \leq 1 \cdot \delta.$$

The same reasoning for $s = 1$ implies the relation $\|t(z)\| \le \|z\|$. Also, $\langle ix, z\rangle = \langle x, tz\rangle$, i. e., $t = i^*$.

Let us consider arbitrary elements $x, y \in A$. Then

$$(i^* ix)^* y = \langle i^* ix, y\rangle = \langle ix, iy\rangle = \langle x, y\rangle = x^* y.$$

Since $y$ is an arbitrary element, we conclude, that $i^* ix = x$ and $i^* i = \mathrm{Id}$. Hence,

$$ii^* ii^* = ii^*,$$

i. e., $p$ is a projection. Since $i^* i = \mathrm{Id}$, $i^*$ is an epimorphism and $\mathrm{Im}\, i = \mathrm{Im}\, p$ (see also [16, Sect. 3]). □

We need some more strong variant of this lemma.

**Lemma 5.2.2** *The injection $J : l_2(A) \to l_2(A)$ under the formula*

$$(a_1, a_2, \ldots) \mapsto \sum_j \sum_i v_{ij} a_j, \qquad \langle v_{ij}, v_{ij}\rangle = \omega_i^2, \qquad v_{ij} \in M_{k(i,j)},$$

$$l_2(A) = M_1 \oplus M_2 \oplus \ldots, \qquad M_r = \{\, (0, \ldots, 0, a_{s(r)}, \ldots, a_{s(r+1)-1}, 0, \ldots)\, \},$$
$$\{k(1,1); k(1,2), k(2,1); k(1,3), k(2,2), k(3,1); \ldots\} = \{1, 2, \ldots\},$$

*remains the inner product and admits an adjoint. In particular, the image is defined by a selfadjoint projection of the form $JJ^*$.*

**Proof:** Let $x = (a_1, a_2, \ldots) \in l_2(A)$, $y = (b_1, b_2, \ldots) \in l_2(A)$. Then

$$\begin{aligned}\langle Jx, Jy\rangle &= \langle \textstyle\sum_j \sum_i v_{ij} a_j, \sum_j \sum_i v_{ij} b_j\rangle = \sum_j \sum_i a_j^* \omega_i^2 b_j = \sum_j a_j^* \left(\sum_i \omega_i^2\right) b_j = \\ &= \textstyle\sum_j a_j^* b_j = \langle x, y\rangle.\end{aligned}$$

In particular, $J$ is bounded. Let us consider operator $T : l_2(A) \to l_2(A)$ of the form

$$T(z) := (t_1, t_2, \ldots), \quad t_j := \sum_i \langle v_{ij}, z\rangle.$$

For this series the Cauchy criterion is carried out: let number $N = N(z)$ be so great, that $\|(1 - p_N)z\| < \delta$ and $m$ be so great, that $s(k(m,j)) > N$ ($j$ is fixed), (by [27], see also [19, 1.2.4])

$$\|\sum_{i=m}^r \langle v_{ij}, z\rangle\| = \|\left\langle \sum_{i=m}^r v_{ij}, (1 - p_N)z \right\rangle\| \le \left\|\left\langle \sum_{i=m}^r v_{ij}, \sum_{i=m}^r v_{ij}\right\rangle\right\|^{1/2} \cdot \|(1 - p_N)z\| \le 1 \cdot \delta.$$

For any $r$ by [27] (see also [19, 1.2.4]) the following inequality holds

$$\sum_{i=1}^r \langle v_{ij}, z\rangle^* \sum_{i=1}^r \langle v_{ij}, z\rangle = \left\langle \sum_{i=1}^r v_{ij}, q_j z\right\rangle^* \left\langle \sum_{i=1}^r v_{ij}, q_j z\right\rangle \le \langle q_j z, q_j z\rangle,$$

where $q_j$ is the orthoprojection on $\bigoplus_i M_{k(i,j)}$. Hence

$$t_j^* t_j \le \langle q_j z, q_j z\rangle, \qquad \langle T(z), T(z)\rangle \le \langle z, z\rangle.$$

So, $T$ is bounded, and the fact, that it is the adjoint for $J$ is obvious.

The proof of the second statement literally repeats the reasoning from the previous lemma. □

Let us consider an operator $F \in \mathrm{GL}$. Then, with the respect to the standard decomposition $l_2(A)$ into the direct sum of $E_i \cong A$, the operator $F$ has a matrix $F_j^i$ with the elements from $\mathbf{LM}(A)$. If $F \in \mathrm{GL}^*$, $F_j^i \in \mathbf{M}(A)$, since $(F^*)_j^i = (F_i^j)^*$. Let us note, that for any $b \in A$ and any $F \in \mathrm{GL}$ holds $\|F_{m_0}^i(b)\| \to 0$ as $i \to \infty$, because $\{F_{m_0}^i(b)\}_{i=1}^\infty = F(e_{m_0} b) \in l_2(A)$. For $F \in \mathrm{GL}^*$ holds $\|F_j^{m_0}(b)\| \to \infty$ as $j \to \infty$ as well, as it is proved in the following lemma.

**Lemma 5.2.3** *For any $F \in \mathrm{GL}^*$, $\varepsilon > 0$ and $e_k\gamma$ there exists a number $m(k)$, such that for any $m \ge m(k)$ and $\varphi \in A$ with $\|\varphi\| \le 1$ holds*

$$\|\langle e_k\gamma, F e_m \varphi\rangle\| < \varepsilon.$$

**Proof:** Let us consider the bounded operator $F^*$. Since $F^* e_k \gamma \in l_2(A)$, there exists a number $m(k)$, such that

$$\|(1 - p_{m(k)})F^* e_k\gamma\| < \varepsilon, \qquad \|Q_m F^* e_k \gamma\| < \varepsilon, \quad (m > m(k)).$$

Hence,

$$\|\langle e_k\gamma, F e_m \varphi\rangle\| = \|Q_m F^* e_k\gamma\| \cdot \|\varphi\| < \varepsilon, \quad (m > m(k)). \qquad □$$

## 5.3 Proof of the Cuntz–Higson theorem

**Lemma 5.3.1** *Let $F_r \in \mathrm{GL}^*$, $r = 1, \ldots, N$, be arbitrary operators, and $\varepsilon > 0$ be any number. Then we can choose such increasing non-intersecting sequences of natural numbers $i(k)$ and $j(k)$, that*

$$\|(1 - p_{j(s)})F_r e_{i(k)}\alpha_k\| < \frac{1}{4} \cdot \frac{\varepsilon}{2^s \cdot 2^k}, \qquad s = k, k+1, \ldots, \quad r = 1, \ldots, N, \tag{79}$$

$$\|\langle F_r e_{i(k)}\alpha_k, e_{j(s)}\alpha_s\rangle\| < \frac{1}{2} \cdot \frac{\varepsilon}{2^s \cdot 2^k}, \qquad s = 1, \ldots k-1, \quad r = 1, \ldots, N \tag{80}$$

**Proof:** Let us take $i(1) := 1$. Let us choose $j(1) > i(1)$ in such a way that

$$\|(1 - p_{j(1)})F_r e_{i(1)}\alpha_1\| < \frac{1}{4} \cdot \frac{\varepsilon}{2^1 \cdot 2^1}, \quad r = 1, \ldots, N.$$

Let us discover $i(2) > j(1)$, such that (in the correspondence with Lemma 5.2.3)

$$\|\langle F_r e_{i(2)}\alpha_2, e_{j(1)}\alpha_1\rangle\| < \frac{1}{2} \cdot \frac{\varepsilon}{2^1 \cdot 2^2}, \quad r = 1, \ldots, N.$$

Let us now choose $j(2) > i(2)$, such that

$$\|(1 - p_{j(2)})F_r e_{i(k)}\alpha_k\| < \frac{1}{4} \cdot \frac{\varepsilon}{2^2 \cdot 2^k}, \qquad k = 1, 2, \quad r = 1, \ldots, N,$$

and such $i(3) > j(2)$, such that

$$\|\langle F_r e_{i(3)}\alpha_3, e_{j(s)}\alpha_s\rangle\| < \frac{1}{2} \cdot \frac{\varepsilon}{2^s \cdot 2^k}, \qquad s = 1, 2, \quad r = 1, \ldots, N.$$

Let us continue the process by induction. Let $i(1), \ldots, i(k-1)$ and $j(1), \ldots, j(k-2)$ be already found in such a manner, that the conditions (79) and (80) hold for them. Let us find $j(k-1) > i(k-1)$, such that

$$\|(1 - p_{j(k-1)})F_r e_{i(m)}\alpha_m\| < \frac{1}{4} \cdot \frac{\varepsilon}{2^{k-} \cdot 2^m}, \qquad m = 1, \ldots k-1, \quad r = 1, \ldots, N,$$

and after that let us find $i(k) > j(k-1)$ in such a manner that

$$\|\langle F_r e_{i(k)}\alpha_k, e_{j(s)}\alpha_s\rangle\| < \frac{1}{2} \cdot \frac{\varepsilon}{2^s \cdot 2^k}, \qquad s = 1, \ldots k-1, \quad r = 1, \ldots, N.$$

By induction we obtain the required statement. □

Let us define now embeddings $J$ and $J'$ similarly to the constructions in Lemma 5.2.2. For the definition of $J$ we shall take some of $e_{i(g)}\alpha_g\omega_s$ as vectors $v_{sj}$, but so that $g = g(s,j) > s + j$, $g > s$, whence $e_{i(g)}\alpha_g\omega_s = e_{i(g)}\omega_s$ and $\langle v_{sj}, v_{sj}\rangle = \omega_s^2$. Let us define similarly $v'_{sm}$ for $J'$, but taking $e_{j(k)}$ instead of $e_{i(k)}$. From the conditions (79) and (80) we obtain

$$\|\langle F_r v_{st}, v'_{nm}\rangle\| = \|\langle F_r e_{i(g(s,t))}\alpha_{g(s,t)}\omega_s, e_{j(h(n,m))}\alpha_{h(n,m)}\omega_n\rangle\| \le \|Q_{j(h(n,m))}F_r e_{i(g(s,t))}\alpha_{g(s,t)}\| \le$$

$$\le \|(1 - p_{j(h(n,m)-1)})F_r e_{i(g(s,t))}\alpha_{g(s,t)}\| < \frac{1}{4} \cdot \frac{\varepsilon}{2^{h-1} \cdot 2^g}, \ h \ge g, \ r = 1, \ldots, N. \tag{81}$$

$$\|\langle F_r v_{st}, v'_{nm}\rangle\| = \|\langle F_r e_{i(g(s,t))}\alpha_{g(s,t)}\omega_s, e_{j(h(n,m))}\alpha_{h(n,m)}\omega_n\rangle\| < \frac{1}{2} \cdot \frac{\varepsilon}{2^h \cdot 2^g}, \ h < g, \ r = 1, \ldots, N. \tag{82}$$

Let us denote through $P$ and $P'$ the correspondent orthoprojections. Then $PP' = P'P = 0$. Let $x = (a_1, a_2, \ldots)$ and $y = (b_1, b_2, \ldots)$ be arbitrary vectors from $l_2(A)$ with the norm 1. Then for any $r = 1, \ldots, N$ by (81,82)

$$\|\langle F_r Jx, J'y\rangle\| = \left\| \left\langle \sum_t \sum_s F_r v_{st} a_t, \sum_m \sum_n v'_{nm} b_m \right\rangle \right\| \le$$

$$\leq \sum_{t,s,n,m} \left( \sum_{h(n,m)\geq g(s,t)} \|\langle F_r v_{st}, v'_{nm}\rangle\| + \sum_{h(n,m)<g(s,t)} \|\langle F_r v_{st}, v'_{nm}\rangle\| \right) \leq \sum_{t,s,n,m} \frac{\varepsilon}{2^{h(n,m)} \cdot 2^{g(t,s)}} < \varepsilon,$$

since $h(n,m) > n+m$, $g(t,s) > t+s$ by the construction. From this we obtain

$$\|P'F_rP\| < \varepsilon, \qquad r = 1, \ldots, N. \tag{83}$$

As it was shown in Lemma 5.1.2, it is sufficient to know how to construct a homotopy of picewise-linear map with the image in a finite polyhedron in $\mathrm{GL}^*$ with vertices $F_1, \ldots, F_N$ into a map in a compact set $\{D(x)\} \subset \mathrm{GL}^*$, such that

$$PD(x) = D(x)P = P \quad \forall\, x \in S.$$

For this purpose we can apply a homotopy of Neubauer type (see Section 5.6). By (83) we have to take care only of that, we have an operator $H_0 : P'(l_2(A)) \to P(l_2(A))$, such that operators $H_0P'$ and $H_0^{-1}P$ admit adjoint. Let us assume $H_0 = JJ'^*$. Then $H_0P' = JJ'^*J'J'^* = JJ'^*$, where $J'^*$ is an isomorphism $P'(l_2(A)) \to l_2(A)$, and $J : l_2(A) \cong P(l_2(A))$.

We have proved the following statement.

**Theorem 5.3.2** [3] *Let $A$ be a $\sigma$-unital $C^*$-algebra. Then $\mathrm{GL}^*(A)$ is contractible with the respect to the norm topology.* □

## 5.4 The case $A \subset \mathcal{K}$

Let algebra $A$ be (for some faithful representation) a subalgebra of algebra $\mathcal{K}$ of compact operators on a separable Hilbert space $H$. Under these restrictions we can prove the following statement.

**Lemma 5.4.1** *Let $a, b \in A$, $(f_1, f_2, \ldots) \in l_2'(A)$. Then*

$$\|af_ib\| \to 0 \qquad (i \to \infty).$$

**Proof:** Since $a^* \in \mathcal{K}$, for any $\varepsilon > 0$ we can find a number $N = N(\varepsilon)$ and base $h_1, h_2, \ldots$ in $H$, such that

$$\|p'_N a^*\| < \frac{\varepsilon}{2 \cdot \sup \|f_i\|}, \qquad H_N = \mathrm{span}_{\mathbf{C}}\langle h_1, \ldots, h_N\rangle, \qquad H'_N = H_N^{\perp},$$

$p_N$ and $p'_N$ are the correspondent projections. Since [8] the partial sums of series $\sum_i f_i f_i^*$ form an increasing uniformly bounded sequence of positive operators in $\mathcal{B}(H)$, $f_if_i^*$ is strong convergent to the zero operator. Hence, for any $h \in H$

$$\|f_i^*h\| = \langle f_i^*h, f_i^*h\rangle = \langle f_if_i^*h, h\rangle \to 0.$$

Thus, $f_i^*$ is strong convergent to 0. Let $i_0$ be so large, that

$$\|f_i^* p_N\| < \frac{\varepsilon}{2\|a\|}.$$

for $i > i_0$. Then

$$\|af_i\| = \|f_i^*p_Na^*\| + \|f_i^*p'_Na^*\| < \frac{\varepsilon}{2\|a\|} \cdot \|a^*\| + \|f_i^*\| \frac{\varepsilon}{2 \cdot \sup \|f_i\|} \leq \varepsilon. \qquad \square$$

Let us remark, that similar properties for matrix elements themselves (which belong $\mathbf{LM}(\mathcal{K}) = \mathcal{B}(H)$) are not valid even for operators from have not $\mathrm{GL}^*$. Moreover, the following example shows, that all matrix elements can have the norm 1.

**Example 5.4.2** (A. V. Buchina) Let $H$ be Hilbert space, $\mathcal{K} = \mathcal{K}(H)$ be the algebra of compact operators on $H$,

$$l_2(\mathcal{K}) = \left\{ (k_1, k_2, k_3, \ldots) \mid k_i \in \mathcal{K}, \|\sum_i k_i^* k_i\| < \infty \right\}.$$

Let us construct a invertible operator $F : L_2(\mathcal{K}) \longrightarrow l_2(\mathcal{K})$ with the matrix elements with the respect to the standard decomposition of $l_2(\mathcal{K})$, satisfying $\|F_{i,j}\| \geq 1$ for all $i$ and $j$. These elements belong to $\mathbf{LM}(\mathcal{K}) = \mathcal{B}(H)$, i. e. to the algebra of all bounded operators.

Let us denote through $\{e_i\}$ a base of $H$, and through $p_i$ the projection onto the span of the correspondent basis vector. Let us take as $F$ an operator with the following matrix

$$\begin{pmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & \cdots \\ p_2 & p_1 & p_4 & p_3 & p_6 & p_5 & p_8 & p_7 & \cdots \\ p_3 & p_4 & p_1 & p_2 & p_7 & p_8 & p_5 & p_6 & \cdots \\ p_4 & p_3 & p_2 & p_1 & p_8 & p_7 & p_6 & p_5 & \cdots \\ p_5 & p_6 & p_7 & p_8 & p_1 & p_2 & p_3 & p_4 & \cdots \\ p_6 & p_5 & p_8 & p_7 & p_2 & p_1 & p_4 & p_3 & \cdots \\ p_7 & p_8 & p_5 & p_6 & p_3 & p_4 & p_1 & p_2 & \cdots \\ p_8 & p_7 & p_6 & p_5 & p_4 & p_3 & p_2 & p_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where in the first row $p_i$ are ordered with the respect to the increase of $i$, the second row is obtained from the first by permutations in pairs $p_{2i-1}$ and $p_{2i}$ for $i = 1, 2, \ldots$, the third row is obtained from the first one by permutations of adjacent pairs, the fourth row is obtained by permutations in pairs in the third one, the fifth is obtained by permutations of 4-tuples in the first one, and so on. Let us enter the following notation: $F_{ij} = p_{\sigma_i(j)}$ and let us remark, that for $i_1 \neq i_2 \quad p_{\sigma_{i_1}(j)} \neq p_{\sigma_{i_2}(j)}$ for any $j$. Let us show, that this operator $F$ satisfies to all given conditions.

**1.** Let us prove, that the image of $(k_1, k_2, k_3 \ldots) \in l_2(\mathcal{K})$ is compact for the action of each row of the operator $F$, i. e. let us verify, that inequality: $\|\sum_j p_{\sigma_i(j)} k_j\| < \infty$ holds for each $i$.

For any $\varepsilon > 0$ we can find $N \in \mathbf{N}$, such that for all $n > N$ and all $p \in \mathbf{N}$ the following inequality holds:

$$\left\|\sum_{j=n}^{n+p} p_{\sigma_i(j)} k_j\right\|^2 = \left\|\left(\sum_{j=n}^{n+p} p_{\sigma_i(j)} k_j\right)^* \left(\sum_{r=n}^{n+p} p_{\sigma_i(r)} k_r\right)\right\| = \left\|\sum_{j=n}^{n+p} k_j^* p_{\sigma_i(j)} k_j\right\| \leq \left\|\sum_{j=n}^{n+p} k_j^* k_j\right\| < \varepsilon,$$

and by the Cauchy criterion from the convergence with the respect to the norm of $\sum_j k_j^* k_j$ it follows, that $\sum_j p_{\sigma_i(j)} k_j$ converges with the respect to the norm. Thus, the series is norm convergent and its terms are compact operators, hence, $\sum_j p_{\sigma_i(j)} k_j$ is compact too.

**2.** Let us verify, that image of a vector $(k_1, k_2, k_3, \ldots) \in l_2(\mathcal{K})$ under the action of an operator $F$ belongs to $l_2(\mathcal{K})$,i. e. let us prove convergence with the respect to the norm of the series

$$\sum_{i=1}^{\infty} \left(\sum_j p_{\sigma_i(j)} k_j\right)^* \left(\sum_r p_{\sigma_i(r)} k_r\right).$$

We have

$$\left\|\sum_{i=1}^{\infty} \left(\sum_j p_{\sigma_i(j)} k_j\right)^* \left(\sum_r p_{\sigma_i(r)} k_r\right)\right\| = \left\|\sum_{i=1}^{\infty} \sum_j k_j^* p_{\sigma_i(j)} \sum_r p_{\sigma_i(r)} k_r\right\| = \left\|\sum_i \sum_j k_j^* p_{\sigma_i(j)} k_j\right\|.$$

Since $\|\sum_j k_j^* k_j\| < \infty$ for any $x \in H$ of norm 1, the followinf inequalities hold:

$$\infty > \left\|\sum_j k_j^* k_j\right\| \geq \left\langle \sum_j k_j^* k_j x, x \right\rangle = \sum_j \langle k_j x, k_j x\rangle =$$

$$= \sum_j \|k_j x\|^2 = \sum_j \sum_i |(k_j x)_{\sigma_i(j)}|^2 = \sum_i \sum_j |(k_j x)_{\sigma_i(j)}|^2.$$

Hence, for any $\varepsilon > 0$ and any $x, ||x|| = 1$, one can find a number $N(x) \in \mathbf{N}$, such that for all $n > N(x)$ and all $p \in \mathbf{N}$ the following inequality holds: $\sum_{i=n}^{n+p} \sum_j |(k_j x)_{\sigma_i(j)}|^2 < \varepsilon$. Thus,

$$\left\langle \sum_{i=n}^{n+p} \sum_j k_j^* p_{\sigma_i(j)} k_j x, x \right\rangle = \sum_{i=n}^{n+p} \sum_j \langle k_j^* p_{\sigma_i(j)} k_j x, x \rangle =$$

$$= \sum_{i=n}^{n+p} \sum_j \langle p_{\sigma_i(j)} k_j x, p_{\sigma_i(j)} k_j x \rangle = \sum_{i=n}^{n+p} \sum_j ||(k_j x)_{\sigma_i(j)}||^2 < \varepsilon.$$

Let us enter operators $B = \sum_j k_j^* k_j$ and $B_n = \sum_{i=1}^{n} \sum_j k_j^* p_{\sigma_i(j)} k_j$, acting on $H$. The first of them is compact, and consequently, the remaining are compact too, as serieses with compact entries converging with the respect to the norm. For a fixed vector $x \in H$ of length 1, the following statements hold:

(i) Inequality holds:

$$\langle B_n x, x \rangle = \sum_j \sum_{i=1}^{n} |k_j x|^2_{\sigma_i(j)} \leq \sum_j \sum_i |(k_j x)_{\sigma_i(j)}|^2 = \langle Bx, x \rangle;$$

(ii) $\lim_{n\to\infty}(B_n x, x) = (Bx, x)$.

Let us consider operator $E_n := B - B_n$. It is easy to see, that:

(i) $\langle E_n x, x \rangle \to 0$ for $n \to \infty$;

(ii) $E_n^* = E_n$;

(iii) $E_n \geq 0$;

(iv) $||E_n|| = ||B - B_n|| \leq ||B||$.

From (i) and (iii) it follows, that $E_n^{1/2} \longrightarrow 0$ with the respect to the strong topology. By (iv) we obtain $E_n \longrightarrow 0$ with the respect to the strong topology, as multiplication is strong continuous on bounded sets in both variables. We have proved the strong convergence of increasing sequence of positive compact operators $B_n$ to a compact operator $B$. Let us choose a finite-dimensional projection $p$, such that $||B(1-p)|| < \varepsilon$, and then $n$, so large that $||(B - B_m)p|| < \varepsilon$ for $m > n$. Then, since the sequence increases,

$$||B - B_m|| \leq ||(B - B_m)p|| + ||p(B - B_m)|| + ||p(B - B_m)p|| + ||(1-p)(B - B_m)(1-p)|| \leq$$

$$\leq 3||(B - B_m)p|| + ||(1-p)B(1-p)|| < 4\varepsilon.$$

Thus, the operators $B_n$ converge to the operator $B$ with the respect to the norm, $||B - B_n|| \to 0$, for $n \to \infty$. Since $||B|| < \infty$, we have $||\sum_i \sum_j k_j^* p_{\sigma_i(j)} k_j|| < \infty$. So, the item 2 is completely proved.

**3.** From the general form of the constructed operator $F$ it is obvious, that $||F_{ij}|| = 1$ for any $i$ and $j$.

**4.** Let us remark, that $F^*F = F^2 = FF^* = \mathrm{Id}$, Therefore, $F$ is invertible.

The constructed operator $F$ by items 1–4 satisfies all necessary conditions.

**Theorem 5.4.3** *The group* $\mathrm{GL}(A)$ *is contractible with the respect to the norm for* $A \subset \mathcal{K}$.

**Proof:** Since Lemma 5.4.1 is the analog of Lemma 5.2.3, the proof can be obtained by the literal repeating of the reasoning from Section 5.3. □

## 5.5 Some other cases

**Definition 5.5.1** Let us tell, that $C^*$-algebra $A$ has *property* (K), if for any functional $f : l_2(A) \to A$, any $\varepsilon > 0$ and any $a \in A$ it is possible to find a vector $x \in l_2(A)$, such that

$$\|f(x)\| < \varepsilon, \qquad \langle x, x\rangle = a^* a.$$

**Definition 5.5.2** A $C^*$-algebra $A$ has *property* (E), if for any functional $f = (f_1, \dots, f_n, \dots) \in l'_2(A)$ and any $\varepsilon > 0$ it is possible to find a another functional $g = (g_1, \dots, g_n \dots) \in l'_2(A)$ and a number $k \in \mathbf{Z}$, such that

$$\|f - g\| < \varepsilon, \qquad f_i = g_i, \quad i = k+1,\ k+2, \dots$$

and $g|_{L_k} : L_k \to A$ is *epimorphism*, where $L_n = \{ (a_1, \dots, a_n, 0, 0, \dots) \}$.

**Example 5.5.3** Let $A$ be the algebra of continuous functions on a smooth $n$-dimensional manifold $M$. Then $A$ has the property (E) (with $k = n + 1$).

For the proof of the following theorem we need

**Lemma 5.5.4** *Let $\mathcal{M}$ be a Hilbert module, $x \in \mathcal{M}$, $\langle x, x\rangle \geq a \geq 0$, $\|a\| \leq 1$. Then one can find an element $y = xb$, $\|b\| \leq 1$, such that $\langle y, y\rangle = a^2$.*

**Proof:** Let us put

$$\gamma := \langle x, x\rangle, \qquad b := \lim_{n\to\infty} \left(\gamma + \frac{1}{n}\right)^{-1/2} a.$$

This (norm) limit exists, as

$$\left[\left(\gamma + \frac{1}{n}\right)^{-1/2} - \left(\gamma + \frac{1}{m}\right)^{-1/2}\right] a^2 \left[\left(\gamma + \frac{1}{n}\right)^{-1/2} - \left(\gamma + \frac{1}{m}\right)^{-1/2}\right] \leq$$

$$\leq \left[\left(\gamma + \frac{1}{n}\right)^{-1/2} - \left(\gamma + \frac{1}{m}\right)^{-1/2}\right]^2 \gamma^2 \to 0,$$

since for any non-negative $z$ holds

$$\frac{z^2}{z + \frac{1}{n}} - \frac{z^2}{z + \frac{1}{m}} = \frac{\frac{1}{m} z^2 - \frac{1}{n} z^2}{(z + \frac{1}{n})(z + \frac{1}{n})} = \left(\frac{1}{m} - \frac{1}{n}\right) \frac{z^2}{(z + \frac{1}{n})(z + \frac{1}{n})} \leq \frac{1}{m} - \frac{1}{n}.$$

Also $\|b\| \leq 1$, as

$$a \left(\gamma + \frac{1}{n}\right)^{-1} a \leq a^{1/2} \gamma \left(\gamma + \frac{1}{n}\right)^{-1} a^{1/2} \leq a \leq 1.$$

The condition $\langle y, y\rangle = a^2$ is obvious now. $\square$

**Theorem 5.5.5** *The property (E) implies the property (K).*

**Proof:** We can suppose $\|a\| = 1$. Let us consider an arbitrary functional $f = (f_1, \dots) \in l'_2(A)$ and $\varepsilon > 0$. Let $g$ and $k$ be as in the condition (E) with the respect to $\varepsilon/2$. Let us put $f' := f|_{L_k^\perp}$. Since $L_k^\perp \cong l_2(A)$, by (E) there exists a functional $g' : L_k^\perp \to A$, such that

$$\|f' - g'\| < \varepsilon/2, \qquad f'_i = g'_i = g_i, \quad i = k'+1,\ k'+2, \dots$$

and $g'|_{L_k^\perp \cap L_{k'}}$ is an epimorphism. Then the functional

$$h := \begin{cases} g & \text{on } L_k; \\ g' & \text{on } L_k^\perp, \end{cases}$$

satisfies to conditions: $\|f-h\|<\varepsilon$, $h$ is an epimorphism on $L_k$ and $L_k^\perp \cap L_{k'}$ separately. Without loss of generality it is possible to suppose, that $\|h\|=1$. Let $x \in L_k$ and $y \in L_k^\perp \cap L_{k'}$ be such that $h(x)=h(y)=a$. Then $h(x-y)=0$, and by [27] (see also [19, 2.1.4]),

$$a^*a = \langle h(x), h(x)\rangle \le \langle x, x\rangle, \qquad a^*a = \langle h(y), h(y)\rangle \le \langle y, y\rangle.$$

By Lemma 5.5.4 it is possible to find $b$, such that $\|b\| \le 1$ and $z=(x-y)b$ satisfies $\langle z, z\rangle = a^2$. Thus $h(z)=h((x-y)b)=0$, and as $\|z\|=1$, $\|f(z)\|<\varepsilon$. □

**Remark 5.5.6** Let $i$ and $i'$ be enclosures admitting adjoint and respecting inner product, and for the correspondent projections $q=ii^*$ and $q'=i'i'^*$ we have $\|qq'\|<\varepsilon$, $\|q'q\|<\varepsilon$. Let us remark, that $qq'=ii^*i'i'^*$, where $i$ is an isometric enclosure and $i'^*$ is an epimorphism with norm 1. Therefore, the indicated inequalities are equivalent to $\|i^*i'\|<\varepsilon$, $\|i'^*i\|<\varepsilon$. Then the map $I := (i, i') : l_2(A)\oplus l_2(A) \to l_2(A)$ is also an enclosure, admitting adjoint $I^*(x) = (i^*(x), i'^*(x))$. Really, $I^*$, given by this formula, is continuous and

$$\langle I(x,y), z\rangle = \langle i(x)+i'(y), z\rangle = \langle x, i^*(z)\rangle + \langle y, i'^*(z)\rangle = \langle (x,y), I^*(z)\rangle.$$

Also,

$$I^*I(x,y) = (i^*(ix+i'y), i'^*(ix+i'y)) = (x,y) + (i^*i'y, i'^*ix),$$

so that

$$\|\mathrm{Id} - I^*I\| < 2\varepsilon \tag{84}$$

and $I^*I$ is invertible. Therefore, $I$ is an enclosure. Let us remark, that for this reasoning we need to have $\varepsilon < 1/2$.

**Theorem 5.5.7** *Let algebra $A$ have the property (K). Then the group* $\mathrm{GL}(A)$ *is norm contractible.*

**Proof:** As above, it is necessary to prove a statement, similar to Lemma 5.2.3. In the present situation we argue as follows. Let $F_1$ be the first row (i. e., a functional) of matrix $F$ with the respect to the standard decomposition $l_2(A)$. Let us remark, that any vector from $l_2(A)$ with any beforehand given exactness $\delta$ belongs to $L_n$ for a sufficient large $n=n(\varepsilon)$. Hence, applying the property (K), it is possible at once to suppose, that $x \in L_n$. Really, let $f(x) < \varepsilon/2$, $\langle x,x\rangle = a \le 1$, $\|f\|=1$. Let us find a number $n$, such that $\|(1-p_n)x\| < \varepsilon/4$, $x' := p_n x$. Then $\langle x', x'\rangle \le \langle x,x\rangle = a$ and

$$\|\alpha\| \le \frac{\varepsilon}{4}, \qquad \text{if} \quad \alpha := \left(\langle x,x\rangle - \langle x',x'\rangle\right)^{1/2}.$$

Let us put $y := x' + e_{n+1}\alpha$. Then $\langle y,y\rangle = a$, $y \in L_{n+1}$ and

$$\|f(y)\| \le \|f(x)\| + \|f(x-x')\| + \|f(x'-y)\| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.$$

By applying the property (K) infinitely many times with constants, decreasing as geometrical progression, we can find a sequence of vectors $x_i \in l_2(A)$, satisfying to conditions

$$x_i \in M_i, \qquad l_2(A) = M_1 \oplus M_2 \oplus \dots, \qquad M_i = \{\,(0,\dots,0,a_{k(i)},\dots,a_{k(i+1)-1},0,\dots)\,\}, \tag{85}$$

$$\langle x_i, x_i\rangle = \alpha_i, \qquad \alpha_i \text{ — approximate unit for } A, \tag{86}$$

$$\|F_1(x_i)\| < \frac{\varepsilon}{2}\cdot\frac{1}{2^i}. \tag{87}$$

Let us remark, that for $k > k(i)$ : $\omega_i = \alpha_k^{1/2} d(i,k)$, $\|d(i,k)\| \le 1$. Therefore, similar to reasonings above, the map

$$J_1 : l_2(A) \to l_2(A), \qquad (a_1, a_2, \dots) \mapsto \sum_j \sum_i x_{k(i,j)} d(i, k(i,j)) a_j,$$

where

$$k(1,1); k(1,2), k(2,1); k(1,3), k(2,2), k(3,1); \dots$$

is some increasing sequence, will be an enclosure admitting an adjoint and preserving the inner product. If we denote $H_1 := \operatorname{Im} J_1$, then by (87)

$$\|F_1|_{H_1}\| < \frac{\varepsilon}{2}.$$

Let $G_1$ be the orthogonal complement to the image of the first copy of $A$ under $J_1$. Let $m(2) > m(1) := 1$ be so large, that $\|(1 - p_{m(2)})Fy(1)\| < \varepsilon/2$, where $y_1 := J_1(\alpha_1^{1/2}, 0, \ldots)$. Let us denote through $F_2$ the restriction of the $m(2)$-th row of the matrix $F$ on $G_1 \cong l_2(A)$, and let us find by the same algorithm a new enclosure $J_2$, such that its image equals to $H_2$ and there exists a correspondent submodule $G_2 \subset H_2$, and

$$\|F_2|_{H_2}\| < \frac{\varepsilon}{2^2}.$$

Let $m(3) > m(2)$ be so large, that

$$\|(1 - p_{m(3)})Fy_i\| < \frac{\varepsilon}{2^3 \cdot 2^i}, \quad i = 1, 2, \qquad y_2 := J_2(\alpha_2^{1/2}, 0, \ldots),$$

$$\|(1 - p_{m(3)})y_i\| < \frac{\varepsilon}{2^3 \cdot 2^i}, \quad i = 1, 2.$$

And so on. We obtain sequences $m(j)$ and $y_i$ such, that

$$\|(1 - p_{m(j)})Fy_i\| < \frac{\varepsilon}{2^j \cdot 2^i}, \qquad i = 1, \ldots, j-1, \tag{88}$$

$$\|(1 - p_{m(j)})y_i\| < \frac{\varepsilon}{2^j \cdot 2^i}, \qquad i = 1, \ldots, j-1, \tag{89}$$

$$\|Q_{m(j)}Fy_i\| < \frac{\varepsilon}{2^j \cdot 2^i}, \qquad j = 1, \ldots, i, \tag{90}$$

Again, using $\omega_j$, we can arrange an enclosure $J$ of the module $l_2(A)$ on a submodule $H$ of the linear span of $y_i$ and an enclosure $J'$ of the module $l_2(A)$ on the submodule $H' := \bigoplus_j E_{m(j)}$. Since these modules are $\varepsilon$-ortogonal, there exist mutually vanishing projectors $p$ and $p'$ on them. More precisely, let us remark first of all, that the enclosure $J$ admits adjoint. Really, the image of each vector $(a_1, a_2, \ldots)$ under $J_1$ is a sum of the form

$$\sum_j \sum_i v_{ij} a_j, \qquad \langle v_{ij}, v_{ij} \rangle = \omega_i^2, \qquad v_{ij} \in M_{k(i,j)}.$$

For construction of the higher $J_s$ the correspondent $v_{ij}^s$ will lay again in direct sums of modules $M_r$, and for $v_{i1}^s$ these sets are not intersecting. We can apply Lemma 5.2.2. The operator $J$ will is defined by the formula

$$J : (a_1, a_2, \ldots) \mapsto \sum_s \sum_i v_{i1}^s a_s, \qquad \sum_i v_{i1}^s a_s = y_s \mu_s a_s. \tag{91}$$

Hence, there are the orthoprojections $q$ and $q'$ on $H$ and $H'$, correspondently. Let us remark, that from this reasoning we can make the following refinement. We, in particular, have shown, that for any $J_s$ and any $m$ there exists no more than one $r$, such that $Q_m J_s Q_r \neq 0$. Therefore, throwing out if necessary, a finite number of canonical summands in $l_2(A)$ and restricting $J_s$ on the remaining module, we can suppose, that

$$Q_{m(j)} J_s = 0, \qquad j = 1, \ldots, s-1, \tag{92}$$

$$Q_{m(j)} y_i = 0, \qquad j = 1, \ldots, i, \tag{93}$$

Also, $\|qq'\| < \varepsilon$, $\|q'q\| < \varepsilon$. Really, let us consider a vector of the form

$$x = \sum_s \sum_i v_{i1}^s a_s = \sum_s y_s \mu_s a_s, \qquad \|\sum_s a_s^* a_s\| \leq 1.$$

It is necessary to show, that $\|q'x\| < \varepsilon$. It follows from (89, 93):

$$\|q'x\| = \left\| \sum_j Q_{m(j)} \sum_s \sum_i v_{i1}^s a_s \right\| \leq \sum_s \left\| \sum_{j>s} Q_{m(j)} \left( \sum_i v_{i1}^s a_s \right) \right\| + \sum_s \sum_{j \leq s} \left\| Q_{m(j)} \left( \sum_i v_{i1}^s a_s \right) \right\| \leq$$

$$\leq \sum_s \left\|(1-p_{m(s)})y_s\mu_s a_s\right\| + \sum_s \sum_{j\leq s} 0 \leq \sum_s \frac{\varepsilon}{2^s} = \varepsilon.$$

Since the projections $q$ and $q'$ are self-adjoint, we obtain and second estimation.

Then by Remark 5.5.6 $H\widetilde{\oplus}H'$ is the image of an enclosure, admitting adjoint, and by [23] (see also [19, Theorem 2.3.3]) the decomposition $l_2(A) = H\widetilde{\oplus}H' \oplus (H^\perp \cap H'^\perp)$ takes place. Let us denote through $p$ and $p'$ projections on $H$ and $H'$ correspondent to this decomposition, so that $pp' = p'p = 0$. Thus

$$\|p-q\| < 3\varepsilon, \qquad \|p'-q'\| < 3\varepsilon, \qquad \|p\| < 1+3\varepsilon < 2, \qquad \|p'\| < 1+3\varepsilon < 2. \tag{94}$$

Really, let $x \in H\widetilde{\oplus}H'$, $\|x\| = 1$, so that $x = II^*Iy$, and by (84) $\|Iy\| \leq 2(1+\varepsilon)$,

$$\|(p-q)x\| = \|(p-q)(ii^*Iy + i'i'^*Iy)\| = \|(p-q)(q+q')Iy\| = \|-qq'Iy\| \leq 2\varepsilon(1+\varepsilon) < 3\varepsilon.$$

Besides, $\|p'Fp\| < 7\|F\|\varepsilon$. In fact,

$$\|p'Fp\| = \|(p'-q')Fp + q'Fp\| < 3\varepsilon\|F\| + \|q'Fp\|,$$

and by (94) it is sufficient to prove, that for $x \in H$, $\|x\| \leq 1$, holds $\|q'Fx\| < 2\varepsilon$. Any such vector $x$ can be presented as

$$\sum_s \sum_i v_{i1}^s a_s = \sum_s y_s\mu_s a_s, \qquad \|\sum_s a_s^* a_s\| \leq 1.$$

Then

$$\|q'Fx\| = \left\|\sum_j Q_{m(j)} \sum_s \sum_i F v_{i1}^s a_s\right\| \leq \sum_s \sum_i \left\|\sum_{j>s} Q_{m(j)} F v_{i1}^s\right\| + \sum_s \sum_{j\leq s} \left\|Q_{m(j)} F \left(\sum_i v_{i1}^s a_s\right)\right\| \leq$$

$$\leq \sum_s \left\|(1-p_{m(s)})Fy_s\mu_s a_s\right\| + \sum_s \sum_{j\leq s} \frac{\varepsilon}{2^j \cdot 2^s} \leq \sum_s \frac{\varepsilon}{2^s} + \varepsilon = 2\varepsilon.$$

Let us remark, that similar statement we can receive not only for one operator $F$ (actually for two: $F$ and $\mathrm{Id}$), but for a finite collection (vertices of a simplicial complex): $F^{(1)}, \ldots, F^{(N)}$. For this purpose it is necessary to conduct reasonings for $F = F^{(1)}$ with a constant $\varepsilon$ and to receive projections $P_1$ and $P_1'$. Then apply algorithm To $P_1'F^{(2)}P_1$ and receive projections $P_2'$ and $P_2$, such that

$$P_1'P_2' = P_2'P_1' = P_2', \quad P_1P_2 = P_2P_1 = P_2, \quad P_2P_1 = P_1P_2 = 0, \quad \|P_2'F^{(1)}P_2\| < \varepsilon, \quad \|P_2'F^{(2)}P_2\| < \varepsilon.$$

And so on. This completes the proof, since now it is possible to apply the Neubauer homotopy. □

Let us complete this paragraph by a discussion on the following example, which is reassuring in relation to size of the class (K).

**Example 5.5.8** In the notation of Example 2.5.6 in [19] let us consider the functional

$$f : l_2(A) \to A, \qquad f : (a_1, a_2, \ldots) \mapsto \sum_i u_i a_i.$$

It has the property (K). Really,

$$\|f(z_n)\| < \frac{1}{n}, \qquad \|z_n\| = 1, \qquad z_n := \Bigg(\underbrace{\frac{1}{n}, \ldots, \frac{1}{n}}_{n^2}, 0, 0, \ldots\Bigg).$$

## 5.6 Neubauer type homotopy

In this section we describe, how to modify the homotopy from [26] for our purposes. Though we work with completely other objects, the construction in [26] is so universal, that proofs can be transferred almost without modifications.

**Lemma 5.6.1** *Let $\mathcal{M}$ be a Hilbert $A$-module , $X$ be a topological space, $T : X \to \mathcal{G} = \mathcal{G}(\mathcal{M})$ be a continuous map, and $P$ and $P'$ be projections from $\mathcal{E} = \mathcal{E}(\mathcal{M})$, such that*

$$PP' = P'P = 0, \qquad H_0 : P'\mathcal{M} \cong P\mathcal{M}, \qquad H_0P' \in \mathcal{E}, \quad H_0^{-1}P \in \mathcal{E}, \qquad P'T(x)P = 0 \quad \forall\, x \in X.$$

*Then there is a homotopy $T \sim D$ in $\mathcal{G}$, such that*

$$PD(x) = D(x)P = P \quad \forall\, x \in X.$$

**Proof:** Le us put $Q := \mathrm{Id} - P$, $Q' := \mathrm{Id} - P'$,

$$\mathcal{P}(x) := T(x)PT(x)^{-1}Q', \qquad \mathcal{Q}(x) := Q' - \mathcal{P}(x).$$

Then $\mathcal{P}(x)$ is a projection on $T(x)P\mathcal{M}$ and there is the decomposition into projections $\mathrm{Id} = \mathcal{Q}(x) + \mathcal{P}(x) + P'$, and $\mathcal{Q}(x)$, $\mathcal{P}(x)$ and $P'$ are mutual vanishing for each $x$. Really,

$$Q'T(x)P = (\mathrm{Id} - P')T(x)P = T(x)P, \quad PT(x)^{-1}Q'T(x)P = P, \quad T(x)P\mathcal{M} \subset \mathcal{P}(x)\mathcal{M} \subset T(x)P\mathcal{M},$$

$$\mathcal{P}(x)\mathcal{P}(x) = T(x)PT(x)^{-1}(\mathrm{Id} - P')T(x)PT(x)^{-1}(\mathrm{Id} - P') =$$
$$= T(x)PT(x)^{-1}T(x)PT(x)^{-1}(\mathrm{Id} - P') = T(x)PT(x)^{-1}(\mathrm{Id} - P') = \mathcal{P}(x),$$
$$\mathcal{Q}(x) + \mathcal{P}(x) + P' = Q' - \mathcal{P}(x) + P(x) + P' = \mathrm{Id}\,,$$
$$\mathcal{P}(x)P' = T(x)PT(x)^{-1}(\mathrm{Id} - P')P' = 0, \qquad P'\mathcal{P}(x) = P'T(x)PT(x)^{-1}(\mathrm{Id} - P') = 0,$$

Hence, $\mathcal{P}(x) + P'$ is a projection, whence $\mathcal{Q}(x) = \mathrm{Id} - (\mathcal{P}(x) + P')$ is a projection too.

Let us define

$$H = -H_0P' + H_0^{-1}P,$$

then, as $P'P = PP' = 0$, $P'H_0 = PH_0^{-1} = 0$ and

$$H^2 = (-H_0P' + H_0^{-1}P)(-H_0P' + H_0^{-1}P) = -(P' + P),$$

$$HP'H = (-H_0P' + H_0^{-1}P)P'(-H_0P' + H_0^{-1}P) = -H_0P'H_0^{-1}P = -H_0H_0^{-1}P = -P,$$
$$Q'HP = HP - P'HP = H_0^{-1}P - H_0^{-1}P = 0,$$
$$Q'HT(x)^{-1}\mathcal{P}(x) = Q'HT(x)^{-1}T(x)PT(x)^{-1}Q' = 0,$$
$$\mathcal{P}(x)T(x)HP' = T(x)PT(x)^{-1}Q'T(x)HP' = T(x)PT(x)^{-1}(1 - P')T(x)(-H_0P') =$$
$$= T(x)PT(x)^{-1}(1 - P')T(x)P(-H_0P') = T(x)PT(x)^{-1}T(x)P(-H_0P') = T(x)P(-H_0P') = T(x)HP'.$$

Let's assume

$$G(x) := HT(x)^{-1}\mathcal{P}(x) + T(x)HP',$$

Then

$$G(x)^2 = (HT(x)^{-1}\mathcal{P}(x) + T(x)HP')(HT(x)^{-1}\mathcal{P}(x) + T(x)HP') =$$
$$= HT(x)^{-1}\mathcal{P}(x)HT(x)^{-1}\mathcal{P}(x) + T(x)(-P)T(x)^{-1}\mathcal{P}(x) + HT(x)^{-1}\mathcal{P}(x)T(x)HP' + T(x)HP'T(x)HP' =$$
$$= HT(x)^{-1}T(x)PT(x)^{-1}Q'HT(x)^{-1}\mathcal{P}(x) + T(x)(-P)T(x)^{-1}T(x)PT(x)^{-1}Q' +$$
$$+ HT(x)^{-1}T(x)HP' + T(x)HP'T(x)HP' =$$
$$= 0 - T(x)PT(x)^{-1}Q' - P' + T(x)(-H_0P')T(x)(-H_0P') = 0 - \mathcal{P}(x) - P' + 0 = -(P' + \mathcal{P}(x)),$$
$$G(x)\mathcal{Q}(x) = 0, \qquad \mathcal{Q}(x)G(x) = (Q' - \mathcal{P}(x))(HT(x)^{-1}\mathcal{P}(x) + T(x)HP') =$$
$$= Q'HT(x)^{-1}\mathcal{P}(x) + (\mathrm{Id} - P')T(x)(-PH_0P') - \mathcal{P}(x)HT(x)^{-1}\mathcal{P}(x) - \mathcal{P}(x)T(x)HP' =$$
$$= 0 + T(x)HP' - (T(x)PT(x)^{-1}Q')(-H_0P' + H_0^{-1}P)T(x)^{-1}(T(x)PT(x)^{-1}Q') - T(x)HP' =$$
$$= (T(x)PT(x)^{-1}Q'H_0[P'P]T(x)^{-1}Q') - T(x)PT(x)^{-1}[Q'P']H_0^{-1}PT(x)^{-1}(T(x)PT(x)^{-1}Q') = 0.$$

Hence, for

$$U(s, x) := \mathcal{Q}(x) + (1 - s)(\mathcal{P}(x) + P') + sG(x)$$

we obtain

$$U(s,x)^{-1} = \mathcal{Q}(x) + \frac{1}{s^2+(1-s)^2}[(1-s)(\mathcal{P}(x)+P') - sG(x)].$$

Therefore, $U(s,x)T(x)$ defines a homotopy in $\mathcal{G}$

$$U(0,x)T(x) = \mathrm{Id} \circ T(x) \sim U(1,x) \circ T(x).$$

Thus, as $\mathcal{P}(x)T(x)P = T(x)P$,

$$\begin{aligned} U(1,x)T(x)P &= \mathcal{Q}(x)\mathcal{P}(x)T(x)P + G(x)\mathcal{P}(x)T(x)P = \\ &= 0 + HT(x)^{-1}\mathcal{P}(x)T(x)P = HT(x)^{-1}T(x)P = HP. \end{aligned}$$

Since $H(P+P') = (P+P')H = H$, for

$$V(s) := QQ' + (1-s)(P+P') - sH$$

we have

$$V(s)^{-1} = QQ' + \frac{1}{s^2+(1-s)^2}[(1-s)(P+P') + sH].$$

Besides, $V(0) = QQ' + P + P' = \mathrm{Id}$ . Therefore, the following homotopy is defined

$$R(x) := V(1)\,U(1,x)\,T(x) \sim U(1,x)\,T(x) \qquad \text{в} \qquad C(X,\mathcal{G}(\mathcal{M})),$$

and

$$\begin{aligned} R(x)P &= V(1)\,U(1,x)\,T(x)P = \\ &= V(1)\,HP = QQ'HP - H^2P = 0 + (P+P')\,P = P. \end{aligned}$$

Let us put

$$R(s,x) := R(x) - sPR(x)Q.$$

Let for some $e \in \mathcal{M}$ the equality $R(s,x)e = 0$ hold. Then

$$0 = R(s,x)e = R(x)(P+Q)e - sPR(x)Qe = Pe + R(x)Qe - sPR(x)Qe, \quad 0 = QR(s,x)e = QR(x)Qe.$$

Let $f = PR(x)Qe$, so that $f = Pf$. Then

$$PR(x)(Qe - Pf) = f - Pf = 0, \qquad QR(x)Pf = 0.$$

Therefore, $R(x)(Qe - Pf) = 0$, $Qe = Pf = f = 0$ and $PR(s,x)e = Pe = 0$, $e = 0$. Also

$$R(x)\mathcal{M} = \mathcal{M}, \quad R(x)P = P, \quad QR(x)Q\mathcal{M} = QR(x)(1-P)\mathcal{M} = QR(x)\mathcal{M} = Q\mathcal{M}.$$

Therefore, with the respect to the decomposition $\mathcal{M} = P\mathcal{M}\widetilde{\oplus}Q\mathcal{M}$ the operator $R(s,x)$ has the matrix

$$\begin{pmatrix} \mathrm{Id} & \star \\ 0 & QR(x)Q \end{pmatrix}, \qquad QR(x)Q\mathcal{M} = Q\mathcal{M},$$

hence, $R(s,x)$ is an epimorphism, and $R(s,x) \in \mathcal{G}(\mathcal{M})$ as an epimorphism without kernel. It is sufficient to put $D(x) := R(1,x)$. □

**Lemma 5.6.2** *Let $\mathcal{M}$ be a Hilbert A-module , Xbe a compact set, $T : X \to \mathcal{G}(\mathcal{M})$ be a continuous map with $0 < \varepsilon < \min \|T(x)^{-1}\|^{-1}$,, and $P$ and $P'$ be such projections from $\mathcal{E} = \mathcal{E}(\mathcal{M})$, that*

$$\|P'T(x)P\| \leq \varepsilon \quad \forall\, x \in X.$$

*Then there exists a homotopy $S(s,x)$ in $\mathcal{G}$, such that*

$$S(0,x) = T(x), \qquad P'S(1,x)P = 0 \quad \forall\, x \in X.$$

**Proof:** Let us put $S(s,x) := T(x) - sP'T(x)P$. Since

$$\|S(s,x) - T(x)\| \leq \varepsilon,$$

$S(s,x) \in \mathcal{G}(\mathcal{M})$. □

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